Probability-Truth Logic

Youssef Prince Abed

Department of Mathematical Statistics,
Institute of Statistical Studies and Research, Cairo, Egypt

ABSTRACT— ***In this paper, a new kind of infinite logic will be introduced. The elements of this logic are truth function that assigns truth value belongs to the range [0,1] to any statement representing degree of truth and probability function that assigns a value ranging [0,1] to any random event representing degree of likelihood (both truth and probability are measures satisfying the classical axioms of probability theory). New axioms and definitions will be added to the axioms and definitions of probability theory to represent intersection, union, implication and double implication between truth statements, probabilistic events and truth statements and probabilistic events. By this, we can evaluate degree of truth and degree of likelihood and the interaction between them.***

Keywords— *Membership function, Measure, Conditional distribution, Bi-conditional distribution, Elementary event, Bayes rule.*

1. INTRODUCTION

The entire motivation behind developing the system came from fuzzy logic and probability, in binary logic we can say that A → B but statement A or statement B takes 0 or 1 as truth values, then comes fuzzy logic replacing the idea of binary truth by introducing membership functions for A and for B by then degree of truth of statement A or statement B takes values belong to the interval [0, 1] representing different degrees of truth [1] [2]. However, membership function doesn't represent likelihood. We will define t a truth function that assigns truth value belongs to [0,1] to any statement, value from a scale of deterministic variable or any function of deterministic variable to represent different degrees of truth, probability function p to assign value from the range [0,1] to any event or random variable to represent degree of likelihood, i.e., consider two persons, first person is outdoors and temperature degree represents moderate temperature so we can assign 0.5 to temperature degree, while the second person is indoors and he doesn't know what is the degree of temperature outdoors but there is 0.5 chance that the degree of temperature will be moderate. For first person 0.5 represents degree of truth but for second person 0.5 represents degree of likelihood. Existence function Ȩ will be a [0,1] transformation to represent degree of existence "given information about degree of truth and/or degree of probability" of union, intersection, conditionality or bi-conditionality between truth statements, probabilistic events or truth statements and probability events. Both, truth and probability functions satisfy the classical axioms of probability theory. Hence, by those definitions we can handle the concept of truth and likelihood.

2. AXIOMS

The axioms will be an extension to probability and fuzzy theory axioms [3]. For simplicity, We will use the notation of probability measure p for all axioms, definitions and theorems but in case of applying truth for all axioms, definitions and theorems, we replace probability measure p by truth measure t, Ω_1 by Ω_2, A_i by B_i, a_{ij} by b_{ij} and x and y by m and z where, Ω_1 represents the set of all possible outcomes of interest, Ω_2 represents the set of all possible statements of interest, A_i represents the ith collection from Ω_1, B_i represents the ith collection from Ω_2, a_{ij} represents the jth single element in the ith collection A_i, b_{ij} represents the jth single element in the ith collection B_i [4], x and y represents values from random variables X and Y respectively and m and z represent values from deterministic variables M and Z respectively, Then

1. $p(A_i) \geq 0$ for all $A_i \in \Omega_1$
2. $p(\Omega_1) = 1$
3. $p(A_1 \cup A_2) = \sum_i p(A_i) \; if \;\; p(A_1 \cap A_2) = \emptyset$
4. $Ȩ(A_1 \cup A_2) = Max\left(\sum_{a_{1j}\in A_1} p(a_{1j}), \sum_{a_{2j}\in A_2} p(a_{2j})\right)$
5. $Ȩ(A_1 \cap A_2) = Min\left(\sum_{a_{1j}\in A_1} p(a_{1j}), \sum_{a_{2j}\in A_2} p(a_{2j})\right)$ for all $a_{1j}, a_{2j} \in A_1 \cap A_2$

Where, $Ȩ(A_1 \cup A_2 \cup \ldots)$ and $Ȩ(A_1 \cap A_2 \cap \ldots)$ are [0, 1] transformations representing the degree of existence of union and intersection respectively between probabilistic events. Note that, Existence function Ȩ is different from probability measure p, the following example on axiom number 5 will clarify the idea of probability function and existence function, i.e., suppose that some person will go out to meet his friends depending on moderate humidity (say between 30 and 35 with assigned probability 0.3) and moderate temperature (say between 20 and 25 with assigned probability 0.4). Now the decision to go out can take two different forms, first form "he will go out if the probability of having moderate humidity and moderate temperature is greater than certain value" while second form "he will go out if a measure (existence function) of both moderate humidity with probability of 0.3 and moderate temperature with probability of 0.4 is greater than certain value". The first decision is based on joint probability distribution of both temperature and humidity while the second decision is based on marginal probability of temperature and marginal probability of humidity. Existence function is defined to evaluate situations when we need to consider variables and their assigned probabilities and/or truth degrees, existence function shows how those variables intersects, unionize, conditionalize and bi-conditionalize on each other. By adding existence function to axioms of probability theory and by defining truth function, we can evaluate complex decisions based on variables with assigned degrees of truth and/or probability, this can be generally applied in many fields like decision theory, statistical inference, data analysis and artificial intelligence.

3. DEFINITION: Ȩ-CONDITIONALITY

The conditional existence of probabilistic event A_1 given probabilistic event A_2 is the quotient of the minimum of probability of event A_1 and probability of event A_2 to probability of event A_2 [5]. Expressed as follows;

$$Ȩ(A_1|A_2) = \frac{Ȩ(A_1 \cap A_2)}{P(A_2)} = \frac{Min(\sum_{a_{1j} \in A_1} p(a_{1j}), \sum_{a_{2j} \in A_2} p(a_{2j}))}{\sum_{a_{2j} \in A_2} p(a_{2j})} \tag{1}$$

For discrete values x and y from random variables X and Y

$$Ȩ(X|Y) = \frac{Min(\sum_x p(x), \sum_y p(y))}{\sum_y p(y)}$$

For continuous values x and y from random variables X and Y

$$Ȩ(X|Y) = \frac{Min(\int_x^{x+\Delta x} p(x)\,dx, \int_y^{y+\Delta y} p(y)\,dy)}{\int_y^{y+\Delta y} p(y)\,dy}$$

Where $Ȩ(X|Y)$ is a [0, 1] transformation representing the degree of existence of conditionality between probabilistic variable X and probabilistic variable Y.

4. DEFINITION: Ȩ-INDEPENDENCE AND Ȩ-MUTUALITY

The concept of Ȩ-Independent events means that probabilistic event A_1 given probabilistic event A_2 has the same probability of probabilistic event A_1

$$Ȩ(A_1|A_2) = \frac{Ȩ(A_1 \cap A_2)}{P(A_2)} = P(A_1) \tag{2}$$

From equation (2) the intersection of probabilistic events

$$Ȩ(A_1 \cap A_2) = P(A_1)P(A_2) \tag{3}$$

Note that, truth is independent of truth of other statements and probabilistic events, as truth degree of a statement is a fact and it is not affected by the change of other degree of truth or probability. Hence, existence function of the intersection of truth statements B_1, B_2

$$Ȩ(B_1 \cap B_2) = t(B_1)t(B_2)$$

Existence function of intersection of truth statement B_1 and probabilistic event A_1

$$Ȩ(B_1 \cap A_1) = t(B_1)p(A_1)$$

Ȩ-Mutuality between probabilistic events is defined as follows;

$$Ȩ(A_1 \cap A_2) = 0 \quad if \quad A_1 \cap A_2 = \emptyset$$

5. DEFINITION: Ḝ-BI-CONDITIONALITY

In classical logic the bi-conditional operator is frequently used on two conditional statements. It represents dual implication as B_1 implies B_2 and B_2 implies B_1, but there is no counterpart for dual implication in the context of probability. Bi-conditionality between event A_1 and event A_2 Will be expressed in a probabilistic terms as follows;

$$p(A_1||A_2) = \frac{p(A_1 \cap A_2)}{p(A_1 \cup A_2)} \tag{4}$$

Formally as conditional probability [6] we can define bi-conditional probability, let a_{ij} be an elementary event, A_1, A_2 be any arbitrary events. For a_{ij} we need to satisfy the following conditions

$$a_{ij} \in A_1 \cup A_2 : p(a_{ij} | A_1 \cup A_2) = \alpha p(a_{ij})$$
$$a_{ij} \notin A_1 \cap A_2 : p(a_{ij} | A_1 \cup A_2) = 0$$
$$\sum_{a_{ij} \in \Omega} p(a_{ij} | A_1 \cup A_2) = 1$$

Now we can say

$$\begin{aligned}\sum_{a_{ij} \in \Omega} p(a_{ij} | A_1 \cup A_2) &= \sum_{a_{ij} \in A_1 \cup A_2} p(a_{ij} | A_1 \cup A_2) + \sum_{a_{ij} \notin A_1 \cup A_2} p(a_{ij} | A_1 \cup A_2) \\ &= \sum_{a_{ij} \in A_1 \cup A_2} \alpha p(a_{ij}) \\ &= \alpha p(A_1 \cup A_2)\end{aligned}$$

Now consider bi-conditionality between A_1 and A_2

$$\begin{aligned}p(A_1||A_2|A_1 \cup A_2) = p(A_1||A_2) &= \sum_{a_{ij} \in A_1 \cap A_2} p(a_{ij} | A_1 \cup A_2) + \sum_{a_{ij} \notin A_1 \cap A_2} p(a_{ij} | A_1 \cup A_2) \\ &= \sum_{a_{ij} \in A_1 \cap A_2} \frac{p(a_{ij})}{p(A_1 \cup A_2)} = \frac{p(A_1 \cap A_2)}{p(A_1 \cup A_2)}\end{aligned}$$

The existence degree of bi-conditionality between probabilistic event A_1 and probabilistic event A_2

$$Ḝ(A_1||A_2) = \frac{Min(\sum_{a_{1j} \in A_1} p(a_{1j}), \sum_{a_{2j} \in A_2} p(a_{2j}))}{Max\left(\sum_{a_{1j} \in A_1} p(a_{1j}), \sum_{a_{2j} \in A_2} p(a_{2j})\right)} \tag{5}$$

For discrete values x and y from random variables X and Y

$$Ḝ(X||Y) = \frac{Min(\sum_x p(x), \sum_y p(y))}{Max(\sum_x p(x), \sum_y p(y))}$$

For continuous values x and y from random variables X and Y

$$Ḝ(X||Y) = \frac{Min(\int_x^{x+\Delta x} p(x)\,dx, \int_y^{y+\Delta y} p(y)\,dy)}{Max(\int_x^{x+\Delta x} p(x)\,dx, \int_y^{y+\Delta y} p(y)\,dy)}$$

Where $Ḝ(X||Y)$ is a [0, 1] transformation representing the degree of existence of bi-conditionality between probabilistic variable X and probabilistic variable Y.

6. THEOREM: Ḝ-BAYES RULE

In our new context, the Ḝ-Bayes rule can be derived from the definition of conditional existence of A_1 and A_2

$$Ḝ(A_1|A_2) = \frac{Min\left(\sum_{a_{1j} \in A_1} p(a_{1j}), \sum_{a_{2j} \in A_2} p(a_{2j})\right)}{\sum_{a_{2j} \in A_2} p(a_{2j})} \tag{6}$$

From equation (6)

$Min(\sum_{a_{1j}\in A_1} p(a_{1j}), \sum_{a_{2j}\in A_2} p(a_{2j})) = Ȩ(A_1|A_2)P(A_2)$ and $Min(\sum_{a_{1j}\in A_1} p(a_{1j}), \sum_{a_{2j}\in A_2} p(a_{2j})) = Ȩ(A_2|A_1)P(A_1)$

From the two preceding equations Ȩ-Bayes rule for A_1 and A_2 is

$$Ȩ(A_1|A_2) = \frac{Ȩ(A_2|A_1)P(A_1)}{P(A_2)}$$

7. CONCLUSION

Through the introduction of this kind of logic it is clear to see that by extending the axioms and definitions of fuzzy and probability theory we can represent measures of the degree of truth, likelihood and how both concepts interact with themselves and each other.

8. REFERENCES

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