Analytical Construction of Uniformly Convergent Method for Convection Diffusion Problem

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ABSTRACT— ***In this paper, we study the uniformly convergent method on equidistant meshes for the convection-diffusion problem of type;***

$$Lu = -\varepsilon u'' + bu' + c\,u = f(x),\;\; u(0) = 0,\;\; u(1) = 0$$

where L^* the formal adjoint operator of L.
At the end of the this paper we will generate the scheme;

$$-\frac{e^{\rho_i}}{e^{\rho_i}+1}U_{i-1} + U_i - \frac{1}{e^{\rho_i}+1}U_{i+1} = (f_i - c_i\,U_i)\,\frac{h}{b}\left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right)$$

Keywords— Local Green's function, convection-diffusion problem, boundary value problem, singular points.

1. INTRODUCTION

We consider a uniformly convergent method, called Il'in-Allen-Southwell Scheme. First of all, we will show how to construct such a method. Then, numerical results can be done similar manner in future study.

2. CONSTRUCTION OF UNIFORMLY CONVERGENT METHOD

We describe a way of constructing a uniformly convergent difference scheme. We start with the standard derivation of an exact scheme for the convection-diffusion problem given above. The formal adjoint of L is defined by

$$L^* = -\varepsilon\frac{d^2}{dx^2} - b\frac{d}{x}$$

$$Lu = -\varepsilon u'' + bu' + c\,u = f(x),\;\; u(0) = 0,\;\; u(1) = 0 \tag{1}$$

Consider the following problem on the unit interval $\Omega = (0,1)$. Let g_i be local Green's function of L^* with respect to the point x_i; that is

$$L^*g_i = -\varepsilon\, g_i'' - b\, g_i' = 0 \;\;\text{in } (x_{i-1}, x_i) \cup (x_i, x_{i+1}) \tag{2}$$

Let us impose boundary conditions

$$g_i\,(x_{i-1}) = g_i(x_{i+1}) = 0$$

And impose additional conditions

$\varepsilon\,(g_i'(x_i^-) - \; g_i'(x_i^+)) = 1.$

Now

$$\int_{x_{i-1}}^{x_{i+1}} (L\,u)\, g_i\, dx = \int_{x_{i-1}}^{x_{i+1}} f\, g_i\, dx$$

$$\int_{x_{i-1}}^{x_{i+1}} (L\,u)\, g_i\, dx = \int_{x_{i-1}}^{x_i} (-\varepsilon u'' + bu')\, g_i\, dx \;+ \int_{x_i}^{x_{i+1}} (-\varepsilon u'' + bu')\;\, g_i\, dx$$

Then apply integrating by parts

$$\int_{x_{i-1}}^{x_i} (-\varepsilon u'' + bu') g_i \, dx + \int_{x_i}^{x_{i+1}} (-\varepsilon u'' + bu') \, g_i \, dx$$

$$= (-\varepsilon u' + \mathrm{bu}) g_i(x)|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} (-\varepsilon u' + \mathrm{bu}) g_i' \, dx + (-\varepsilon u' + \mathrm{bu}) g_i(x)|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} (-\varepsilon u' + \mathrm{bu}) g_i' \, dx$$

$$= \left[\left(-\varepsilon u'(x_i^-) + \mathrm{bu}(x_i)\right) g_i(x_i) - \left(-\varepsilon u'(x_{i-1}) + \mathrm{bu}(x_{i-1})\right) g_i(x_{i-1}) \right]$$
$$+ \left[\left(-\varepsilon u'(x_{i+1}) + \mathrm{bu}(x_{i+1})\right) g_i(x_{i+1}) - \left(-\varepsilon u'(x_i^+) + \mathrm{bu}(x_i)\right) g_i(x_i) \right]$$
$$- \int_{x_{i-1}}^{x_i} (\mathrm{bu}) g_i' \, dx - \int_{x_i}^{x_{i+1}} (\mathrm{bu}) g_i' \, dx + \int_{x_{i-1}}^{x_i} (\varepsilon u') g_i' \, dx + \int_{x_i}^{x_{i+1}} (\varepsilon u') g_i' \, dx$$

From properties of Green's function ($g_i\,(x_{i-1}) = g_i(x_{i+1}) = 0$) and $-\varepsilon\, g_i'' - b\, g_i' = 0$)

$$\int_{x_{i-1}}^{x_{i+1}} (L\,u)\, g_i \, dx = \int_{x_{i-1}}^{x_{i+1}} f\, g_i \, dx = -\varepsilon u'(x_i^-) g_i(x_i) + \varepsilon u'(x_i^+) g_i(x_i) + \left(\varepsilon \mathrm{u(x)}\right) g_i(x)|_{x_{i-1}}^{x_i} + \left(\varepsilon \mathrm{u(x)}\right) g_i(x)|_{x_i}^{x_{i+1}}$$
$$+ \int_{x_{i-1}}^{x_i} (-\varepsilon g_i'' - b g_i') \, \mathrm{u} \, dx + \int_{x_i}^{x_{i+1}} (-\varepsilon g_i'' - b g_i') \, \mathrm{u} \, dx$$

Since u' is continuous on (x_{i-1}, x_{i+1}) then we have

$$= [\varepsilon\, u(x_i) g_i'(x_i^-) - \varepsilon\, u(x_{i-1}) g_i'(x_{i-1}^+)] + [\varepsilon\, u(x_{i+1}) g_i'(x_{i+1}^-) - \varepsilon\, u(x_i) g_i'(x_i^+)]$$

The identity can be written as

$$-\varepsilon g_i'(x_{i-1})\, u_{i-1} + u_i + \varepsilon g_i'(x_{i+1})\, u_{i+1} = (f - cu) \int_{x_{i-1}}^{x_{i+1}} g_i \, dx$$

General solution of second order differential equation (2) is given by
$-\varepsilon\, g_i'' - b\, g_i' = 0$ is

$$g_i(x_i^-) = c_1 + c_2 \left(-\frac{\varepsilon}{b}\right) e^{-\frac{bx}{\varepsilon}} \in (x_{i-1}, x_i) \tag{3}$$

$$g_i(x_i^+) = c_1' + c_2' \left(-\frac{\varepsilon}{b}\right) e^{-\frac{bx}{\varepsilon}} \in (x_i, x_{i+1}) \tag{4}$$

We have four unknowns c_1, c_2, c_1' and c_2' , hence we need four equations:
$g_i\,(x_{i-1}) = 0$, $g_i(x_{i+1}) = 0$, $\varepsilon\,(g_i'(x_i^-) - g_i'(x_i^+)) = 1$ and from continuity of g_i at x_i
$g_i(x_i^-) = g_i(x_i^+)$
Imposing boundary conditions

$$g_i(x_{i-1}) = c_1 + c_2 \left(-\frac{\varepsilon}{b}\right) e^{-\frac{bx_{i-1}}{\varepsilon}} = 0$$

$$g_i(x_{i+1}) = c_1' + c_2' \left(-\frac{\varepsilon}{b}\right) e^{-\frac{bx_{i+1}}{\varepsilon}} = 0$$

By taking derivative of the equations (3) and (4)

$$g_i'(x_i^-) = c_2 \left(-\frac{b}{\varepsilon}\right)\left(-\frac{\varepsilon}{b}\right) e^{-\frac{bx_{i-1}}{\varepsilon}} = c_2 e^{-\frac{bx_i}{\varepsilon}}$$

$$g_i'(x_i^+) = c_2' \left(-\frac{b}{\varepsilon}\right)\left(-\frac{\varepsilon}{b}\right) e^{-\frac{bx_i}{\varepsilon}} = c_2'\, e^{-\frac{bx_i}{\varepsilon}}$$

Apply for the equation $\varepsilon\,(g_i'(x_i^-) - g_i'(x_i^+)) = 1$ and get

$$\varepsilon \left(c_2 e^{-\frac{bx_i}{\varepsilon}} - c_2'\, e^{-\frac{bx_i}{\varepsilon}}\right) = 1 \Rightarrow c_2 - c_2' = \frac{1}{\varepsilon}\, e^{\frac{bx_i}{\varepsilon}}$$

We have written $g_i(x_i^-) = g_i(x_i^+)$ from continuity of g_i at $x = x_i$

$$g_i(x_i^-) - g_i(x_i^+) = 0$$

$$c_1 + c_2 \left(-\frac{\varepsilon}{b}\right) e^{-\frac{bx_i}{\varepsilon}} - \left(c_1' + c_2' \left(-\frac{\varepsilon}{b}\right) e^{-\frac{bx_i}{\varepsilon}}\right) = 0$$

and then we have

$$(c_1 - c_1') + (c_2 - c_2')\left(-\frac{\varepsilon}{b}\right) e^{-\frac{bx_i}{\varepsilon}} = 0$$

Let us simplify that

$$\alpha_i = \frac{bx_i}{\varepsilon}$$

$$\rho_i = \frac{bh}{\varepsilon}$$

We can write

$e^{\frac{bx_{i+1}}{\varepsilon}} = e^{\frac{bx_i+h}{\varepsilon}} = e^{\frac{bx_i}{\varepsilon}}.e^{\frac{bh}{\varepsilon}} = e^{\alpha_i+\rho_i}$ and $e^{\frac{bx_{i-1}}{\varepsilon}} = e^{\frac{bx_i-h}{\varepsilon}} = e^{\frac{bx_i}{\varepsilon}}.e^{-\frac{bh}{\varepsilon}} = e^{\alpha_i-\rho_i}$

We transform the equations

$$c_1 + c_2\left(-\frac{\varepsilon}{b}\right)e^{-\alpha_i+\rho_i} = 0 \tag{5}$$

$$c_1' + c_2'\left(-\frac{\varepsilon}{b}\right)e^{-\alpha_i-\rho_i} = 0 \tag{6}$$

$$c_2 - c_2' = \frac{1}{\varepsilon}e^{\alpha_i} \tag{7}$$

$$(c_1 - c_1') + (c_2 - c_2')\left(-\frac{\varepsilon}{b}\right)e^{-\alpha_i} = 0 \tag{8}$$

Plug the equation (7) into the equation (8) we get

$$(c_1 - c_1') = \frac{1}{b} \tag{9}$$

After using symbolic programming MATHEMATICA we obtain

$$c_1 = \frac{1}{b}\left(\frac{e^{\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right), \quad c_2 = \frac{e^{\alpha_i}}{\varepsilon}\left(\frac{1-e^{-\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right) \tag{10}$$

$$c_1' = \frac{1}{b}\left(\frac{e^{-\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right), \quad c_2' = \frac{e^{\alpha_i}}{\varepsilon}\left(\frac{1-e^{\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right) \tag{11}$$

Let us impose the equations (10) and (11), then we can rewrite the equations (3) and (4) as follows

$$g_i(x_i^-) = c_1 + c_2\left(-\frac{\varepsilon}{b}\right)e^{-\frac{bx}{\varepsilon}} = \frac{1}{b}\left(\frac{e^{\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right) + \frac{e^{\alpha_i}}{\varepsilon}\left(\frac{1-e^{-\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right)\left(-\frac{\varepsilon}{b}\right)e^{-\frac{bx}{\varepsilon}} \tag{12}$$

$$g_i(x_i^+) = c_1' + c_2'\left(-\frac{\varepsilon}{b}\right)e^{-\frac{bx}{\varepsilon}} = \frac{1}{b}\left(\frac{e^{-\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right) + \left(-\frac{\varepsilon}{b}\right)e^{-\frac{bx}{\varepsilon}}\ \frac{e^{\alpha_i}}{\varepsilon}\left(\frac{1-e^{\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right) \tag{13}$$

Taking derivative equations (12) and (13),

$$g_i'(x_i^-) = \frac{1}{\varepsilon}\left(\frac{1-e^{-\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right)$$
$$g_i'(x_i^+) = \frac{1}{\varepsilon}\left(\frac{1-e^{\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right)$$

where $\alpha_i = \frac{bx_i}{\varepsilon}$ and $\rho_i = \frac{bh}{\varepsilon}$. We can easily obtain

$$g_i'(x_{i-1}^-) = \frac{1}{\varepsilon}\left(\frac{e^{\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right)$$
$$g_i'(x_{i+1}^-) = \frac{1}{\varepsilon}\left(\frac{e^{-\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right)$$

Now, we can calculate the integral using g_i^- and g_i^+

$$(f-cu)\int_{x_{i-1}}^{x_{i+1}} g_i\,dx = (f-cu)\left(\int_{x_{i-1}}^{x_i} g_i^-\,dx + \int_{x_i}^{x_{i+1}} g_i^+\,dx\right)$$

$$= (f-cu)\left(\int_{x_{i-1}}^{x_i}\left(\frac{1}{b}\left(\frac{e^{\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right) + \frac{e^{\alpha_i}}{\varepsilon}\left(\frac{1-e^{-\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right)\left(-\frac{\varepsilon}{b}\right)e^{-\frac{bx}{\varepsilon}}\right)dx + \int_{x_i}^{x_{i+1}}\left(\frac{1}{b}\left(\frac{e^{-\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right) + \frac{e^{\alpha_i}}{\varepsilon}\left(\frac{1-e^{\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right)\left(-\frac{\varepsilon}{b}\right)e^{-\frac{bx}{\varepsilon}}\right)dx\right)$$

İntegrate with respect to x;

$$= (f-cu)\left[\frac{1}{b}\left(\frac{e^{\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right).x\Big|_{x_{i-1}}^{x_i} + \frac{e^{\alpha_i}}{\varepsilon}\left(\frac{1-e^{-\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right)\left(-\frac{\varepsilon}{b}\right).\left(-\frac{b}{\varepsilon}\right)e^{-\frac{bx}{\varepsilon}}\Big|_{x_{i-1}}^{x_i}\right.$$

$$\left. + \frac{1}{b}\left(\frac{e^{-\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}\right).x\Big|_{x_i}^{x_{i+1}} + \frac{e^{\alpha_i}}{\varepsilon}\left(\frac{1-e^{-\rho_i}}{e^{\rho_i}-e^{-\rho_i}}\right)\left(-\frac{\varepsilon}{b}\right).\left(-\frac{b}{\varepsilon}\right)e^{-\frac{bx}{\varepsilon}}\Big|_{x_i}^{x_{i+1}}\right]$$

Remember that $x_{i+1} = x_i + h$, $x_{i-1} = x_i - h$ and $x_{i+1} - x_{i-1} = h$.

Finally, it can be written as follows

$$(f-cu)\int_{x_{i-1}}^{x_{i+1}} g_i\,dx = (f-cu)\left(\frac{h}{b}\right)\left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right)$$

$$-\varepsilon g_i'(x_{i-1})\,u_{i-1} + u_i + \varepsilon g_i'(x_{i+1})\,u_{i+1} = (f-cu)\int_{x_{i-1}}^{x_{i+1}} g_i\,dx$$

$$-\frac{e^{\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}U_{i-1} + U_i - \frac{1-e^{-\rho_i}}{e^{\rho_i}-e^{-\rho_i}}U_{i+1} = (f_i - c_i\,U_i)\,\frac{h}{b}\left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right)$$

Re-arranging above equation and get we get

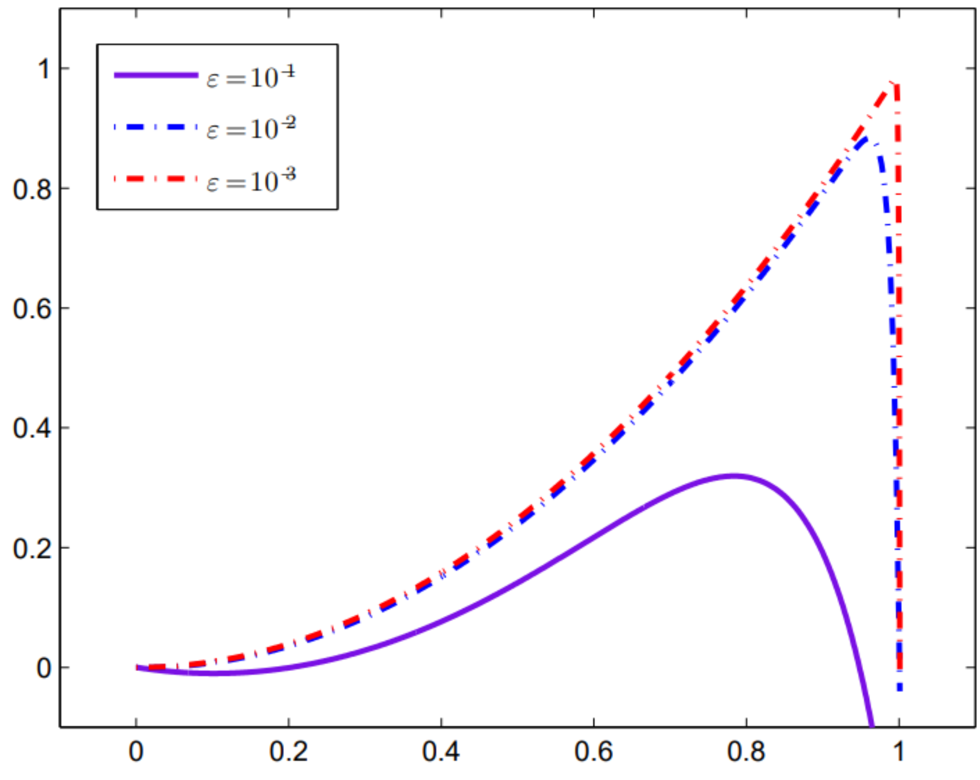
$$-\frac{e^{\rho_i}}{e^{\rho_i}+1}U_{i-1} + U_i - \frac{1}{e^{\rho_i}+1}U_{i+1} = (f_i - c_i\,U_i)\,\frac{h}{b}\left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right)$$

This result is exactly same a El-Mistikawy-Werle scheme in [6].

Example 1: In equation (1), if we choose b=1, c=0, u(0)=0, u(1)=0 and $f(x) = 2x$ we get exact solution

$$u(x) = \frac{1 - e^{\frac{x}{\varepsilon}} - x^2 + x^2\,e^{\frac{1}{\varepsilon}} + 2\varepsilon - 2e^{\frac{x}{\varepsilon}} - 2\varepsilon x + 2\varepsilon x\,e^{\frac{1}{\varepsilon}}}{-1 + e^{\frac{1}{\varepsilon}}}$$

Above, the solution changes rapidly in the interval $(1-\varepsilon, 1)$. That is, there is an ε wide boundary layer around $x = 1$ as $\varepsilon \to 0$. This shows that the boundary layer thickness gets thinner as ε gets smaller. Therefore, it remains important to develop effective algorithms for the numerical solution of such problems.



Graphics 1: Exact solution of equation (1) for different ε values.

3. CONCLUSION

In fact, we have obtained Analytical Construction of Uniformly Convergent Method for Convection Diffusion Problem.

4. REFERENCES

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