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A Common Fixed Point Theorem for Asymptotically Regular in Cone Metric Spaces

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ABSTRACT— ***In this paper, we prove a unique common fixed point theorem for asymptotically regular self-maps in cone metric spaces without appealing to commutativity condition and which extends metric space into cone metric space. These results extend and generalize several well-known comparable results in the literature.***

Keywords— Asymptotically regular, coincidence points, common fixed point, cone metric space.

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1. INTRODUCTION AND PRELIMINARIES

In 2007 Huang and Zhang [7] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [7], [12] and the references mentioned therein). Isak Altun, G.Durmaz,[3] have proved some fixed point theorems on ordered cone metric spaces. In 2009, M.Abbas, B.E.Rhoades [2] have proved common fixed point theorems for mappings without appealing to commutativity conditions in cone metric spaces. S.L.Singh et al. [16] have obtained coincidence point result for two mappings in metric spaces. In this paper, we prove a common fixed point theorem for asymptotically regular self-mappings in cone metric spaces without appealing to commutativity condition and which is an extension of metric spaces into cone metric spaces.

Through this section R_+,denotes the set of all nonnegative real numbers, B, is a real Banach space, N={1,2,3,…,}, the set of all natural numbers. For the mapping f, g: X→ X, let C (f, g) denote the set of coincidence points of f and g, that is C(f, g) ={z∈X: fz = gz}.

The following definitions and lemma are due to Huang and Zhang [7].

Definition 1.1. Let B be a real Banach space and P a subset of B. The set P is called a cone if and only if:

(a). P is closed , non–empty and P ≠ { 0 } ;

(b). a, b $\in R$, a, b≥0, x, y $\in P$ implies ax+by $\in P$;

(c). x∈P, -x $\in P \Rightarrow$ x = 0.

Definition 1.2. Let P be a cone in a Banach space E, define partial ordering '$\leq$' with respect to P by x ≤ y if and only if y-x $\in P$ .We shall write x<y to indicate x ≤y but x ≠ y while x<<y will stand for y-x∈Int P, where Int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1.3. Let B be a Banach space and P⊂B be an order cone .The order cone P is called normal if there exists L>0 such that for all x, y∈B,

$0 \leq x \leq y$ implies $\|x\| \leq L\|y\|$.

The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.4. Let X be a nonempty set of B .Suppose that the map d: X $\times$ X→B satisfies:

(d1). $0 \leq d(x,y)$ for all x, y $\in X$ and
d(x, y) = 0 if and only if x = y ;

(d2). d(x, y) = d(y, x) for all x, y $\in X$;

(d3). d(x, y) $\leq$ d(x, z) +d(z, y) for all x, y, z $\in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Example 1.5. ([7]). Let $B = R^2$, $P = \{(x, y) \in B$ such that: $x, y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to B$ such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant.Then (X, d) is a cone metric space.

Definition 1.6. Let (X, d) be a cone metric space .We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in B with c >>0,there is N such that for all n, m>N, $d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for any c>>0,there is an N such that for all n>N, $d(x_n, x) \ll c$, for some fixed x in X .We denote this $x_n \to x$ (as $n \to \infty$).

Lemma 1.1.[7] Let (X, d) be a cone metric space, and let P be a normal cone with normal constant L .Let $\{x_n\}$ be a sequence in X .Then
(i). $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ $(n \to \infty)$.
(ii). $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

The following definitions are due to Sastry et al. [14] and S.L.Sing et al. [15].

Definition 1.7. Let T, f be maps on X with values in a cone metric space (X, d).The pair (T, f) is said to be asymptotically regular at $x_0 \in X$ if there exists $\{x_n\}$ in X Such that $Tx_n = fx_{n+1}$, $n = 0,1,2,\ldots$ and

$$\lim_{n \to \infty} d(fx_n, fx_{n+1}) = 0.$$

If f is identity map on X, then we get the usual definition of asymptotic regular for a map T due to Browder and Pteryshyn [6].

2. COMMON FIXED POINT THEOREM

In this section, we prove a unique common fixed point theorem for asymptotically regular self-mappings in cone metric spaces without appealing to commutativity condition, and also which extends metric space to cone metric space.

The following theorem extends and generalizes the Theorem 2.5 [16].

Theorem 2.1. Let (X, d) be cone metric space and P be a normal cone with normal constant L. And T, f be self-mappings, (T, f) is asymptotically regular at $x_0 \in X$, then

(A1): $d(Tx, Ty) \leq \varphi(g(x, y))$ for all $x, y \in X$,
where, $g(x, y) = d(fx, fy) + \gamma [d(fx, Ty) + d(fy, Ty)]$, $0 \leq \gamma \leq 1$ and $\varphi: R_+ \to R_+$ continuous.

If the range of f contains the range of T and T(X) or f(X) is a complete subspace of X, (T, f) have a coincidence point in X, then T, f have a unique common fixed point in X.

Proof: Let $x_0 \in X$ define a sequence $\{y_n\}$ by
$y_{n+1} = Tx_n = fx_{n+1}$, $n = 0,1,2,\ldots$, this can be done since the range of f contains the range of T. Since the pair (T, f) is asymptotically regular, $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$.

First we shall show that $\{y_n\}$ is Cauchy sequence. Suppose $\{y_n\}$ is not Cauchy sequence. Then there exists $\mu > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that For all $n \leq m_k < n_k$,

$d(y_{m_k}, y_{n_k}) \geq \mu$ and $d(y_{m_k}, y_{n_{k-1}}) < \mu$.

By the triangle inequality,
$d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{n_{k-1}}) + d(y_{n_{k-1}}, y_{n_k})$.

Making $k \to \infty$,

$d(y_{m_k}, y_{n_k}) < \mu$.

Thus, $d(y_{m_k}, y_{n_k}) \to \mu$ as $k \to \infty$. Now by (A1),

$$d(y_{m_{k+1}}, y_{n_{k+1}}) = d(Tx_{m_k}, Tx_{n_k})$$
$$\leq \varphi\,(g(x_{m_k}, x_{n_k}))$$
$$= \varphi\,(d(fx_{m_k}, fx_{n_k}) + \gamma[d(fx_{m_k}, Tx_{m_k}) + d(fx_{n_k}, Tx_{n_k})]\,).$$

Letting $k \to \infty$,

$$\mu \leq \varphi\,(\mu) < \mu, \quad \text{a contradiction.}$$

Therefore, $\{y_n\}$ is a Cauchy sequence. Suppose f(X) is complete. Then $\{y_n\}$ being contained in f(X) has a limit in f(X). Call it z. Let $u \in f^{-1}z$. Then fu = z.

Using (A1),

$d(Tu, Tx_n) \leq \varphi\,(d(fu, fx_n) + \gamma\,[d(Tu, fu) + d(Tx_n, fx_n)])$.

Letting $n \to \infty$,

$d(Tu, z) \leq \varphi\,(\gamma\, d(Tu, z)) < d(Tu, z)$, a contradiction.

Therefore, Tu = z = fu, u is a coincidence point of T and f. … (1)

Now using (A1), and triangle inequality

$$d(TTu, Tu) \leq d(TTu, y_{n+1}) + d(y_{n+1}, Tu),$$
$$\leq d(TTu, y_{n+1}) + d(Tx_n, Tu),$$
$$\leq d(TTu, y_{n+1}) + \varphi\,(d(fx_n, fu) + \gamma\,[d(fx_n, Tx_n) + d(fu, Tu)]).$$

From (1.3),

$$\|d(TTu, Tu)\| \leq L(\|d(TTu, y_{n+1}) + \varphi\,(d(fx_n, fu) + \gamma\,[d(fx_n, Tx_n) + d(fu, Tu)]\,))\|)$$

$$\leq L(\|d(TTu, y_{n+1})\| + \varphi(\|d(fx_n, fu)\| + \gamma\,[\|d(fx_n, Tx_n)\| + \|d(fu, Tu)\|])\,),$$

as $n \to \infty$, we get that

$$\|d(TTu, Tu)\| \leq L(\|d(TTu, z)\| + \varphi\,(\|d(z, z)\| + \gamma\,[\|d(z, z)\| + \|d(z, z)\|]))$$

$$\leq L\|d(TTu, Tu)\| < \|d(TTu, Tu)\|, \text{ which is a contradiction.}$$

Therefore, TTu = Tu(= z). … (2)

And $d(fu, ffu) = d(Tu, Tfu)$ (since, $fu = Tu$),

$\leq \varphi\, (d(fu, ffu) + \gamma[d(fu, Tu) + d(ffu, Tfu)])$,

$\leq \varphi\, (d(fu, ffu) + \gamma[d(z, z) + d(Tfu, Tfu)]\,)$ (since, $fu = Tu$),

$\leq \varphi(d(fu, ffu)) < d(fu, ffu)$, which is a contradiction.

Therefore, $ffu = fu\ (=z)$. … (3)

From (2) and (3) it follows that,

T and f have common fixed point.

Uniqueness, let z_1 be another common fixed point of T and f.

$d(z, z_1) = d(Tz, Tz_1)$

$\leq \varphi\, (g(z, z_1))$

$\leq \varphi\, (d(fz, fz_1) + \gamma\, [d(fz, Tz) + d(fz_1, Tz_1)])$,

$\leq \varphi\, (d(z, z_1) + \gamma\, [d(z, z) + d(z_1, z_1)])$,

$\leq \varphi\, (d(z, z_1)) < d(z, z_1)$, which is a contradiction.

Hence, $z = z_1$.

Therefore, T and f have a unique common fixed point.

3. CONCLUSION

Finally we conclude that in cone metric spaces we can get common fixed points with out using the commutativity condition with the help of normal cone with normal constant.

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