# Random Fixed Point Result of Weak Contraction

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**ABSTRACT----** ***We define Cone random metric space and find some fixed point results for weak contraction condition.***

**Keywords----**Random operator, Cone Random Metric Space , Cauchy Sequence, Random Fixed Point

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## 1. INTRODUCTION

Random fixed point theorem for contraction mappings in Random metric spaces and in probabilistic functional analysis random fixed point theorems have fundamental importance. The structure of common random fixed points and random coincidence points of a pair of compatible random operators have studied by Beg and Shahzad[2]. Huang and Zhang [4] generalized the concept of metric spaces and replaced the set of real numbers by an ordered Banach space and hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. Subsequently, various authors [1], [6] have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cones. Recently Sumitra, V.R., Uthariaraj,R. Hemavathy[6] gave some results for normal cones. There exist a lot of work involving points used the Banach contraction principle. Recently Dhagat et. al.[3] given some results for random operators and Mehta, Singh , Sanodia and Dhagat [5] has given some results in random cone metric space.

## 2. PRELIMINARY

**Definition 2.1:** Let $(E, \tau)$ be a topological vector space and P a subset of E, P is called a cone if

1. P is non-empty and closed, $P \neq \{0\}$,
2. For $x, y \in P$ and $a, b \in R \Rightarrow ax + by \in P$ where $a, b \geq 0$
3. If $x \in P$ and $-x \in P \Rightarrow x = 0$
   For a given cone $P \subseteq E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$, $x < y$ if $x \leq y$ and $x \neq y$, while $x << y$ will stand for $y - x \in \text{int } P$, int P denotes the interior of P.

A cone $P \subset E$ is called normal if there is a number K>0 such that for all $x,y \in E$ $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

The least positive number satisfying the above inequality is called the normal constant of P. It is clear that $K \geq 1$. We know that that there exists an ordered Banach Space E with cone P which is not normal but int $P \neq \emptyset$.

**Definition 2.2 : Measurable function** : Let $(\Omega, \sum)$ be a measurable space with $\sum$ a sigma algebra of subsets of $\Omega$ and M a non-empty subset of a metric space X = (X, d). Let $2^M$ be the family of all non-empty subsets of M where C(M) the family of all nonempty closed subsets of M. A mapping $G : \Omega \to 2^M$ is called measurable if, for each open subset U of M, $G^{-1}(U) \in \Sigma$, where $G^{-1}(U) = \{\omega \in \Omega : G(\omega) \cap U \neq \phi\}$.

**Definition 2.3: Measurable selector:** A mapping $\xi : \Omega \to M$ is called a measurable selector of a measurable mapping $G : \Omega \to 2^M$ if $\xi$ is measurable and $\xi(w) \in G(w)$ for each $w \in \Omega$.

**Definition 2.4: Random operator:** Mapping T : $\Omega \times M \rightarrow X$ is said to be a random operator if, for each fixed $x \in M$, $T(., x) : \Omega \rightarrow X$ is measurable.

**Definition 2.5:Continuous Random operator:** A random operator T : $\Omega \times M \rightarrow X$ is said to be continuous random operator if, for each fixed $x \in M$, $T(., x) : \Omega \rightarrow X$ is continuous .

**Definition 2.6: Random fixed point:** A measurable mapping $\xi : \Omega \rightarrow M$ is a random fixed point of a random operator T : $\Omega \times M \rightarrow X$ if $\xi(\omega) = T(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

**Definition 2.7:** Let M be a nonempty set and the mapping d: $\Omega \times M \rightarrow X$ and $P \subset X$ be a cone, $\omega \in \Omega$ be a selector, satisfies the following conditions:

2.7.1) $d(x(\omega), y(\omega)) > 0 \; \forall \; x(\omega), y(\omega) \in \Omega \times X \Leftrightarrow x(\omega) = y(\omega)$

2.7.2) $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega)) \;\; \forall x, y \in X, \omega \in \Omega \; and \; x(\omega), y(\omega) \in \Omega \times X$

2.7.3) $d(x(\omega), y(\omega)) = d(x(\omega), z(\omega)) \; + d(z(\omega), y(\omega)) \, \forall x, y \in X$

$and \; \omega \in \Omega \, be \, a \, selector.$

2.7.4) $For \, any \, x, y \in X, \omega \in \Omega, \; d(x(\omega), y(\omega))$

$is \, non-increasing \; and \; left \; continuous \; in \, \alpha.$

Then d is called cone random metric on M and (M, d) is called a cone random metric space.

## 3. MAIN RESULTS

**Theorem 3.1** Let (M,d) be a complete cone random metric space with regular cone P such that $d(x(\omega),y(\omega)) \in int(\Omega xP)$ for $x(\omega),y(\omega) \in M$ with $x(\omega) \neq y(\omega)$. Let T : $\Omega \times M \rightarrow M$ be a mapping satisfying the inequality

$d(T(\omega, x(\omega)), T(\omega, y(\omega))) \leq d(x(\omega),y(\omega)) - \varphi(d(x(\omega),y(\omega)))$, for $x(\omega),y(\omega) \in M$

where $\varphi: int(\Omega xP) \cup \{\omega\} \rightarrow int(\Omega xP) \cup \{\omega\}$ is a continuous and monotonic increasing function with

(I) $\varphi(t(\omega)) = \omega$ if and only if $t(\omega) = (\omega)$;

(II) $\varphi(t(\omega)) << t(\omega)$, for $t(\omega) \in int \, \Omega xP$;

(III) either $\varphi(t(\omega)) \leq d(x(\omega),y(\omega))$ or $d(x(\omega),y(\omega)) \leq \varphi(t(\omega))$,
for $t(\omega) \in int \, \Omega xP \cup \{\omega\}$ and $x(\omega),y(\omega) \in M$

Then T has a unique random fixed point in M.

**Proof:** Let $x_0(\omega) \in \Omega xM$. We construct the sequence $\{x_n(\omega)\}$, as $x_n(\omega) = T(\omega, x_{n-1}(\omega))$ $n \geq 1$. If $x_{n+1}(\omega) = x_n(\omega)$, for some n, then trivially T has a random fixed point .

If $x_{n+1}(\omega) \neq x_n(\omega)$ for $n \in N$. Then by inequality

$d(x_n(\omega), x_{n+1}(\omega)) = d(T(\omega, x_{n-1}(\omega)), T(\omega, x_n(\omega)))$

$\leq d(x_{n-1}(\omega), x_n(\omega)) - \varphi(d(x_{n-1}(\omega), x_n(\omega)))$

By the property of $\varphi$, that $0 \leq \varphi(t(\omega))$, for all $t(\omega) \in int \, \Omega xP \cup \{\omega\}$,we have

$d(x_n(\omega), x_{n+1}(\omega)) \leq d(x_{n-1}(\omega), x_n(\omega))$.

It follows that the sequence $\{ d(x_n(\omega), x_{n+1}(\omega))\}$ is monotonic decreasing. Since random cone P is regular and $0 \leq d(x_n(\omega), x_{n+1}(\omega))$ for all $n \in N$, there exists $\varepsilon \geq 0$ such that

$d(x_n(\omega), x_{n+1}(\omega)) \rightarrow \varepsilon$ as $n \rightarrow \infty$

Since $\varphi$ is continuous and

$d(x_n(\omega), x_{n+1}(\omega)) \leq d(x_{n-1}(\omega), x_n(\omega)) - \varphi(d(x_{n-1}(\omega), x_n(\omega)))$

On taking $n \rightarrow \infty$

$\varepsilon \leq \varepsilon - \varphi(\varepsilon)$, which is contradiction unless $\varepsilon = 0$.

Therefore $d(x_n(\omega), x_{n+1}(\omega)) \rightarrow 0$ as $n \rightarrow \infty$.

Let $c \in E$ with $0 \ll c$ be arbitrary. Since $d(x_n(\omega), x_{n+1}(\omega)) \to 0$ as $n \to \infty$, there exists $m \in N$ such that $d(x_n(\omega), x_{n+1}(\omega)) \ll \varphi(\varphi(c/2))$.

Let $B(x_m(\omega),c) = \{x(\omega) \in \Omega xX: d(x_m(\omega),x(\omega)) \ll c\}$. Clearly, $x_m(\omega) \in B(x_m(\omega),c)$.

Therefore , $B(x_m(\omega),c)$ is nonempty.

Now we will show that

$T(\omega, x(\omega)) \in B(x_m(\omega),c)$ for $x(\omega) \in B(x_m(\omega),c)$.

Let $x(\omega) \in B(x_m(\omega),c)$, by property (III) of $\varphi$, we have

**Case (i)** $d(x(\omega),x_m(\omega)) \leq \varphi(c/2)$ and **Case(ii)** $\varphi(c/2) < d(x(\omega),x_m(\omega)) \ll c$.

Therefore

**Case (i):**

$$\begin{aligned} d(T(\omega, x(\omega)), x_m(\omega)) &\leq d(T(\omega, x(\omega)), T(\omega, x_m(\omega)) + d(x_m(\omega), T(\omega, x_m(\omega))) \\ &\leq d(x(\omega), x_m(\omega)) - \varphi(d(x(\omega), x_m(\omega))) + d(x_m(\omega), x_{m+1}(\omega)) \\ &\leq \varphi(c/2) + \varphi(c/2) \ll c/2 + c/2 = c. \end{aligned}$$

$$\begin{aligned} d(T(\omega, x(\omega)), x_m(\omega)) &\leq d(T(\omega, x(\omega)), T(\omega, x_m(\omega)) + d(x_m(\omega), T(\omega, x_m(\omega))) \\ &\leq d(x(\omega), x_m(\omega)) - \varphi(d(x(\omega), x_m(\omega))) + d(x_m(\omega), x_{m+1}(\omega)) \\ &\leq d(x(\omega), x_m(\omega)) - \varphi(\varphi(c/2)) + \varphi(\varphi(c/2)). \\ &\leq d(x(\omega), x_m(\omega)) \ll c. \end{aligned}$$

Thus T is a self mapping of $B(x_m(\omega),c)$.

Since $x_m(\omega) \in B(x_m(\omega),c)$ and $x_n(\omega) = T(\omega, x_{n-1}(\omega))$, $n \geq 1$.

It follows that $x_n(\omega) \in B(x_m(\omega),c)$ for all $n \geq m$. Again c is an arbitrary, so $\{x_n(\omega)\}$ is a Cauchy sequence and by the completeness of X, there exists $x(\omega)$ such that $x_n(\omega) \to x(\omega)$ as $n \to \infty$. Now,

$$\begin{aligned} d(x_n(\omega), T(\omega, x(\omega))) &= d(T(\omega, x_{n-1}(\omega)), T(\omega, x(\omega)) \\ &\leq d(x_{n-1}(\omega), x(\omega)) - \varphi(d(x_{n-1}(\omega), x(\omega))). \end{aligned}$$

On taking $n \to \infty$,we get $d(x(\omega), T(\omega, x(\omega))) \leq 0$.

Thus $d(x(\omega), T(\omega, x(\omega))) = 0. \Rightarrow x(\omega) = T(\omega, x(\omega))$.

Hence $x(\omega)$ is random fixed point of T.

**Uniqueness:** Let $y(\omega) \in \Omega xX$ be another random fixed point of T. Then by the condition We have,

$$\begin{aligned} d(x(\omega),y(\omega)) &= d(T(\omega, x(\omega)), T(\omega, y(\omega))) \\ &\leq d(x(\omega),y(\omega)) - \varphi(d(x(\omega),y(\omega))) \end{aligned}$$

Now we have that $\varphi(d(x(\omega),y(\omega))) \leq 0$, which is contradiction to the property of $\varphi$.

Hence $x(\omega)$ is unique random fixed point of T.

**Example:**-Let E = R and $P = \{x \in E: x \geq 0\}$, also $\Omega = [0,1]$ and $\Sigma$ be the sigma algebra of Lebesgue's measurable subset of [0,1]. Let $X = [0, \infty)$ and define a mapping $d: (\Omega xX)x(\Omega xX) \to E$ by $d(x(\omega),y(\omega)) = |x(\omega) - y(\omega)|$. Then (X,d) is a cone random metric space. Define random operator T from $\Omega xX$ to M as $T(\omega,x(\omega)) = (1-\omega)x(\omega)+\omega(1-\omega^2)$. Let us define $\varphi: int(\Omega xP) \cup \{\omega\} \to int(\Omega xP) \cup \{\omega\}$ as $\varphi(t(\omega)) = \omega t(\omega)$

$$\begin{aligned} & d(T(\omega, x(\omega)), T(\omega, y(\omega)))) \\ &= \|(1-\omega)x(\omega) + \omega(1-\omega^2) - \{(1-\omega)y(\omega) + \omega(1-\omega^2)\}\| \\ &= \|(x(\omega) - y(\omega)) + (-\omega)(x(\omega) - y(\omega))\| \\ &\leq \|(x(\omega) - y(\omega))\| + |-\omega|\|x(\omega) - y(\omega)\| \\ &= \|(x(\omega) - y(\omega))\| + \omega\|x(\omega) - y(\omega)\| \\ &= d(x(\omega),y(\omega)) - \varphi(d(x(\omega),y(\omega))). \end{aligned}$$

Define measurable mapping $x : \Omega \to M$ as $x(\omega) = (1-\omega^2)$for every $\omega \in \Omega$. T Satisfies all condition of the theorem and hence $(1-\omega^2)$ is random fixed point of the space.

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