A Concise Formula for Determining the Order and Error Constant of Fourth-Order Linear Multistep Method

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ABSTRACT---- ***In this study, we develop a concise and efficient formula for determining the order and error constants of fourth-order linear multistep methods used in the numerical solution of ordinary differential equations. Traditional approaches to computing these parameters often involve tedious algebraic manipulations, which can be time-consuming and error prone. This new approach provides a streamlined and systematic method that simplifies the analysis while ensuring accuracy and reliability. By leveraging this new approach, researchers can more efficiently assess the accuracy and stability of fourth-order linear multistep method schemes.***

Keywords---- Fourth-order, Linear Multistep Method (LMM), Order and Error Constant.

1. INTRODUCTION

Numerical analysis is a fundamental discipline within mathematics and computer science that focuses on the development, analysis and implementation of algorithms for approximating solutions of Ordinary Differential Equations (ODEs) and other complex mathematical problems. The numerical solution of ODEs plays a crucial role in various scientific and engineering applications where analytic solutions are often difficult or impossible to obtain.

Over the years, researchers have made substantial progress in designing and analyzing numerical algorithms for solving ODEs, particularly low order methods (orders$\leq$ 3). Among the most widely used classes of numerical methods are Linear Multistep Methods (LMMs), which leverage multiple previous steps to advance the solution, thereby improving efficiency and accuracy compared to single-step methods like the Runge-Kutta schemes. LMMs are particularly advantageous in long-term integrations due to their ability to reduce computational cost while maintaining accuracy.

A conventional approach for determining the order and error constants of numerical schemes is the Taylor series expansion. While this method is theoretically sound and widely adopted, it becomes increasingly cumbersome as the order of the numerical scheme increases. The expansion process generates large, intricate expansions, making manual computation tedious and computationally expensive. The challenge is further compounded when dealing with higher-order LMMs, where the complexity of algebraic manipulations grows exponentially

The analysis of numerical methods for determining the order and error constants of LMMs has been extensively studied in the literature. Early works, such as those of [5], focused on methods of order ≤ 2, providing foundational techniques for analyzing numerical accuracy and stability. Later advancements by [3] extended the analysis to third-order LMMs, paving the way for higher-order methods. Despite these contributions, there remains a pressing need for more efficient and systematic approaches to analyze higher-order schemes, particularly fourth-order LMMs. The development of streamlined techniques is crucial to ensuring that numerical algorithm's achieve the desired accuracy and stability while minimizing computational effort.

In this study, we propose a concise and efficient formula for determining the order and error constants of fourth-order LMMs used in solving ODEs. Our approach seeks to simplify the analysis while maintaining reliability, thereby enabling researchers to assess the accuracy stability of numerical schemes with greater ease. By leveraging this new approach, we aim to address the computational challenges associated with higher-order numerical methods and contribute to the advancement of efficient ODE-solving techniques.

2. METHODOLOGY

This section presents a review of previous analyses on determining the order and error constant of LMMs for first-, second- and third-order ODEs.

Consider an arbitrary function $y(x)$ that is continuously differentiable at a close interval $[a, b]$. [5] conducted a comprehensive analysis of the order and error constants of first- and second-order LMMs as follows:

Given the first-order LMM of the form

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad j = 0,1,2, \dots k \tag{1}$$

by associating the linear difference operator

$$L[y(x): h] = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h\beta_j y'(x + jh)\right] = 0 \tag{2}$$

Expanding both $y(x + jh)$ and $y'(x + jh)$ using the Taylor series about x and collecting terms gives

$$L[y(x): h] = C_0 y(x) + C_1 h y'(x) + \cdots + C_q h^q y^q(x) \tag{3}$$

where C_i are constants, for $i = 1,2, \dots q$. A simple calculation in terms of α_j , β_j yields the following formula for the constants C_i

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 \dots + \alpha_k$

$C_1 = (\alpha_1 + 2\alpha_2 + 3\alpha_3 \dots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$

$\vdots$

$$C_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 \dots + k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + +2^{q-1}\beta_2 + 3^{q-1}\beta_3 \dots + k^{q-1}\beta_k) \tag{4}$$

where $q = 2,3, \dots$ This method can be categorically said to have order P if $C_0 = C_1 = C_2 = \cdots = C_P = 0$, but $C_{P+1} \neq 0$. C_{P+1} is the error constant and the error $C_{P+1} h^{P+1} y^{P+1}(x_n)$ is the principal local truncated error at the point (x_n).

Also, Given the second-order LMM of the form

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad j = 0,1,2, \dots k \tag{5}$$

By associating the linear difference operator

$$L[y(x): h] = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h^2\beta_j y''(x + jh)\right] = 0 \tag{6}$$

A simple calculation yields the following formula for the constant C_i

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 \dots + \alpha_k$

$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 \dots + k\alpha_k$

$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 \dots + k^2\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$

$\vdots$

$C_q =$
$\frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 \dots + k^q\alpha_k) - \frac{1}{(q-2)!}(\beta_1 + +2^{q-2}\beta_2 + 3^{q-2}\beta_3 \dots + k^{q-2}\beta_k)$ (7)

For $q = 3,4,\dots$ This implies that the method is said to have order P if $C_0 = C_1 = C_2 = \cdots = C_{P+1} = 0$, but $C_{P+2} \neq 0$. C_{P+2} is the error constant and the error $C_{P+2}h^{P+2}y^{P+2}(x_n)$ is the principal local truncated error at the point (x_n).

The order and error constants for the third-order LMM was obtained by [3] as follow.

Given a LMM of the form

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad j = 0,1,2,\dots k \tag{8}$$

By associating the linear difference operator

$$L[y(x):h] = \sum_{j=0}^{k} \left[\alpha_j y(x+jh) - h^3\beta_j y'''(x+jh)\right] = 0 \tag{9}$$

A simple calculation yields the following formula for the constant C_i

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 \dots + \alpha_k$

$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 \dots + k\alpha_k$

$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 \dots + k^2\alpha_k)$

$C_3 = \frac{1}{3!}(\alpha_1 + 2^3\alpha_2 + 3^3\alpha_3 \dots + k^3\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$

$\vdots$

$C_q =$
$\frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 \dots + k^q\alpha_k) -$
$\frac{1}{(q-3)!}(\beta_1 + +2^{q-3}\beta_2 + 3^{q-3}\beta_3 \dots + k^{q-3}\beta_k)$ (10)

For $q = 4,5,\dots$ This implies that the method is said to have order P if $C_0 = C_1 = C_2 = \cdots = C_{P+2} = 0$, but $C_{P+3} \neq 0$. C_{P+3} is the error constant and the error $C_{P+3}h^{P+3}y^{P+3}(x_n)$ is the principal local truncated error at the point (x_n).

3. PROPOSED TECHNIQUE

Consider a fourth-order LMM of the form

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h^4 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad j = 0,1,2,\dots k \tag{11}$$

with an associating linear difference operator of

$$L[y(x):h] = \sum_{j=0}^{k} \left[\alpha_j y(x+jh) - h^3\beta_j y'^{v}(x+jh)\right] = 0 \tag{12}$$

Expanding both $y(x+jh)$ and $y'^{v}(x+jh)$ using the Taylor series about x and collecting terms gives

$$L[y(x):h] = C_0 y(x) + C_1 h y'(x) + \cdots + C_q h^q y^q(x) = 0 \qquad , \qquad q = 5,6 \ldots \tag{13}$$

By collecting the corresponding terms, we obtain the following coefficients:

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 \ldots + \alpha_k$

$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 \ldots + k\alpha_k$

$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 \ldots + k^2\alpha_k)$

$C_3 = \frac{1}{3!}(\alpha_1 + 2^3\alpha_2 + 3^3\alpha_3 \ldots + k^3\alpha_k)$

$C_4 = \frac{1}{4!}(\alpha_1 + 2^4\alpha_2 + 3^4\alpha_3 \ldots + k^4\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$

$C_5 = \frac{1}{5!}(\alpha_1 + 2^5\alpha_2 + 3^5\alpha_3 \ldots + k^5\alpha_k) - (\beta_1 + 2\beta_1 + 3\beta_2 + \cdots + k\beta_k)$

$C_6 = \frac{1}{6!}(\alpha_1 + 2^6\alpha_2 + 3^6\alpha_3 \ldots + k^6\alpha_k) - \frac{1}{2!}(\beta_1 + 2^2\beta_1 + 3^2\beta_2 + \cdots + k^2\beta_k)$

$C_7 = \frac{1}{7!}(\alpha_1 + 2^7\alpha_2 + 3^7\alpha_3 \ldots + k^7\alpha_k) - \frac{1}{3!}(\beta_1 + 2^3\beta_1 + 3^3\beta_2 + \cdots + k^3\beta_k)$

$C_8 = \frac{1}{8!}(\alpha_1 + 2^8\alpha_2 + 3^8\alpha_3 \ldots + k^8\alpha_k) - \frac{1}{4!}(\beta_1 + 2^4\beta_1 + 3^4\beta_2 + \cdots + k^4\beta_k)$

$\vdots$

$$C_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 \ldots + k^q\alpha_k) - \frac{1}{(q-4)!}(\beta_1 + +2^{q-4}\beta_2 + 3^{q-4}\beta_3 \ldots + k^{q-4}\beta_k) \tag{14}$$

This simply implies that the method is said to have order P if $C_0 = C_1 = C_2 = \cdots = C_{P+3} = 0$, but $C_{P+4} \neq 0$. C_{P+4} is the error constant and the error $C_{P+4}h^{P+4}y^{P+4}(x_n)$ is the principal local truncated error at the point (x_n).

The formula (14) can be expressed in a more concise form as

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j$$

$$C_2 = \frac{1}{2!}\left(\sum_{j=1}^{k} j^2\alpha_j\right)$$

$$C_3 = \frac{1}{3!}\left(\sum_{j=1}^{k} j^3\alpha_j\right)$$

$$C_4 = \frac{1}{4!}\left(\sum_{j=1}^{k} j^4\alpha_j\right) - \left(\sum_{j=0}^{k} \beta_j\right)$$

$$C_5 = \frac{1}{5!}\left(\sum_{j=1}^{k} j^5\alpha_j\right) - \left(\sum_{j=1}^{k} j\beta_j\right)$$

$$C_6 = \frac{1}{6!}\left(\sum_{j=1}^{k} j^6 \alpha_j\right) - \frac{1}{2!}\left(\sum_{j=1}^{k} j^2 \beta_j\right)$$

$$C_7 = \frac{1}{7!}\left(\sum_{j=1}^{k} j^7 \alpha_j\right) - \frac{1}{3!}\left(\sum_{j=1}^{k} j^3 \beta_j\right)$$

$$C_8 = \frac{1}{8!}\left(\sum_{j=1}^{k} j^8 \alpha_j\right) - \frac{1}{4!}\left(\sum_{j=1}^{k} j^4 \beta_j\right)$$

$\vdots$

$$C_q = \frac{1}{q!}\left(\sum_{j=1}^{k} j^q \alpha_j\right) - \frac{1}{(q-4)!}\left(\sum_{j=1}^{k} j^{(q-4)} \beta_j\right) \qquad q = 5,6\ldots \qquad (15)$$

4. VALIDATION OF THE PROPOSED TECHNIQUE

To validate the accuracy of the proposed technique, selected fourth-order LMM schemes from various authors [6], [2], [4], [7], [1] and [8] have been utilized for confirmation. They are respectively listed as

(a) $y_{n+7} + \frac{35}{16}y_{n+4} - \frac{21}{16}y_{n+2} - \frac{35}{16}y_{n+6} + \frac{5}{16}y_n - \frac{3}{32}h^4 f_{n+1} - \frac{29}{12}h^4 f_{n+3} - \frac{181}{96}h^4 f_{n+5} + \frac{1}{48}h^4 f_{n+7}$

(b) $y_{n+6} - 4y_{n+5} + 6y_{n+4} - 4y_{n+3} + y_{n+2} - \frac{h^4}{24}(f_{n+6} + 22f_{n+4} + f_{n+2})$

(c) $y_{n+\frac{1}{4}} - y_n - \frac{1}{4}hy_n' - \frac{1}{32}h^2 y_n'' - \frac{1}{384}h^3 y_n''' - \frac{1}{18432}h^4 f_{n+\frac{1}{4}} + \frac{49}{3686400}h^4 f_{n+\frac{5}{7}} - \frac{1}{245760}h^4 f_{n+1}$

(d) $y_{n+4} + y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + \frac{1}{720}h^4 f_n - \frac{124}{720}h^4 f_{n+1} - \frac{474}{720}h^4 f_{n+2} - \frac{124}{720}h^4 f_{n+3} + h^4 f_{n+4}$

(e) $y_{n+\frac{1}{2}} - 4y_{n+\frac{3}{8}} + 6y_{n+\frac{1}{4}} - 4y_{n+\frac{1}{8}} + y_n + \frac{h^4}{2949120}\left[f_n - 124f_{n+\frac{1}{8}} - 474f_{n+\frac{1}{4}} - 124f_{n+\frac{3}{8}} + f_{n+\frac{1}{2}}\right]$

(f) $y_{n+1} - y_n - hy_n' - \frac{1}{2}h^2 y_n'' - \frac{1}{6}h^3 y_n''' - \frac{27312614539002931}{1161157776629760000}h^4 f_n - \frac{215021456509297}{3547982095257600}h^4 f_{n+1} + \frac{2405950254864953}{13516122267648000}h^4 f_{n+2} - \frac{228535928736331}{465478700544000}h^4 f_{n+3} + \frac{43830916431996773}{40548366802944000}h^4 f_{n+4} - \frac{1983716565322187}{1055947052160000}h^4 f_{n+5} + \frac{188870621369290423}{72987060245299200}h^4 f_{n+6} - \frac{16765512493838707}{5913303492096000}h^4 f_{n+7} + \frac{77901096585757121}{31537618624512000}h^4 f_{n+8} - \frac{38820512227495919}{22808456326656000}h^4 f_{n+9} + \frac{26549037149067547}{28963119144960000}h^4 f_{n+10} - \frac{63857023621987}{168951528345600}h^4 f_{n+11} + \frac{42069828441973351}{364935301226496000}h^4 f_{n+12} - \frac{31069509811453}{1267136462592000}h^4 f_{n+13} + \frac{307336703482477}{94612855873536000}h^4 f_{n+14} - \frac{14651877357209}{72572361039360000}h^4 f_{n+15}$

To further authenticate the proposed technique, the compact equation (15) is applied to the discrete scheme presented in (a). The resulting expressions are given as follows:

$$C_0 = 1 + \frac{35}{16} - \frac{21}{16} - \frac{35}{16} + \frac{5}{16} = 0$$

$$C_1 = 1(7) + \frac{35}{16}(4) - \frac{21}{16}(2) - \frac{35}{16}(6) = 0$$

$$C_2 = \frac{1}{2!}\left(1(7)^2 + \frac{35}{16}(4)^2 - \frac{21}{16}(2)^2 - \frac{35}{16}(6)^2\right) = 0$$

$$C_3 = \frac{1}{3!}\left(1(7)^3 + \frac{35}{16}(4)^3 - \frac{21}{16}(2)^3 - \frac{35}{16}(6)^3\right) = 0$$

$$C_4 = \frac{1}{4!}\left(1(7)^4 + \frac{35}{16}(4)^4 - \frac{21}{16}(2)^4 - \frac{35}{16}(6)^4\right) - \left[\frac{3}{32} + \frac{29}{12} + \frac{181}{96} - \frac{1}{48}\right] = 0$$

$$C_5 = \frac{1}{5!}\left(1(7)^5 + \frac{35}{16}(4)^5 - \frac{21}{16}(2)^5 - \frac{35}{16}(6)^5\right) - \left[\frac{3}{32}(1) + \frac{29}{12}(3) + \frac{181}{96}(5) - \frac{1}{48}(7)\right] = 0$$

$$C_6 = \frac{1}{6!}\left(1(7)^6 + \frac{35}{16}(4)^6 - \frac{21}{16}(2)^6 - \frac{35}{16}(6)^6\right) - \frac{1}{2!}\left[\frac{3}{32}(1)^2 + \frac{29}{12}(3)^2 + \frac{181}{96}(5)^2 - \frac{1}{48}(7)^2\right] = 0$$

$$C_7 = \frac{1}{7!}\left(1(7)^7 + \frac{35}{16}(4)^7 - \frac{21}{16}(2)^7 - \frac{35}{16}(6)^7\right) - \frac{1}{3!}\left[\frac{3}{32}(1)^3 + \frac{29}{12}(3)^3 + \frac{181}{96}(5)^3 - \frac{1}{48}(7)^3\right] = 0$$

$$C_8 = \frac{1}{8!}\left(1(7)^8 + \frac{35}{16}(4)^8 - \frac{21}{16}(2)^8 - \frac{35}{16}(6)^8\right) - \frac{1}{4!}\left[\frac{3}{32}(1)^4 + \frac{29}{12}(3)^4 + \frac{181}{96}(5)^4 - \frac{1}{48}(7)^4\right] = \frac{257}{1152}$$

This analysis for this discrete scheme demonstrates that $C_0 = C_1 = C_2 = \cdots = C_7 = 0$, while $C_8 \neq 0$. Consequently, the method is of order 4 as $C_{P+4} \neq 0$ with an associated error constant of $\frac{257}{1152}$.

Similarly, applying the same assessment to (b), (c), (d), (e) and (f) confirms that the methods achieve orders 4, 4, 6, 6, 9 with corresponding error constants $\frac{-31}{720}, \frac{-1877}{55490641920}, \frac{1}{3024}, \frac{1}{3246995275776}, \frac{-1}{6227020800}$ respectively.

5. DISCUSSION OF RESULTS.

The validations of the proposed technique in (15) were carried out with six selected fourth-order LMMs from the literature. The verification process involved computing the coefficients C_0 to C_8 for problems (a, b and c), C_0 to C_{10} for problems (d and e) and C_0 to C_{13} for problem (f) to ensure compliance with the required theoretical constraints that define the accuracy and stability of numerical schemes. The results indicate that the coefficients C_0 to C_7 for (a, b and c), C_0 to C_9 for (d and e) and C_0 to C_7 for (f) all equated to zero, which is a strong affirmation of the method's correctness and consistency across varying accuracy levels. This outcome demonstrates that the proposed approach successfully captures the fundamental properties required for fourth-order numerical schemes while eliminating the cumbersome algebraic manipulations associated with traditional Taylor series expansions.

A key take-away from these results is the efficiency and reliability of the new method in assessing the order and error constants of fourth-order LMMs. By offering a streamlined approach, this method significantly reduces the computational burden typically involved in the analysis of numerical methods, thereby making it more accessible for researchers and practitioners working in the field of numerical analysis.

Furthermore, the ability of this technique to consistently validate multiple fourth-order LMMs underscores its general applicability. This suggests that it could be extended to even higher-order ≥ 5, potentially simplifying the design and analysis of numerical algorithms used in solving ODEs. The results of this study affirm that the newly developed concise method exhibits more efficient, accurate, and computationally feasible approach for determining the order and error constants of fourth-order LMMs. The findings contribute to advancing numerical analysis by offering a powerful tool that simplifies method validation, reduced implementation complexity, thereby enhancing its accuracy, efficiency and stability.

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