Semitotal Blocks in Fuzzy Graphs

Mohiddin Shaw[1], B. Narayana[2], D. Srinivasulu[3] and A. Sudhakaraiah[4]

[1]Department of Basic Science and Humanities,
Narasaraopeta Engineering College, Narasaraopet, A.P., India.

[2]Department of Basic Science and Humanities,
St. Mary's Women's Engineering College, Guntur, A.P. India.

[3]Department of Basic Science and Humanities,
NRI Institute of Technology, Vijayawada, A.P. India.

[4]Department of Mathematics,
Sri Venkateswara University, Tirupathi, A.P., India

ABSTRACT – This paper is a study of semitotal blocks in fuzzy graphs. During the study some interesting results regarding the semitotal blocks in fuzzy graphs are obtained. It is observed that when 'B' is a block of a given fuzzy graph G:(V, σ, μ), then degree of the vertex B in semi total block fuzzy graph $T_{STB}F(G)$ is equal to the sum of the membership grade of the vertices in that block and the number of edges in $T_{STB}F(G)$ related to block B is $|V(B)|$ with membership grade minimum of σ(u), σ(B). Finally, the result is $|E_{STB}F(G)| = |EF(G)| + |V(B_1)| + |V(B_2)| + \ldots + |V(B_k)|$.

Keywords:- Graph, fuzzy graph, Degree of vertex in fuzzy graphs, Semitotal-block in fuzzy graph.

---

1. INTRODUCTION

A finite graph G = (V, E) consists of a finite nonempty set of objects, $V = \{v_1, v_2, \ldots\}$ called vertices and another finite set, $E = \{e_1, e_2, \ldots\}$ of elements called edges such that each edge e_k is identified with an unordered pair $\{v_i, v_j\}$ of vertices. An edge associated with a vertex pair $\{v_i, v_i\}$ is called a self- loop. The number of edges associated with the vertex is the degree of the vertex, and d(v) denotes the degree of the vertex v. If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges (or) multiple edges. A graph that has does not have self-loop or parallel edges called a simple graph. Two vertices are said to be adjacent if they are the end vertices of the same edge. A finite alternating sequence of vertices and edges (no repetition of edge allowed) beginning and ending with vertices such that each edge is incident with the vertices preceding and following it, is called a walk and an open walk in which no vertex appears more than once, is called a path. A graph said to be connected if there is at least one path between every pair of vertices in G, otherwise it is called disconnected. In a connected graph, a vertex whose removal disconnects the graph is called a cut-vertex.

In this paper, we consider only finite simple graphs.

Rosenfeld [11] considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. Rosenfeld has obtained the fuzzy analogues of several basic graph-theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties. Later on several authors Nagoor Gani and Radha [10], M.S. Sunitha and Mini Tom [7] were studied about fuzzy graphs. Fuzzy graph theory is now finding numerous applications in modern science and technology especially in the fields of information theory, neural networks, expert systems, cluster analysis, medical diagnosis, control theory etc.

2. FUZZY GRAPHS

In this section some necessary literature is collected which presents the basic concept of this paper.

2.1 Definition: A fuzzy graph(f-graph) [7] is a triplet G : (V ,σ,μ) where V the vertex set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ such that $\mu(u, v) \le \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.

2.2 Note: (i) We assume that V is finite and non empty, μ is reflexive and symmetric. (ii) In all the examples σ is chosen suitably. Also we denote the underlying crisp graph [7] by $G^* :(\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. Here we assume $\sigma^* = V$.

2.3 Definition: A fuzzy graph $H : (V, \tau, \nu)$ is called a partial fuzzy sub-graph of $G : (V, \sigma, \mu)$ if $\tau(u) \leq \sigma(u)$ for all $u, v \in \tau^*$ and $\nu(u, v) \leq \mu(u, v)$ for all edges $(u, v) \in \nu^*$. In particular, we call $H : (V, \tau, \nu)$ a fuzzy sub-graph of $G: (V, \sigma, \mu)$.

2.4 Definition: If $\tau(u) = \sigma(u)$ $\forall u \in \tau^*$ and $\nu(u, v) = \mu(u, v)$ $\forall (u, v) \in \nu^*$ and if in addition $\tau^* = \sigma^*$, then H is called a spanning fuzzy sub-graph of G.

2.5 Definition: A weakest edge of a fuzzy graph $G: (V, \sigma, \mu)$ is an edge with least membership value.

2.6 Definition: A path P of length n is a sequence of distinct nodes $u_0, u_1, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1,2,3 \ldots n$ and the degree of membership of a weakest edge in the path is defined as its strength.

2.7 Definition: If $u_0 = u_n$ and $n \geq 3$, then P is called a cycle and a cycle P is called a fuzzy cycle (f-cycle) if it contains more than one weakest edge.

2.8 Definition: The strength of connectedness between two nodes u and v is defined as the maximum of the strengths of all paths between u and v and is denoted by $CONN_G(u, v)$.

A $u - v$ path P is called a strongest $u - v$ path if its strength equals $CONN_G(u, v)$.

2.9 Definition: A fuzzy graph $G : (V, \sigma, \mu)$ is connected if for every u, v in σ^*, $CONN_G(u, v) > 0$.
Throughout this paper, it is assumed that fuzzy graph G is connected.

2.10 Definition [7]**:** An edge of a fuzzy graph is called strong if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted and a $u - v$ path is called a strong path if it contains only strong edges. In a fuzzy graph the strongest path need not be a strong path and a strong path need not be the strongest path [7].

2.11 Definition: An edge (u, v) is a fuzzy bridge (f-bridge) of G if deletion of (u, v) reduces the strength of connectedness between some pair of nodes [7]. Equivalently, (u, v) is a fuzzy bridge if and only if there exist x, y such that (u, v) is an edge on every strongest $x - y$ path.

2.12 Definition: A node is a fuzzy cutnode (f-cutnode) of G if removal of it reduces the strength of connectedness between some other pair of nodes [7]. Equivalently, w is a fuzzy cutnode if and only if there exist u, v distinct from w such that w is on every strongest $u - v$ path. A connected fuzzy graph $G: (V, \sigma, \mu)$ is a block if G has no fuzzy cutnode.

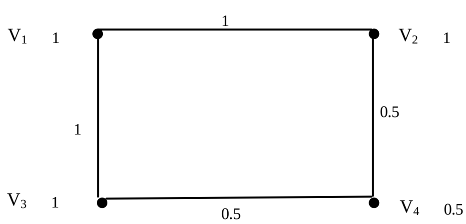
2.13 Definition: A connected fuzzy graph $G : (V, \sigma, \mu)$ is a fuzzy tree (f-tree) if it has a spanning fuzzy sub-graph $F : (V, \sigma, \nu)$, which is a tree, where for all edges (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$.

3. BLOCKS IN FUZZY GRAPHS

This section explains blocks in fuzzy graphs with some examples which will render better comprehension. The definition of ring sum of two fuzzy graphs is introduced with required examples. It is also observed that the ring sum of two fuzzy graphs is also a fuzzy graph.

3.1Definition: A connected non-trivial fuzzy graph having no fuzzy cut vertex is a block in fuzzy graph.

3.2 Example: Consider the fuzzy graph $G:(V, \sigma, \mu)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $\sigma(v_1) = 1$, $\sigma(v_2) = 1$, $\sigma(v_3) = 1$, $\sigma(v_4) = 0.5$ and $\mu(v_1, v_2) = 1$, $\mu(v_2, v_4) = 0.5$, $\mu(v_3, v_4) = 0.5$, $\mu(v_1, v_3) = 1\}$. In this fuzzy graph there are no fuzzy cut vertices. Therefore, this fuzzy graph is a block in fuzzy graph $G:(V, \sigma, \mu)$



3.3 Note: A block of a fuzzy graph is a fuzzy sub-graph that is a block and is maximal with respect to the property in Definition 3.1.

3.4 Example: The set of all Blocks in fuzzy graph G is denoted by SBF(G)

3.5 Example: Consider the following fuzzy graph $G:(V, \sigma, \mu)$. Where $V = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9, V_{10}, V_{11}\}$ with $\sigma(v_1) = 0.8$, $\sigma(v_2) = 1$, $\sigma(v_3) = 0.7$, $\sigma(v_4) = 0.8$, $\sigma(v_5) = 0.8$, $\sigma(v_6) = 0.8$, $\sigma(v_7) = 0.8$, $\sigma(v_8) = 0.8$, $\sigma(v_9) = 0.7$, $\sigma(v_{10}) = 0.8$, $\sigma(v_{11}) = 0.8$ and $\mu(v_1, v_2) = 0.4$, $\mu(v_2, v_3) = 0.2$, $\mu(v_1, v_3) = 0.2$, $\mu(v_3, v_4) = 0.5$, $\mu(v_3, v_5) = 0.2$, $\mu(v_3, v_6) = 0.2$, $\mu(v_5, v_6) = 0.5$, $\mu(v_6, v_7) = 0.2$, $\mu(v_7, v_8) = 0.5$, $\mu(v_8, v_9) = 0.2$, $\mu(v_8, v_{11}) = 0.2$, $\mu(v_8, v_{10}) = 0.2$, $\mu(v_{10}, v_{11}) = 0.2\}$.

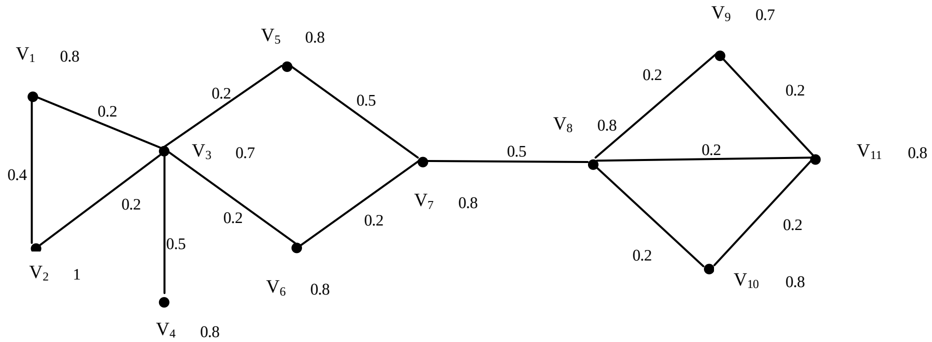


Fig-3.5 G:(V, σ, μ)

3.6 Example: (i). Consider the fuzzy graph $G:(V, \sigma, \mu)$ in figure -3.5(above). There are five blocks in fuzzy graph $G:(V, \sigma, \mu)$. They are $B_1F(G)$, $B_2F(G)$, $B_3F(G)$, $B_4F(G)$, $B_5F(G)$. These blocks are shown in figure 2. Where $\mathbf{B_1F(G)}$ is a block having vertex set $\{V_1, V_2, V_3\}$ with $\sigma(v_1) = 0.8$, $\sigma(v_2) = 1$, $\sigma(v_3) = 0.7$ and $\mu(v_1, v_2) = 0.4$, $\mu(v_2, v_3) = 0.2$, $\mu(v_1, v_3) = 0.2$, $\mathbf{B_2F(G)}$ is a block having vertex set $\{V_3, V_4\}$ with $\sigma(v_3) = 0.7$, $\sigma(v_4) = 0.8$ and $\mu(v_3, v_4) = 0.5$, $\mathbf{B_3F(G)}$ is a block having the vertex set $\{V_3, V_5, V_6, V_7\}$ with $\sigma(v_3) = 0.7$, $\sigma(v_5) = 0.8$, $\sigma(v_6) = 0.8$, $\sigma(v_7) = 0.8$, and $\mu(v_3, v_5) = 0.2$, $\mu(v_3, v_6) = 0.2$, $\mu(v_5, v_6) = 0.5$, $\mu(v_6, v_7) = 0.2$, $\mathbf{B_4F(G)}$ is block in fuzzy graph having the vertex set $\{V_7, V_8\}$ with $\sigma(v_7) = 0.8$, $\sigma(v_8) = 0.8$, and $\mu(v_7, v_8) = 0.5$, $\mathbf{B_5F(G)}$ is block in fuzzy graph having the vertex set $\{V_8, V_9, V_{10}, V_{11}\}$ with $\sigma(v_8) = 0.8$, $\sigma(v_9) = 0.7$, $\sigma(v_{10}) = 0.8$, $\sigma(v_{11}) = 0.8$ and $\mu(v_8, v_9) = 0.2$, $\mu(v_8, v_{11}) = 0.2$, $\mu(v_8, v_{10}) = 0.2$, $\mu(v_{10}, v_{11}) = 0.2\}$.

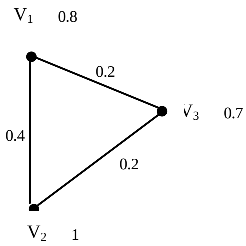


Fig-3.6 $B_1F(G)$

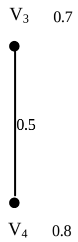


Fig-3.6 $B_2F(G)$

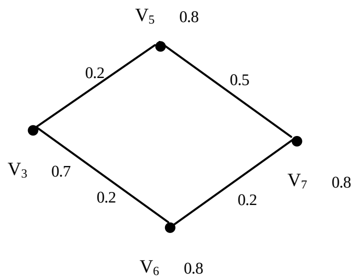


Fig-3.6 $B_3F(G)$

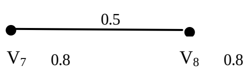


Fig-3.6 $B_4F(G)$

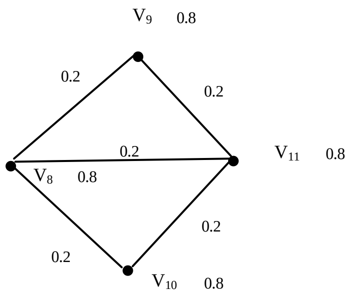


Fig-3.6 $B_5F(G)$

In this example, SBF(G) = { $B_1F(G)$, $B_2F(G)$, $B_3F(G)$, $B_4F(G)$, $B_5F(G)$}

3.7 Note: All blocks in the fuzzy graph are fuzzy sub-graphs of the given fuzzy graph

Here an operation on fuzzy graphs is introduced.

3.8 Definition: Let $G_1:(V_1, \sigma_1, \mu_1)$, $G_2:(V_2, \sigma_2, \mu_2)$ be two fuzzy graphs. Then the ring sum of these two fuzzy graphs is a fuzzy graph $G:(V, \sigma, \mu)$ defined as $V = V_1 \cup V_2$,

$$\sigma(u) = \begin{cases} \sigma_1(u) & if \quad u \in V_1 \\ \sigma_2(u) & if \quad u \in V_2 \\ \sigma_1(u) \ \wedge \ \sigma_2(u) & if \quad u \in V_2 \cap V_2 \end{cases}$$ and $$\mu(\overline{uv}) = \begin{cases} \mu_1(\overline{uv}) & if \quad \overline{uv} \in (E_1 - E_2) \\ \mu_2(\overline{uv}) & if \quad \overline{uv} \in (E_2 - E_1) \\ 0 & if \quad \overline{uv} \in (E_1 \cap E_2) \end{cases}$$. Where $\overline{uv}$ denotes the edge between the vertex u and the vertex v.

3.9 Example: Let $G_1:(V_1, \sigma_1, \mu_1)$, $G_2:(V_2, \sigma_2, \mu_2)$ be two fuzzy graphs as shown below:

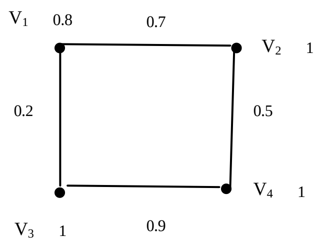


Fig-3.9 $G_1:(V_1, \sigma_1, \mu_1)$

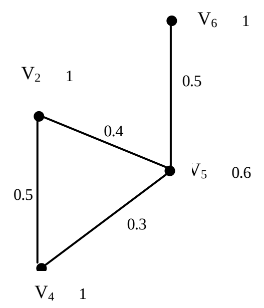


Fig-3.9 $G_2:(V_2, \sigma_2, \mu_2)$

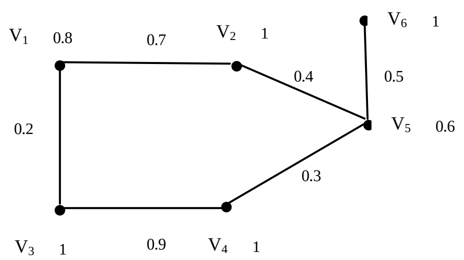


Fig-3.9 $G = G_1 \oplus G_2$

3.10 Note: The fuzzy graph obtained by the ring sum operation is also a fuzzy graph.

4. SEMI-TOTAL BLOCK FUZZY GRAPHS

This section gives the definition of semitotal block fuzzy graph $T_{STB}F(G)$ of a fuzzy graph and observation of some interesting results. Finally, it is proved that "$|E_{STB}F(G)| = |E F(G)| + |V(B_1)| + |V(B_2)| + \ldots + |V(B_k)|$".

4.1 Definition: The semi-total block fuzzy graph denoted by $T_{STB}F(G)$ of a fuzzy graph $G:(V_{STBFG}, \sigma_{STBFG}, \mu_{STBFG})$ is defined as the fuzzy graph having the vertices set $V_{STBFG} = V(F(G)) \cup B(F(G))$ and two fuzzy vertices are adjacent if they corresponds to two adjacent vertices of F(G) or one corresponds to a block B of F(G) and other to fuzzy vertex v of F(G) and the membership grade of those new vertices and edges are defined as follows:

$$\sigma_{STBFG}(u) = \begin{cases} \sigma_{FG}(u) & if \quad u \in \ V(F(G)) \\ \max \quad \{\sigma(u) & / \quad if \quad u \in B_i, 1 \le i \le n \end{cases}$$ and $$\mu_{STBFG}(\overline{UV}) = \begin{cases} \mu_{FG}(\overline{uv}\ ) \ if \quad \overline{uv} \in \ V \ F(G(\quad)) \\ \min\{ \ \sigma_{STBFG}(u), \quad \sigma_{STBFG}(B_i) \ \ , 1 \le i \le n\} \end{cases}$$

where B_i denotes the i^{th} block in Fuzzy Graph.

4.2 Example: Consider the following fuzzy graph $G:(V, \sigma, \mu)$, Where $V = \{V_1, V_2, V_3, V_4\}$ with $\sigma(v_1) = 0.8$,

$\sigma(v_2) = 0.9$, $\sigma(v_3) = 0.7$, $\sigma(v_4) = 0.8$, and $\mu(v_1, v_2) = 0.8$, $\mu(v_2, v_3) = 0.2$, $\mu(v_1, v_3) = 0.2$, $\mu(v_3, v_4) = 0.5$. Then **$B_1F(G)$** is a block having vertex set $\{V_1, V_2, V_3\}$ with $\sigma(v_1) = 0.8$, $\sigma(v_2) = 0.9$, $\sigma(v_3) = 0.7$ and $\mu(v_1, v_2) = 0.8$, $\mu(v_2, v_3) = 0.2$, $\mu(v_1, v_3) = 0.2$, **$B_2F(G)$** is a block having vertex set $\{V_3, V_4\}$ with $\sigma(v_3) = 0.7$, $\sigma(v_4) = 0.8$ and $\mu(v_3, v_4) = 0.5$.

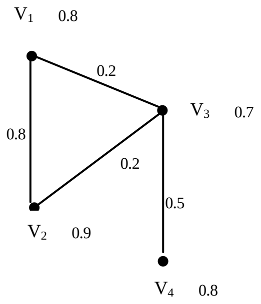


Fig-4.2 G:(V, σ, μ)

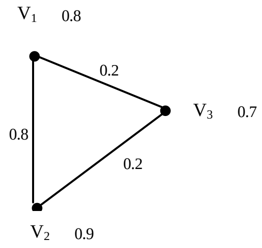


Fig-4.2 $B_1F(G)$

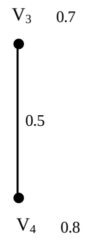


Fig-4.1, $B_2F(G)$

Then the semi total block fuzzy graph (STBFG) of above blocks is a fuzzy graph $G:(V_{STBFG}, \sigma_{STBFG}, \mu_{STBFG})$ having the vertex set $V_{STBFG} = \{V_1, V_2, V_3, V_4, B_1F(G), B_2F(G)\}$ with $\sigma(v_1) = 0.8$, $\sigma(v_2) = 0.9$, $\sigma(v_3) = 0.7$, $\sigma(v_4) = 0.8$, $\sigma(B_1F(G)) = 0.9$, $\sigma(B_2F(G)) = 0.8\}$ and $\mu(v_1, v_2) = 0.8$, $\mu(v_2, v_3) = 0.2$, $\mu(v_1, v_3) = 0.2$, $\mu(v_3, v_4) = 0.5$, $\mu(v_1, B_1F(G)) = 0.8$, $\mu(v_2, B_1F(G)) = 0.9$, $\mu(v_3, B_1F(G)) = 0.7$, $\mu(v_3, B_2F(G)) = 0.7$, $\mu(v_4, B_2F(G)) = 0.8$.

We denote $B_1F(G)$, $B_2F(G)$ as B_1, B_2 respectively. Then for this example $SBF(G) = \{ B_1F(G), B_2F(G)\} = B_1, B_2\}$

Now the semi total block fuzzy graph of $T_{STB}F(G)$ of the fuzzy graph shown in the above figure is

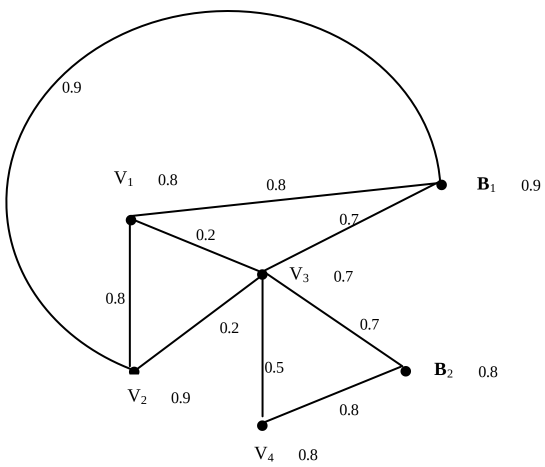


Fig-4.2,

4.3 Note: Since every edge of fuzzy graph G:(V, σ, μ) is also an edge in the semi total block fuzzy graph with same μ -value (membership grade), every vertex of fuzzy graph G:(V, σ, μ) in the semi total block fuzzy graph with same μ -value (membership grade), the fuzzy graph is a fuzzy sub-graph of semi total block fuzzy graph. That is G:(V, σ, μ) contained in $G:(V_{STBFG}, \sigma_{STBFG}, \mu_{STBFG})$.

4.4 Definition: Let G:(V, σ, μ) be a fuzzy graph and B be a block. Then a fuzzy edge in semi total block fuzzy graph is said to be fuzzy edge related to block B if one the end vertex of that fuzzy bridge is B.

4.5 Example: In example 4.2, the fuzzy edge v_2B_1, v_1B_1, v_3B_1 are the fuzzy edges related to block B_1, and v_3B_2, v_4B_2

are the fuzzy edges related to block B_2.

4.6 Lemma: Let B be a block of a given fuzzy graph G:(V, σ, μ). Then,

(i) degree of the vertex B in semi total block fuzzy graph $T_{STB}F(G)$ is equal to the sum of the membership grade of the vertices in that block.

(ii) The number of edges in $T_{STB}F(G)$ related to block B is $|V(B)|$ with membership grade minimum of σ(u), σ(B).

Proof: Let B be a block in fuzzy graph G:(V, σ, μ) **.** Suppose block B contains k vertices say, $\{v_1, v_2, \dots, v_k\} = V(B)$. Since for each v_i, $(1 \leq i \leq k)$ in block B, there is an edge between the vertex v_i and vertex B in $T_{STB}F(G)$ with membership grade as minimum of $\sigma(v_i)$ and σ(B). This is true for all vertices v_i, in block B. Thus the degree of the vertex B in $T_{STB}F(G)$ is equal to sum of the membership grade of the vertices in block B. (ii) Since block B contains k vertices, so there are k edges with one end vertex B in $T_{STB}F(G)$ with the membership grade as minimum of $\sigma(v_i)$ and σ(B)}. Therefore, the number of edges related to block B is $k = |V(B)|$.

4.7 Theorem [ ]: $|E_{STB}F(G)| = |E F(G)| + |V(B_1)| + |V(B_2)| + \dots + |V(B_k)|$.

Proof: Let G:(V, σ, μ) be a fuzzy graph and $B_1, B_2, \dots, B_k$, be the blocks in fuzzy graph G:(V, σ, μ). Suppose $T_{STB}F(G)$ the semitotal block graph of fuzzy graph G:(V, σ, μ). Write $SBF(G) = \{B_1F(G), B_2F(G), \dots B_mF(G)\}$. If 'e' is any fuzzy edge between two vertices in fuzzy graph G:(V, σ, μ), then 'e' is an edge in E(F(G)). Otherwise, the edge formed in between a vertex and a block $B_iF(G)$ in semitotal block fuzzy graph $T_{STB}F(G)$. Suppose B be a block with k vertices. Then by above lemma the number of edges in $T_{STB}F(G)$ related to block B is $|V(B)|$ with membership grade minimum of σ(u), σ(B). This is true for all blocks in fuzzy graph G:(V, σ, μ). Thus $\sum_{i=1}^{k} |v(B_i)|$ is the number of edges that are related to different blocks in fuzzy graph G:(V, σ, μ). Hence the number of edges in semitotal block fuzzy graph $T_{STB}F(G)$ is $|E_{STB}F(G)| = |E F(G)| + |V(B_1)| + |V(B_2)| + \dots + |V(B_k)|$.

5. ACKNOWLEDGEMENT

Authors owe their gratitude to the reviewers for their valuable suggestions for improvement of the quality of the paper. They are also grateful to their respective managements for the constant encouragement and providing them with the necessary facilities.

6. REFERENCES

[1] Arumugam S. and Ramachandran S. "Invitation to Graph Theory", Scitech Publications (India) Pvt. Ltd, Chennai, (2001).

[2] Balakrishnan. R and A.Selvam," k-neighborhood regular graphs", Proceedings of the National Seminar on Graph Theory, 1996, pp. 35-45.

[3] Bondy J. A. and Murty U. S. R. "Graph Theory with Applications", The Macmillan Press Ltd, (1976).

[4] Devadoss Acharya and E.Sampathkumar, "Graphoidal covers and graphoidal covering numbers", Indian J. Pure Appl. Maths., 18(10) (1987), 882-90.

[5] Frank Harary, Graph Theory, Narosa / Addison Wesley, Indian Student Edition, 1988.

[6] Kulli V. R. "Minimally Non outer planar Graphs: A survey", (in the Book: "Recent Studies in Graph theory" ed: V. R. Kulli), Vishwa International Publication, (1989) 177-189.

[7] Mini To and Sunitha M.S., "On strongest paths, Edges, and Blocks in Fuzzy Graphs", World Applied Sciences Journal, 22(2013), 10-17.

[8] NarsingDeo "Graph Theory with Applications to Engineering and Computer Science", Prentice Hall of India Pvt. Ltd, New Delhi (1997).

[9] Nagoor Gani. A. and V.T.Chandrasekaran, "A First Look at Fuzzy Graph Theory", Allied Publishers, 2010.

[10] Nagoor Gani. A. and K.Radha, On Regular Fuzzy Graphs, Journal of Physical Sciences, Vol. 12, 33 – 40

(2010).

[11] Rosenfeld . A, in: L.A. Zadeh, K.S.Fu, K.Tanaka and M.Shimura, eds, "Fuzzy sets and their applications to cognitive and decision process", Academic press, New York (1975) 75-95.

[12] Satyanarayana Bh. and Syam Prasad K. "An Isomorphism Theorem on Directed Hypercubes of Dimension n", Indian J. Pure & Appl. Math 34 (10) (2003) 1453-1457.

[13] Satyanarayana Bh. and Syam Prasad K.. "Discrete Mathematics and Graph Theory", Prentice Hall of India, New Delhi, 2009.ISSN:978-81-203-3842-5.

[14] Sunitha M.S., Vijayakumar. A, " Blocks in Fuzzy Graphs", The journal of fuzzy Mathematics, Vol.13, No.1, 13-23.