Fuzzy Coding Theorem on Generalized Fuzzy Cost Measure

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ABSTRACT--- In the literature of information theory several types of coding theorems involving fuzzy entropy functions exists. In this paper, some new fuzzy coding theorems have been obtained by considering fuzzy cost measure involving utilities. The fuzzy coding theorems obtained here are not only new but also generalizes some well known results available in the literature.

Keywords--- Fuzzy Set, Fuzzy Entropy Function, Fuzzy useful Entropy Function, Fuzzy Cost measure function and Fuzzy Coding Theorem.

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1. INTRODUCTION

The notion of fuzzy sets was proposed by Zadeh (1965) with a view to tackling problems in which indefiniteness arising from a sort of intrinsic ambiguity plays a significant role. Fuzziness, a feature of uncertainty, results from the lack of sharp distinction of the boundary of a set, i.e., an individual is neither definitely a member of the set nor definitely not a member of it. The first to qualify the fuzziness was made by Zadeh (1968), who based on probabilistic framework introduced the entropy combining probability and membership function of a fuzzy event as weighted Shannon entropy.

Definition 1: Let $X = (x_1, x_2, \ldots, x_n)$ be a discrete universe of discourse. A fuzzy set 'A' on X is defined by a characteristic function $\mu_A(x_i) = \{x_1, x_2, x_{3,\ldots,}x_n\} \longrightarrow [0,1]$.The value of $\mu_A(x)$ of A at $x \in X$ stands for the degree of membership of x in A. If every element of the set 'A'is '0' or '1', there is no uncertainty about it and a set is said to be crisp set.

Definition 2: A fuzzy set A^* is called the sharpened version of fuzzy set A, if the following conditions are satisfied.

$$\mu_{A^*}(x) \le \mu_A \qquad \text{if } \mu_A(x) \le 0.5;\ \ \forall i$$

and

$$\mu_{A^*}(x) \ge \mu_A(x), \quad \text{if } \mu_A(x) \ge 0.5;\ \ \forall i$$

De Luca and Termini (1972) formulated a set of four properties and these properties are widely accepted as criterion for defining any fuzzy entropy. In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity in making a decision whether an element belongs to a set or not. So, a measure of average fuzziness in fuzzy set $H(A)$ should have the following properties to be valid fuzzy entropy.

$\mathbf{P_1}$(Sharpness): $H(A)$ is minimum if and only if A is a crisp set i.e., $\mu_A(x) = 0$ or 1; $\forall$ x.

$\mathbf{P_2}$(Maximality): $H(A)$ is maximum if and only if A is most fuzzy set i.e., $\mu_A(x) = 0.5$; $\forall$ x.

$\mathbf{P_3}$(Resolution): $H(A) \ge H(A^*)$, where A^* is sharpened version of A.

$\mathbf{P_4}$(Symmetry): $H(A) = H(\overline{A})$, where $\overline{A}$ is the complement of A i.e., $\mu_{\overline{A}}(x_i) = 1 - \mu_A(x_i)$.

2. BASIC CONCEPTS

Let X be discrete random variable taking on a finite number of possible values $X = (x_1, x_2, \ldots, x_n)$ with respective membership function $A = \{\mu_A(x_1), \mu_A(x_2), \ldots \mu_A(x_n)\} \rightarrow [0,1], \mu_A(x_i)$ give of the element s the degree of belongingness x_i to the set A.The function $\mu_A(x_i)$ associates with each $x_i \in R^n$ a grade of membership to the set A and is known as membership function.

Denote

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_2 \\ \mu_A(x_1) & \mu_A(x_2) & \cdots & \mu_A(x_n) \end{bmatrix} \tag{2.1}$$

We call the scheme (2.1) as a finite fuzzy information scheme. Every finite scheme describes a state of uncertainty. De Luca and termini (1972) introduced a quantity which, in a reasonable way to measures the amount of uncertainty (fuzzy entropy) associated with a given finite scheme. This measure is given by

$$H(A) = -\sum_{i}^{n} \left[\mu_A(x_i) \log \mu_A(x_i) + \left(1-\mu_A(x_i)\right) \log\left(1-\mu_A(x_i)\right)\right] \tag{2.2}$$

The measure (2.2) serve as a very suitable measure of fuzzy entropy of the finite information scheme(2.1).

Let a finite source of n source symbols $X = (x_1, x_2, \dots, x_n)$ be encoded using alphabet of D symbols, then it has been shown by Feinstein (1958) that there is a uniquely decipherable/ instantaneous code with lengths $l_1, l_2 \dots, l_n$ iff the following Kraft (1949] inequality is satisfied

$$\sum_i^n D^{-l_i} \le 1 \tag{2.3}$$

Belis and Guiasu (1968) observed that a source is not completely specified by the probability distribution P over the source alphabet X in the absence of qualitative character. So it can be assumed (Belis and Guiasu (1968)) that the source alphabet letters are assigned weights according to their importance or utilities in view of the experimenter.

Let $U = (u_1, u_{2,} \dots, u_n)$ be the set of positive real numbers, u_i is the utility or importance of x_i. The utility, in general, is independent of probability of encoding of source symbol x_i, i.e, p_i. The information source is thus given by

$$X = \begin{bmatrix} X_1 & X_2 \dots & X_n \\ p_1 & p_2 \dots & p_n \\ u_1 & u_2 \dots & u_n \end{bmatrix}, u_i > 0\ p_i \ge 0, \sum_i^n p_i = 1 \tag{2.4}$$

Belis and Guiasu (1968] introduced the following quantitative- qualitative measure of information

$$H(P,U) = -\sum_i^n u_i p_i \log p_i \tag{2.5}$$

Which is a measure for the average of quantity of 'variable' or 'useful' information provided by the information source(2.4).

Guiasu and Picard (1971) considered the problem of encoding the letter output by the source (2.4) by means of a single letter prefix code whose codeword's $c_1, c_2, \dots, c_n$ are of lengths $l_1, l_2, \dots, l_n$ respectively and satisfy the Kraft's inequality(2.3), they included the following 'useful' mean length of the code

$$L(U) = \frac{\sum_i^n u_i p_i l_i}{\sum_i^n u_i p_i} \tag{2.6}$$

Further they derived a lower bound for (2.6). However, Longo (1976) interpreted (2.6) as the average transmission cost of the letters x_i and derived the bounds for this cost function.

Now, corresponding to (2.5) and(2.6), we have the following fuzzy measures

$$H(A,U) = -\sum_{i=1}^n u_i \left\{\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right\} \log\left\{\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right\} \tag{2.7}$$

and

$$L(U) = \frac{\sum_{i=1}^n u_i \left\{\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right\} l_i}{\sum_{i=1}^n u_i \left\{\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right\}} \tag{2.8}$$

respectively.

In the next section, the bounds have been derived in terms of generalized 'useful' fuzzy cost measure and 'useful' fuzzy information measure of order α and type β. The main aim of studying these bounds is to generalize some well known results available in the literature.

3. BOUNDS FOR GENERALIZED MEASURE OF COST

In the derivation of the cost measure (2.8), it is assumed that the cost is linear function of the code length, but this is not always the case. There are occasions when the cost behaves like an exponential function of codeword lengths. Such types of functions occur frequently in market equilibrium and growth models in economics. Thus sometimes it might be more appropriate to choose a code which minimizes the monotonic function of the quantity.

$$C = \sum_{i=1}^{n} u_i^{\beta} \left\{ \mu_A{}^{\beta}(x_i) + \left(1 - \mu_A(x_i)\right)^{\beta} \right\} D^{\left(\frac{1-\alpha}{\alpha}\right) l_i} \tag{3.1}$$

Where $\alpha > 0\ (\neq 1)$, $\beta > 0$ are the [parameters related to cost.

In order to make the result of the paper more comparable with the usual noiseless coding theorem, instead of minimizing(3.1), we minimize

$$L_\alpha^\beta(U) = \frac{1}{2^{1-\alpha} - 1} \left[\left(\frac{\sum_{i=1}^{n} u_i^{\beta} \left\{ \mu_A{}^{\beta}(x_i) + \left(1 - \mu_A(x_i)\right)^{\beta} \right\} D^{\left(\frac{1-\alpha}{\alpha}\right) l_i}}{\sum_{i=1}^{n} \left(u_i \left\{ \mu_A(x_i) + \left(1 - \mu_A(x_i)\right) \right\} \right)^{\beta}} \right)^{\alpha} - 1 \right] \tag{3.2}$$

where, $\alpha > 0\ (\neq 1), \beta > 0$,

Which is monotonic function of C and is the 'useful' fuzzy average code length of order α and type β.

Clearly, if $\alpha \to 1$, $\beta = 1$ (3.2) reduces to (2..8) which further reduces to ordinary mean length corresponding to Shannon (1948] when $u_i = 1,\ \forall\, i = 1,2,\dots,n$. It can also be noted that (3.2) is monotonic non-decreasing function of α and if all the l_i's are same, say $l_i = l,\ \forall\, i = 1,2,\dots,n$ and $\alpha \to 1$, then $L_\alpha{}^{\beta}(U) = l$. This is an important property for any measure of length to posses.

Now, consider a function, which is 'useful' fuzzy information measure of order α and type β

$$H_\alpha^\beta(A, U) = \frac{1}{2^{1-\alpha} - 1} \left[\frac{\sum_{i=1}^{n} u_i^{\beta} \left\{ \mu_A^{\alpha+\beta-1}(x_i) + \left(1 - \mu_A(x_i)\right)^{\alpha+\beta-1} \right\}}{\sum_{i=1}^{n} \left(\mu_i^{\beta} \left\{ \mu_A(x_i) + \left(1 - \mu_A(x_i)\right) \right\} \right)^{\beta}} - 1 \right] \tag{3.3}$$

Where, $\alpha > 0\ (\neq 1),\ \beta > 0\ \ \mu_A(x_i) \geq 0;\ \forall\, i = 1,2,\dots,n;\ \sum_{i=1}^{n} \mu_A(x_i) \leq 1$

Remark 3.1

(1) When $\beta = 1$, (3.3) reduces to the measure of 'useful' fuzzy information corresponding to Hooda and Ram (1998].

(2) When $\alpha \to 1, \beta = 1$, (3.3) reduces to the measure corresponding Belis and Guiasu (1968).

(3) When $\alpha \to 1,\ \beta = 1$ and $u_i = 1, \forall\, i = 1,2,\dots,n$ (3.3) reduces to the Du Luca and Termini (1972].

Also the bounds are obtained for the measure (3.3) under the condition

$$\sum_{i=1}^{n} u_i^{\beta} \left\{ \mu_A^{\beta-1}(x_i) + \left(1 - \mu_A(x_i)\right)^{\beta-1} \right\} D^{-l_i} \leq \sum_{i=1}^{n} u_i^{\beta} \left\{ \mu_A^{\beta}(x_i) + \left(1 - \mu_A(x_i)\right)^{\beta} \right\} \tag{3.4}$$

It may be seen that in case $\beta = 1$ and $u_i = 1, \forall\, i = 1,2,\dots,n$ (3.4) reduces to the Kraft (1949) inequality (2.3). Also, D is the size of the code alphabet.

Theorem 3.1: For all integers D (D≥2), let l_i satisfies (3.4), then the generalized average 'useful' codeword length satisfies

$$L_\alpha^\beta(U) \geq H_\alpha^\beta(A;U) \tag{3.5}$$

Equality holds iff

$$l_i = -\log\left\{\mu_i^\alpha(x_i) + \left(1-\mu_A(x_i)\right)^\alpha\right\} + \log\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}} \tag{3.6}$$

Proof: By Holder's inequality (1967]

$$\sum_{i=1}^n x_i y_i \geq \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \tag{3.7}$$

For all, $x_i, y_i > 0, i = 1,2,\dots,n\,;\ \frac{1}{p}+\frac{1}{q} = 1, p < 1\ (\neq 0), q < 0$ or $q < 1(\neq 0), p < 0$. we see the equality holds iff there exists a positive constant C such that

$$x_i^p = cy_i^q \tag{3.8}$$

Making the substitution $\qquad p = \frac{\alpha-1}{\alpha}, q = 1-\alpha$

$$x_i = \frac{\left(u_i\left\{\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right\}\right)^{\frac{\alpha\beta}{\alpha-1}} D^{-l_i}}{\sum_{i=1}^n\left(u_i\left\{\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right\}\right)^{\frac{\alpha\beta}{\alpha-1}}} \quad \text{and} \quad y_i = \frac{u_i^{\frac{\beta}{1-\alpha}}\left(\mu_A^{\frac{\alpha+\beta-1}{1-\alpha}}(x_i)+\left\{1-\mu_A(x_i)\right\}^{\frac{\alpha+\beta-1}{1-\alpha}}\right)}{\sum_{i=1}^n u_i\left\{\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right\}^{\frac{\beta}{1-\alpha}}}$$

In (3.7), we get

$$\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta-1}\right\}D^{-l_i}}{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta-1}\right\}} \geq$$

$$\left[\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}D^{\left(\frac{1-\alpha}{\alpha}\right)l_i}}{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}}\right]^{\frac{\alpha}{\alpha-1}}\left[\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}}\right]^{\frac{1}{1-\alpha}}$$

Using the condition (3.4), we get

$$\left[\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}D^{\left(\frac{1-\alpha}{\alpha}\right)l_i}}{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}}\right]^{\frac{\alpha}{\alpha-1}} \geq \left[\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}}\right]^{\frac{1}{1-\alpha}}$$

Taking $0 < \alpha < 1$, and raising power both sides $(1-\alpha)$, we get

$$\left[\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}D^{\left(\frac{1-\alpha}{\alpha}\right)l_i}}{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}}\right]^{\alpha} \geq \left[\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}}\right]$$

Multiplying both sides by $\frac{1}{2^{1-\alpha}-1} > 0$ for $0 < \alpha < 1$ and after simplifying, we get

$$L_\alpha^\beta(U) \geq H_\alpha^\beta(A;U)$$

For all $\alpha > 1$, the proof follows along the similar lines.

Theorem 3.2: For every code with lengths $l_1, l_2, \dots, l_n$ satisfies (3.4), $L_\alpha^\beta(U)$ can be made to satisfy the inequality

$$L_\alpha^\beta(U) < H_\alpha^\beta(A;U)D^{1-\alpha} + \frac{D^{1-\alpha}-1}{2^{1-\alpha}-1} \tag{3.9}$$

Proof: Let l_i be the positive integer satisfying the inequality

$$-\log\{\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\}+\log\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)}\le$$

$$l_i<-\log\{\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\}+\log\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)}+1 \tag{3.10}$$

Consider the interval

$$\delta_i=\begin{bmatrix}-\log\left(\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right)+\log\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)},\\ -\log\left(\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right)+\log\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)}+1\end{bmatrix} \tag{3.11}$$

Of length 1. In every δ_i, there lies exactly one positive integer l_i such that

$$\begin{aligned}0&<-\log\left(\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right)+\log\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(\mu_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)}\le l_i<\\ &-\log\left(\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right)+\log\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)}+1\end{aligned} \tag{3.12}$$

We will first show that the sequence $l_1, l_2, \dots, l_n$ thus defined satisfies (3.4). From (3.12), we have

$$-\log\left(\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right)+\log\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)}\le l_i$$

or

$$\frac{\left(\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right)}{\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)}}\ge D^{-l_i}$$

Multiplying both sides by $u_i^\beta\left(\mu_A^{\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta-1}\right)$ and summing over $i=1,2,\dots,n$, we get (3.4).

The last inequality in (3.12) gives

$$l_i<-\log\left(\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right)+\log\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)}+1$$

or

$$l_i<\log\left(\frac{\left(\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right)}{\frac{\sum_{i=1}^n u_i^\beta\left\{\mu_A^{\alpha+\beta-1}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^n\left(u_i^\beta\left\{\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta\right\}\right)}}\right)^{-1}D$$

or

$$D^{l_i} < \left(\frac{\left(\mu_A^{\alpha}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha}\right)}{\frac{\sum_{i=1}^{n} u_i^{\beta}\left\{\mu_A^{\alpha+\beta-1}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^{n}\left(u_i^{\beta}\left\{\mu_A^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}\right)}} \right)^{-1} D$$

For $0 < \alpha < 1$, raising power both sides $\frac{1-\alpha}{\alpha}$, we get

$$D^{l_i\left(\frac{1-\alpha}{\alpha}\right)} < \left(\frac{\left(\mu_A^{\alpha}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha}\right)}{\frac{\sum_{i=1}^{n} u_i^{\beta}\left\{\mu_A^{\alpha+\beta-1}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^{n}\left(u_i^{\beta}\left\{\mu_A^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}\right)}} \right)^{\frac{\alpha-1}{\alpha}} D^{\frac{1-\alpha}{\alpha}}$$

Multiplying both sides by

$$\frac{u_i^{\beta}\left\{\mu_A^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}}{\sum_{i=1}^{n}\left(u_i^{\beta}\left\{\mu_A^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}\right)}$$

And summing over $i = 1,2, \ldots, n$, we get

$$\left[\frac{\sum_{i=1}^{n} u_i{}^{\beta}\left\{\mu_A{}^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\} D^{\left(\frac{1-\alpha}{\alpha}\right)l_i}}{\sum_{i=1}^{n} u_i{}^{\beta}\left\{\mu_A{}^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}}\right] < \left[\frac{\sum_{i=1}^{n} u_i^{\beta}\left\{\mu_A^{\alpha+\beta-1}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^{n} u_i{}^{\beta}\left\{\mu_A{}^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}}\right]^{\frac{1}{\alpha}} D^{\frac{1-\alpha}{\alpha}}$$

or

$$\left(\frac{\sum_{i=1}^{n} u_i{}^{\beta}\left\{\mu_A{}^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\} D^{\left(\frac{1-\alpha}{\alpha}\right)l_i}}{\sum_{i=1}^{n} u_i{}^{\beta}\left\{\mu_A{}^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}}\right)^{\alpha} < \left(\frac{\sum_{i=1}^{n} u_i^{\beta}\left\{\mu_A^{\alpha+\beta-1}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^{n} u_i{}^{\beta}\left\{\mu_A{}^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}}\right) D^{1-\alpha}$$

Since $2^{1-\alpha} - 1 > 0$ for $0 < \alpha < 1$ and after suitable operations, we get

$$\frac{1}{2^{1-\alpha}-1}\left[\left(\frac{\sum_{i=1}^{n} u_i{}^{\beta}\left\{\mu_A^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\} D^{\left(\frac{1-\alpha}{\alpha}\right)l_i}}{\sum_{i=1}^{n} u_i{}^{\beta}\left\{\mu_A{}^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}}\right)^{\alpha} - 1\right] <$$

$$\frac{1}{2^{1-\alpha}-1}\left[\left(\frac{\sum_{i=1}^{n} u_i^{\beta}\left\{\mu_A^{\alpha+\beta-1}(x_i) + \left(1-\mu_A(x_i)\right)^{\alpha+\beta-1}\right\}}{\sum_{i=1}^{n} u_i{}^{\beta}\left\{\mu_A{}^{\beta}(x_i) + \left(1-\mu_A(x_i)\right)^{\beta}\right\}}\right)\right] D^{1-\alpha} + \frac{D^{1-\alpha}-1}{2^{1-\alpha}-1}$$

or we can write

$$L_{\alpha}^{\beta}(U) < H_{\alpha}^{\beta}(A;U) D^{1-\alpha} + \frac{D^{1-\alpha}-1}{2^{1-\alpha}-1}$$

As $D \geq 2$, we have $\frac{D^{1-\alpha}-1}{2^{1-\alpha}-1} > 1$ from which it follows that upper bound $L_\alpha^\beta(U)$ in (3.9) is greater than unity. Also, for $\alpha > 1$, the proof follows along the similar lines.

4. REFERENCE

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