,

# Compactly completeness and finitarily completeness

# on continuous information system

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## Abstract

The plane here,  introduce and study the concepts of  bounded completeness and finitely completeness on continuous information system.  Further more compactly completeness, finitarily completeness and strongly compactly completeness for  continuous information system. Some interactions between these concepts are investigated.   Some corresponding results in posets and domains due to R.Hechmann [5] are generalized.


Key Words :   continuous information system , compactly completeness, finitarily completeness, poset


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## 1    Introduction:

In [5], the concepts of bounded complete posets, bounded complete domains, finitely complete posets, finitely complete domains. It worth to mention that H. Zhang [18] studied some interactions between bounded complete domains and scott-topology and lawson topology.  It is interset to mention that in 1994 [1], S. Abramsky and A. Jung considered the concepts of continuous directed complete posets (continuous domain) and algebraic domains. R. Hekmann considered and studies these concepts in detail in this paper [5]. Continuous posets were introduced and studied independently by R. E. Hoffmann [ 2,6,7,8], J. D. Lawson [11,12] and in more fashion by G. Markowsky [14] and M. Erne [3]. It is worth to mention that J. Nino-Salcedo, considered and studied in moredetails.  Our aims here is devoted to introduce and study concepts of  bounded completeness and finitely completeness on continuous information system.

Further more compactly completeness, finitarily completeness and strongly compactly completeness for continuous information system. Some interactions between these concepts are investigated. Some corresponding results in posets and domains due to R.Hechmann [5] are generalized.

**Definition 1.1.** Let $A \subseteq X$. Then:

(1) The lower (resp. upper) bounded subset in $X$ of $\lambda$ is denoted by $lb(\lambda)$ (resp. $ub(\lambda)$) and defined as follows:

$lb(\lambda) = \{x \in X : \forall y \in \lambda, \ x \leq y)$ (resp. $ub(\lambda) = \{x \in X : \forall y \in \lambda, \ y \leq x)$.

(2) The subset of least (resp. largest) elements of a subset $\lambda$ is denoted by $le(\lambda)$ (resp. $la(\lambda)$) and defined as follows:

$le(\lambda) = \{x \in \lambda : \forall y \in \lambda, \ x \leq y)$ (resp. $la(\lambda) = \{x \in \lambda : \forall y \in \lambda, \ y \leq x)$.Each element in $le(\lambda)$ (resp. $la(\lambda)$) is aclled a least (resp. largest ) element of $\lambda$ [13].

(3) The infimum (resp. supremum) subset in $X$ is denoted by $\bigwedge(\lambda)$ (resp. $\bigvee(\lambda)$) and defined as follows: $\bigwedge(\lambda) = la(lb(\lambda))$ (resp. $\bigvee(\lambda) = le(ub(\lambda))$.Each element in $\bigwedge(\lambda)$ (resp. $\bigvee(\lambda)$) is aclled a infimum (resp. supremum ) element of $\lambda$ [13].

(4) The lower ( resp. upper) closure in $X$ of $\lambda$ is denoted by $\downarrow \lambda$ (resp. $\uparrow \lambda$ ) and defined as follows: $\downarrow \lambda = \{x \in X : \exists \ y \in \lambda \ s.t. \ x \leq y \}$( resp. $\uparrow \lambda = \{x \in X : \exists \ y \in \lambda \ s.t. \ y \leq x \}$)[5];

(5) An upper (resp. a lower) cone of $X$ iff $\exists \ x \in \lambda \ s.t. \ \lambda = \uparrow x$ (resp. $\lambda = \downarrow x$)[5].

**Proposition 1.1.** (Proposition 5.2.1 [5]) For a poset $(X, \leq)$, the following statements are equivalent:

(1) $X$ is upper cone, and $\forall x, y \in X$, then the set $\uparrow x \cap \uparrow y$ is empty or an upper cone;

(2) $X$ has a least element, and every two points with a common upper bound have a common least upper bound;

(3) The set of upper bounds of a finite set is either empty or an upper cone; and

(4) Every finite bounded subset of $X$ has a supremum.

Poset with each of these equivalent properties are called bounded complete.

**Proposition 1.2.** (Proposition 5.2.2 [5]). For a poset $(X, \leq)$, the following statements are equivalent:

(1) Every bounded subset of $X$ has a supremum. and;

(2) Every nonempty subset of $X$ has a infimum..

**Proposition 1.3.** (Proposition 5.2.3 [5]). For a domain $X$, the statements in Proposition 2.1 and Proposition 2.2 above are equivalent. Every domian satisfies one of these statments is called a bounded complete domain.

**Proposition 1.4.** (Proposition 5.2.4 [5]). Arbitrary products of bounded complete domains are bounded complete domains.

**Proposition 1.5.** (Proposition 5.3.1 [5]). For a poset $(X, \leq)$, the following statements are equivalent:

(1) $X$ is upper cone, and $\forall x, y \in X$, then the set $\uparrow x \cap \uparrow y$ is empty or an upper cone;

(2) $X$ has a least element, and every two points have a common least upper bound;

(3) The set of upper bounds of a finite set is an upper cone; and

(4) Every finite subset of $X$ has a supremum.

**Proposition 1.6.** (Proposition 5.3.2 [5]). For a poset $(X, \leq)$, the following statements are equivalent:

(1) $X$ is finitely complete domain;

(2) All finite and all directed subset of $X$ have supermum;

(3) Every subset of $X$ has a supremum;

(4) Every subset of $X$ has a infimum; and

(5) $X$ is bounded complete domains with greatest element.

We call posets which satisfies one of these equivalent properties finitely complete domains. The complete domain is also known as a complete lattice.

**Proposition 1.7.** (Proposition 5.4.1 [5]). For a poset $(X, \leq)$, the following statements are equivalent:

(1) The set of upper bounds f every finite set is finitary;

(2) $X$ is finitary, and for every two points $x$ and $y$, the set $\uparrow x \cap \uparrow y$ is finitary;

(3) $X$ is finitary, and intersection of two finitary upper sets is finitary; and

(4) Finite intersection of finitary sets are finitary.

A poset satisfying one of these equivalent conditions is said to have finitarily complete.

**Proposition 1.8.** (Proposition 5.4.2 [5]). In a poset $(X, \leq)$, the following statements are equivalent:

(1) Every finite poset is finitarily complete; and

(2) Every bounded copmlete poset is finitarily complete.

**Proposition 1.9.** (Proposition 5.4.3 [5]). If $X$ and $Y$ are finitarily set, then $X \times Y$ is finitary set.

**Proposition 1.10.** (Proposition 4.7.4 [5]). If $A$ and $B$ are strongly compact, then $A \cup B$ so is.

**Proposition 1.11.** (Proposition 5.5.1 [5]). For a domain $X$, the following statements are equivalent:

(1) $X$ is strongly compac; and $\forall x, y \in X$, then the set $\uparrow x \cap \uparrow y$ is strongly compact; and

(2) $X$ is finitary and the intersection of two finitary upper sets is strongly compact.

Domains with these equivalent properties are called strongly compactly complete.

**Definition 1.2.** Let $`\leq'$ be a binary relation set on $X \neq \phi$. Then;

(1) $`\leq'$ is called reflexive iff $\forall x \in X, x \leq x$ [13];

(2) $`\leq'$ is called antisymetric iff $\forall x, y \in X, x \leq y$ and $y \leq x \Rightarrow x = y$ [13];

(3) $`\leq'$ is called transitive iff $\forall x, y, z \in X, x \leq y$ and $y \leq z \Rightarrow x = z$[13];

(4) $`\leq'$ is called symetric iff $\forall x, y \in X, x \leq y \Rightarrow y \leq x$ [13];

(5) $`\leq'$ is called interpolative iff $\forall x, z \in X,$ with $x \leq z, \exists y \in X$ s.t. $x \leq y \leq z$ [5, 16].

(6) if $`\leq'$ satisfies the conditions (1), (2) and (3), then $(X, \leq)$ is called Partialy order set (Poset) [13];

(7) if $`\leq'$ satisfies the conditions (1), and (3), then $(X, \leq)$ is called pre-orderd set (Quasi set)[13];

(8) if $`\leq'$ satisfies the conditions (1), (2), (3) and (4), then $(X, \leq)$ is called an equvalence set,

(9) if $`\leq'$ satisfies the conditions (3) and (5), then $(X, \leq)$ is a continuous information system [10,16].

(10) if ' $\leq'$ satisfies the conditions (3), and $\forall\ x \in X$, and for every finite subset $\lambda$ of $X$ the following axiom holds: if $\forall\ y \in \lambda,\ y \leq x$ then $\exists\ z \in X\quad s.t.\ \forall\ y \in \lambda,\ y \leq z$ and $z \leq x,$ then $(X, \leq)$ is abstract basis [17].

**Definition 1.3.** (1) A poset $(X, \leq)$ is called domain iff for every directed subset $\lambda$ of $X$, $\bigvee(\lambda)$ exists [5].

(2) $\lambda$ is called directed subset of $X$ iff $\lambda \neq \phi$ and $\forall\ x, y \in \lambda,\ \exists\ z \in \lambda\quad s.t.\ x \leq z$ and $y \leq z$ [5];

**Definition 1.4.** Let $A \subseteq X$. Then:

(1) A subset $A$ of the domain [3] (resp. Poset ) $X$ is called directed closed ( $d$-closed for short) iff $\forall$ directed subset $D$ of $A,\ \bigvee(D) \in A$;

(2) A subset $A$ of the Poset $X$ is called Scott-closed iff $A$ is d-closed lower subset of $X$ [11] ;

(3) A is called d-(resp. Scott-) open iff $A^c$ d-(resp. Scott-) closed [3,11];

**Definition 1.5.** For any poset $X$ consider the following topologies:

(1) $\delta_d = \{A \subseteq X :\ A$ is $d$-open$\}$ is a topology on $X$ ( see proposition 3.5.2 [3]) in the case of $X$ is a domain) and is called the directed topology ) ($d$-topology for short );

(2) $\delta_{A|x} = \{A \subseteq X : A$ is an upper subset $\}$ is a topology on $X$ ([3]) in the case of $X$ is a domain) and is called the Alexandroff topology ) (Alex-topology for short );

(3) $\delta_S = \{A \subseteq X : A$ is Scott open subset $\}$ is a topology on $X$ ( see [3,6,11]) ) and is called the Scott-topology );

(4) The upper topology on $X$ is denoted by $\delta_U$ and is the topology generated by subbasis $\{X- \downarrow x : x \in X\}$[6];

(5) TheLower topology on $X$ is denoted by $\delta_L$ and is the topology generated by subbasis $\{X- \uparrow x : x \in X\}$[6];

(6) The interval topology $\delta_I$ on $X$ is the supremum of $\delta_U$ and $\delta_L$ i.e., $\delta_I = \delta_U \bigvee \delta_L$[6] ;

(7) The Lowson topology $\delta_{LS}$ on $X$ is the supremum of $\delta_S$ and $\delta_L$ i.e., $\delta_{LS} = \delta_L \bigvee \delta_S$ [6].

(8) Let $(X, \delta)$ be a topological space, and let $A \subseteq X$, then the closure of $\lambda$ denoted by $cl_\delta(\lambda)$ defined as follows $cl_\delta(\lambda) = \cap \{F \subseteq X :\ F$ is $\delta$-closed and $\lambda \subseteq F\}$ [9].

**Proposition 1.5** (Proposition 4.6.8 [5]).Let $(X, \leq)$ be a domain. Then every compact open set in $(X, \delta_S)$ is finitary.

**Definition 1.6.**[5] Let $(X, \leq)$ be a poset. A subset $\lambda$ of $X$ is called finitary iff $\exists.$a finite subset $F$ of $A$ with $\lambda \subseteq \uparrow(F)$

**Definition 1.7.** [5] Let $(X, \leq)$ be a poset . A subset $A \subseteq X$ is called strongly compact iff $\forall\ \bigcirc\ \in \delta_S$ with $A \subseteq \bigcirc,\ \exists$ a finitary set $F$ with $A \subseteq F \subseteq \bigcirc$.

## 2. Bounded complete continuous information system and bounded complete domain continuous information system

**Definition 2.1.** An continuous information system $(X, \leq)$ is bounded complete iff $X$ is upper cone, and $\forall\ x, y \in X,\ ub(\{x, y\})$ is empty or an upper cone.

**Theorem 2.1.** For a continuous information system $(X, \leq)$, the following statements are equivalent:

    (1) $X$ is bounded complete continuous information system;

    (2) $le(X) \neq \phi$ and $\forall x, y \in X$, with $ub(\lambda) = (\{x, y\}) \neq \phi$, $\bigvee(\{x, y\}) \neq \phi$;

    (3) If $\lambda$ is a finite bounded subset from above, then $\bigvee(\lambda) \neq \phi$; and

    (4) If $\lambda$ is a finite subset of $X$, then $ub(\lambda)$ is either $\phi$ or an upper cone.

**Proof** $(1) \Rightarrow (2)$ : Because $X$ is bounded complete, then $X$ is an upper cone, and $\exists \, a \in X$ s.t., $\uparrow a = X$. So, $\{\lambda\} \subseteq lb(X)$. If $ub(\{x, y\}) \neq \phi$, then $ub(\{x, y\})$ is an upper cone. There exists $\mu \in ub(\{x, y\})$ s.t., $\uparrow \mu = ub(\{x, y\})$. So, $\mu \in lb(ub(\{x, y\}))$, i.e., $\bigvee(\{x, y\}) \neq \phi$;

$(2) \Rightarrow (3)$ : $\phi$ is a finite bounded subset from above, Since $ub(\phi) = X \neq \phi$. Since $le(X) = le(ub(\phi)) \neq \phi$, $\bigvee(\phi) \neq \phi$. Let $\lambda$ be a non-empty finite bounded subset from above. If $\lambda = \{z\}$ and $ub(\{z\}) \neq \phi$, then $\bigvee(\lambda) \neq \phi$. Let $\lambda = \{x_1, x_2, x_3, \ldots, x_n\}$ and $ub(\lambda) \neq \phi$. Now $\lambda_{1,2} = \{x_1, x_2\}$ and $ub(\lambda_{1,2}) \neq \phi$, $\bigvee(\lambda_{1,2}) \neq \phi$. Take $u_{1,2} \in \bigvee(\lambda_{1,2})$ and consider $\lambda_{1,2,3} = \{u_{1,2}, x_3\}$. Then $\bigvee(\lambda_{1,2,3}) \neq \phi$. because $ub(\lambda_{1,2,3}) \neq \phi$. We can proceed until consider the set $\mu = \{u_{1,2,\ldots\ldots\ldots n-1}, x_n\}$. Since $ub(\mu) \neq \phi$, then $\bigvee(\mu) \neq \phi$. Now $\forall l \in \bigvee(\mu)$, $l \in ub(\lambda)$. Since $m \in ub(\lambda)$, one can deduce that $l \leq m$. Then $l \in \bigvee(\lambda)$. So, $\bigvee(\lambda) \neq \phi$;

$(3) \Rightarrow (4)$ : Let $\lambda$ be a finite subset of $X$. If $\lambda$ is not bounded from above, then $ub(\lambda) = \phi$. Let $\lambda$ is a finite bounded subset from above. Then $\bigvee(\lambda) = le(ub(\lambda)) \neq \phi$. Then $\exists \, x \in ub(\lambda)$ s.t., $\uparrow x = ub(\lambda)$;

$(4) \Rightarrow (1)$ : Now, $X = ub(\phi)$ and so $X$ is an upper cone. Since $\forall x, y \in X$, $\{x, y\}$ is finite. Then $ub(\{x, y\}) = \phi$ or $ub(\{x, y\})$ is an upper cone. .

**Note 2.1.** We refere that Theorem 2.1 is a generalization of the corresponding result in Proposition 1.1 (Proposition 5.2.1 [5]).

The following Lemma is a generalization of the corresponding result in Proposition 1.2 (Proposition 5.2.2 [5]).

**Lemma 2.1.** For a continuous information system $(X, \leq)$, the following statements are equivalent:

    (1) If $\lambda$ is a finite bounded subset from above, then $\bigvee(\lambda) \neq \phi$; and

    (2) If $\lambda$ is a non-empty subset of $X$, then $\bigwedge(\lambda) \neq \phi$.

**Proof** $(1) \Rightarrow (2)$ : Let $\lambda$ is a non-empty subset of $X$ and let $\mu = lb(\lambda)$, Now $ub(\mu) \supseteq \lambda \neq \phi$. Then $\bigvee(\mu) \neq \phi$; let $x \in \bigvee(\mu)$. Now, $x \in ub(\mu)$. Then $\forall \, a \in \lambda$, $x \leq a$. Then $x \in la(lb(\lambda)) = \bigwedge(\lambda)$. Hence $\bigwedge(\lambda) \neq \phi$.

$(2) \Rightarrow (1)$ : Let $\lambda$ is a finite bounded subset of $X$ from above. and let $\mu = ub(\lambda) \neq \phi$. Then $\bigwedge(\mu) \neq \phi$; let $x \in \bigwedge(\mu)$. Since, $\lambda \subseteq lb(\mu)$ and $x \in \bigwedge(\mu)$, then $\forall \, a \in \lambda$, $a \leq x$. So, $x \in le(ub(\lambda)) = \bigvee(\lambda)$. Hence $\bigvee(\lambda) \neq \phi$.

**Definition 2.2.** An continuous information system $(X, \leq)$ is bounded complete domain iff it is bounded complete and domain.

**Theorem 2.2.** For a domain continuous information system $(X, \leq)$, the following statements are equivalent:

    (1) $X$ is bounded complete continuous information system;

    (2) $le(X) \neq \phi$ and $\forall x, y \in X$, with $ub(\lambda) = (\{x, y\}) \neq \phi$, $\bigvee(\{x, y\}) \neq \phi$;

(3) If $\lambda$ is a finite bounded subset from above, then $\bigvee(\lambda) \neq \phi$;

(4) If $\lambda$ is a finite subset of $X$, then $ub(\lambda)$ is either $\phi$ or an upper cone;

(5) If $\lambda$ is bounded subset from above, then $\bigvee(\lambda) \neq \phi$; and

(6) If $\lambda$ is a non-empty subset of $X$, then $\bigwedge(\lambda) \neq \phi$.

**Proof** We refere that Theorem 2.1 and Lemma 2.1, it rests to proof that (3) and (5) are equivalent.

(3) $\Rightarrow$ (5) : Let $\lambda$ be bounded subset of $X$ from above. Let $D = \{x : x \text{ is fixed point of } \bigvee F \text{ for every}$ finite subset $F$ of $\lambda\}$. Since $\bigvee(\phi) = \phi$ and $\forall\, y \in \bigvee(F_1 \cup F_2), \;\; y \in ub(\bigvee(F_1) \cup (\bigvee(F_2)))$, where $F_1, F_2$ are finite subsets of $\lambda$  then is directed. Then $\bigvee(D) = \phi$. Now $\forall\, l \in \bigvee(D),\; l \in ub(\lambda)$. Let $z \in ub(\lambda)$. Then $\forall\, m \in \lambda,\; m \leq z$, so  $z \in ub(\lambda)$ Thus $l \leq z$ so that $l \in \bigvee(\lambda)$. Hence$\bigvee(D) \subseteq \bigvee(\lambda)$ so that$\bigvee(\lambda) \neq \phi$.

(5) $\Rightarrow$ (3) : Obvious.

**Note 2.2.** We refere that Theorem 2.2 is a generalization of the corresponding result in Proposition 1.3. (Proposition 5.2.3 [5]).

**Definition 2.3.** [13]  Let $(X_i, \leq_i : i \in I)$ be a family of posets. The Cartesian product relation $' \leq '$ on $\prod\limits_{i \in I} X_i$  of $\{\leq_i : i \in I\}$ is defined as follows: $(x_i)_{i \in I} \leq (y_i)_{i \in I}$ iff  $x_i \leq_i y_i \;\forall\, i \in I$

**Theorem 2.3.** Let $(X_i : i \in I)$ be a family of continuous information system. If $\forall\, i \in I,\; X_i$ is bounded complete domain, then $\prod\limits_{i \in I} X_i$ so is.

**Proof**  Let $\lambda$ be a subset of $\prod\limits_{i \in I} X_i$.bounded from above by a point $u = (u_i)_{i \in I}$. Let $\lambda_i = \prod \lambda_i \;\forall\, i \in I$. Then $\lambda_i$ is a bounded subset of $X_i$ from a bove by $u_i$ , $\forall\, i \in I$. Then $\bigvee(\lambda_i) \neq \phi \;\forall\, i \in I$. Let $k_i \in \bigvee(\lambda_i) \;\forall\, i \in I$ so that $(k_i)_{i \in I} \in \bigvee(\lambda)$. Hence from  Theorem 2.2, $\prod\limits_{i \in I} X_i$.is bounded complete domain.

**Note 2.3.** We refere that Theorem 2.3 is a generalization of the corresponding result in Proposition 1.4. (Proposition 5.2.4 [5]).

## 3. Finitely complete continuous information system and bounded complete domain continuous information system

**Definition 3.1.** An continuous information system $(X, \leq)$ is  finitely complete iff  $X$ is upper cone, and $\forall\, x, y \in X,\; ub(\{x, y\})$ is an upper cone.

One can easily deduce that any Finitely complete continuous information system is a bounded complete continuous information system.

**Theorem 3.1.** For a continuous information system $(X, \leq)$,  the following statements are equivalent:

(1) $X$ is finitley complete continuous information system;

(2) $X$ has a least element and $\forall x, y \in X,\; \bigvee(\{x, y\}) \neq \phi$;

(3) If $\lambda$ is a finite subset of $X$, then$\bigvee(\lambda) \neq \phi$; and

(4) If $\lambda$ is a finite subset of $X$, then $ub(\lambda)$ is an upper cone.

**Proof** $(1) \Rightarrow (2)$ : Since $X$ is an upper cone, and $\exists\, a \in X$ s.t., $\uparrow a = X$. So, $a \in le(X)$.let $x, y \in X$. Then $\exists\, z \in ub(\{x, y\})$ s.t. $\uparrow z \in ub(\{x, y\})$. So, $z \in \bigvee(\{x, y\})$;

$(2) \Rightarrow (3)$ : $\phi$ is a finite set, Since $\exists\, x \in le(X)$, then $\exists\, x \in \bigvee(\phi)$. Let $\lambda = \{x_1, x_2, x_3, ....., x_n\}$, i.e., $\lambda$ is a finite set. Now, $\lambda_{1,2} = \{x_1, x_2\}$, then $\exists\, u_{1,2} \in \bigvee(\lambda_{1,2})$. Put $\lambda_{1,2,3} = \{u_{1,2}, x_3\}$ so that $\exists\, u_{1,2,3} \in \bigvee(\lambda_{1,2,3})$.We can proceed until consider the set $\mu = \{u_{1,2,3,........,n-1}, x_n\}$ so that $\exists\, l \in \bigvee(\mu)$. Then $l \in ub(\lambda)$. Let $m \in ub(\lambda)$. one can deduce that $l \leq m$. Then $l \in \bigvee(\lambda)$.

$(3) \Rightarrow (4)$ : Let $\lambda$ be a finite subset of $X$, then $\bigvee(\lambda) = \phi$. i.e., $\exists\, l \in le(ub(\lambda))$ so that $\uparrow l = ub(\lambda)$; i.e., $ub(\lambda)$ is an upper cone;

$(4) \Rightarrow (1)$ : Since $\phi$ is a finite set, and $X = ub(\phi)$, then $\lambda$) is an upper cone. For every $\{x, y\} \in X$, the set $\{x, y\}$ is finite, then $ub(\{x, y\})$ is an upper cone.

**Note 3.1.**We refere that Theorem 3.1 is a generalization of the corresponding result in Proposition 1.5 (Proposition 5.3.1 [5]).

**Definition 3.2.** An continuous information system $(X, \leq)$ is complete domain iff $X$ is finitely complete and domain.

**Theorem 3.2.** For a continuous information system $(X, \leq)$, the following statements are equivalent:
(1) $X$ is complete domain;
(2) $X$ is bounded complete domain with $la(X) \neq \phi$;
(3) If $\lambda$ is a finite subset of $X$, then $\bigwedge(\lambda) \neq \phi$; and
(4) If $\lambda$ is a finite subset of $X$, then $\bigvee(\lambda) \neq \phi$; and
(5) If $\lambda$ is a finite subset of $X$ or a directed subset of $X$, then $\bigvee(\lambda) \neq \phi$.

**Proof** $(1) \Rightarrow (2)$ : It is clear that any complete domain is bounded complete domain. Since $\forall\, x, y \in X$. $ub(\{x, y\})$ is an upper cone so that $ub(\{x, y\}) \neq \phi$. Then $X$ is directed which imolies that $la(X) = \bigvee(\lambda) \neq \phi$;

$(2) \Rightarrow (3)$ : Let $\lambda$ be a finite subset of $X$. First, if $\lambda = \phi$, then $lb(\phi)$. Since $le(X) \neq \phi$, then $\exists\, l \in \bigwedge(\phi)$. Secound, if $\lambda \neq \phi$, then from Theorem 2.1(6) $\bigwedge(\phi) \neq \phi$.

$(3) \Rightarrow (4)$ : Let $\lambda$ be a finite subset of $X$, Since $\bigwedge(\phi) = \phi$. i.e., $\exists\, l \in la(\lambda)$ so that every subset of $X$ is bounded from above. From Lemma 2.1, $\bigvee(\lambda) \neq \phi$;

$(4) \Rightarrow (5)$ : Obvious; and

$(5) \Rightarrow (1)$ : Since for every directed subset $\lambda$ of $X$, $\bigvee(\lambda) \neq \phi$, then $X$ is a continuous information system. From Theorem 3.1(3), $X$ is a finitely complete continuous information system.

**Note 3.2.**We refere that Theorem 3.2 is a generalization of the corresponding result in Proposition 1.6 (Proposition 5.3.2 [5]).

## 4. **Finitarily complete continuous information system and compactly complete continuous information system**

**Definition 4.1.** Let $(X, \leq)$ be a continuous information system. A subset $\lambda$ of $X$ is called finitary iff $\exists.a$ finite subset $F$ of $A$ with $\lambda \subseteq \uparrow (F)$

**Theorem 4.1.** Let $(X, \leq)$ be a continuous information system. and let $\{\lambda_i : i \in \{1, 2, 3, ....., n\}\}$ be a family of finitary subset of $X$. Then $\bigcup_j = 1^n \lambda_j$ is a finitary subset.

**Proof**   Since $\forall \, i \in \{1, 2, 3, ....., n\}$, $\exists$ a finite subset $K_i$ s.t., $K_i \subseteq \lambda_i \subseteq \uparrow (K_i)$, then
$\bigcup_{j=1}^n K_i \subseteq \bigcup_{j=1}^n \lambda_i \subseteq \bigcup_{j=1}^n \uparrow (K_i) \subseteq \uparrow \bigcup_{j=1}^n (K_i)$. Since $\bigcup_{j=1}^n K_i$ is finite, then it is clear that $\bigcup_{j=1}^n \lambda_i$ is finitary.

**Definition 4.2.** A continuous information system. $(X, \leq)$ is called finitarily complete iff $X$ is called finitary, $\forall x, y \in X$, $ub(\{x, y\})$ is finitary.

**Theorem 4.2.** For a continuous information system $(X, \leq)$, the following statements are equivalent:
    (1) $X$ is finitarily complete;
    (2) $X$ is finitary and if $\lambda$ and $\mu$ are finitary upper sets, then $\lambda \cap \mu$ is finitary;
    (3) If $\lambda_1, ......., \lambda_n$ are finitary subsets, then $\bigcap_{i=1}^n \lambda_i$ is finitary;
    (4) If $\mu$ is a finite subset of $X$, then $ub(\mu)$ is finitary.

**Proof**  $(1) \Rightarrow (2)$ : If is finitarily complete, then $X$ is finitary. If $\lambda$ is finitary upper set, then there exists a finite set $F_1 \subseteq \lambda$ s.t., $\lambda \subseteq \uparrow (F_1)$ and $\uparrow \lambda \subseteq \lambda$. Hence $\lambda = \uparrow (F_1)$ and if $\mu$ is finitary upper set, then there exists a finite set $F_2 \subseteq \mu$ s.t., $\mu \subseteq \uparrow (F_2)$ and $\uparrow \mu \subseteq \mu$. Hence $\mu = \uparrow (F_2)$. Thus
$\lambda \cap \mu = \uparrow (F_1) \cap \uparrow (F_2) = (\bigcup_{a \in F_1} \uparrow a) \bigcap (\bigcup_{b \in F_2} \uparrow b) = \bigcup_{a \in F_1, \; b \in F_2} (\uparrow a \bigcap \uparrow b)$, i.e., $\lambda \cap \mu$ is a finite union of finitary sets. So that $\lambda \cap \mu$ is a finitary;
$(2) \Rightarrow (3)$ : By indication. The empty intersection is $X$;
$(3) \Rightarrow (4)$ : If $\mu$ is a finite, then $ub(\mu) = (\bigcup_{e \in B} \uparrow e)$. upper cones are finitary;
$(4) \Rightarrow (1)$ : $X$ is the set of upper bounds of $\phi$, and $\uparrow x \bigcap \uparrow y$ is the set of upper bounds of $\{x, y\}$.

**Note 4.1.** We refere that Theorem 4.2 is a generalization of the corresponding result in Proposition 1.7. (Proposition 5.4.1 [5]).

**Theorem 4.3.** Let $(X, \leq)$ be a continuous information system, then;
    (1) Every finite continuous information system is finitarily complete; and
    (2) Every bounded complete continuous information system is finitarily complete continuous information system.

**Proof**  Follow directly from Theorem 2.2 and the fact all finite sets are finitary.

**Note 4.2.** We refere that Theorem 4.3 is a generalization of the corresponding result in Proposition 1.8. (Proposition 5.4.2 [5]).

**Theorem 4.4.** Let $(X, \leq)$ be a continuous information system. If $X$ and $Y$ are finitarily complete, then $X \times Y$ so is.

**Proof** Product of finiary sets are finitary. Hence $X \times Y$ is finitary. furthermore,

$\uparrow (x_1, y_1) \bigcap \uparrow (x_2, y_2) = (\uparrow x_1 \times \uparrow y_1) \bigcap (\uparrow x_2 \times \uparrow y_2) = (\uparrow x_1 \bigcap \uparrow y_1) \times (\uparrow x_2 \bigcap \uparrow y_2)$ holds.

The final outcome is finitary as product of finitary sets.

**Note 4.3.** We refere that Theorem 4.4 is a generalization of the corresponding result in Proposition 1.9. (Proposition 5.4.3 [5]).

**Definition 4.3.** Let $(X, \leq, \delta)$ be a topological continuous information system. A subset $\lambda$ of $X$ is called strongly compact $\forall \bigcirc \in \delta$ s.t., $\lambda \subseteq \bigcirc$, $\exists$ a finitary subset $F$ with $\lambda \subseteq F \subseteq \bigcirc$.

**Theorem 4.5.** Let $(X, \leq, \delta)$ be a topological continuous information system such that each member of $\delta$ is an upper subset. If a subset $\lambda$ of $X$ is strongly compact, then $\lambda$ is compact.

**Proof** Let $U$ be an open cover of $\lambda$, i.e., $\lambda \subseteq \bigcup_{\mu \in U} \mu$ and $U \subseteq \delta$. Put $\bigcup_{\mu \in U} \mu = G$. Then $\lambda \subseteq G \subseteq \delta$. Since $\lambda$ of $X$ is strongly compact, then there exists a finitary subset $K$ of $G$ s.t., $\lambda \subseteq K \subseteq G$ so that there exists a finite subset $F$ of $K$ s.t., $K \subseteq \uparrow (F)$, Then $\forall x \in F$, $\exists B_x \in U$ s.t., $x \in B_x$ so that $F \subseteq \bigcup_{x \in F} B_x$ so that $\lambda \subseteq K \subseteq \uparrow (F) \subseteq \uparrow (\bigcup_{x \in F} B_x) = \bigcup_{x \in F} B_x$. Hence $\lambda$ is compact.

**Corollary 4.1.** (1) Let $\delta_{A|x}$ is Alexandroff topology induced by $'\leq'$, let $\lambda$ of $X$ is strongly compact, then $\lambda$ is compact,

(2) Let $\delta_S$ is Scott- topology induced by $'\leq'$, let $\lambda$ of $X$ is strongly compact, then $\lambda$ is compact

**Theorem 4.6.** Let $(X, \leq, \delta)$ be a topological continuous information and let $\{\lambda_i : i \in \{1, 2, 3, ....., n\}\}$ be a strongly compact subsets of $X$. Then $\bigcup_{i=1}^{n} \lambda_i$ is strongly compact subsets of $X$.

**Proof** Let $\bigcirc \in \delta$ s.t., $\bigcup_{i=1}^{n} \lambda_i \subseteq \bigcirc$. Then $\forall i \in I$, $\exists$ a finitary subset $K_i$ s.t. $\lambda_i \subseteq K_i \subseteq \bigcirc$ so that $\bigcup_{i=1}^{n} \lambda_i \subseteq \bigcup_{i=1}^{n} K_i \subseteq \bigcirc$. From Theorem 4.1 $\bigcup_{i=1}^{n} K_i$ is finitary, then $\bigcup_{i=1}^{n} \lambda_i$ is strongly compact.

**Note 4.4.** We refere that Theorem 4.6 is a generalization of the corresponding result in Proposition 1.10. (Proposition 4.7.4 [5]).

**Definition 4.4.** Let $(X, \leq, \delta)$ be a topological continuous information system. A subset $\lambda$ of $X$ is called strongly compact complete continuous information system iff $X$ is strongly compact and let $\forall x, y \in X$, $ub(\{x, y\})$ is strongly compact.

**Theorem 4.7.** Let $(X, \leq, \delta)$ be a continuous information system, If
(1) $X$ is strongly compactly complete; and
(2) $X$ is finitary and the intersection of two finitary upper sets is strongly compact. Then

(i) $(1) \Rightarrow (2)$,

(ii) If $' \leq '$ is reflexive, $(2) \Rightarrow (1)$.

**Proof** (i) Since $X$ is strongly compac and open, then there exists a finitary $\mu$ of $X$ s.t. $X \subseteq \mu \subseteq X$ so that $X$ is is finitary. Let $\lambda$ and $\mu$ bbe two finitary upper sets. Then there are finite sets $E$ and $F$ s.t. $\lambda \subseteq \uparrow (E) \subseteq \uparrow (\lambda) \subseteq \lambda$ and $\mu \subseteq \uparrow (F) \subseteq \uparrow (\mu) \subseteq \mu$. So $\lambda \subseteq \uparrow (E)$ and $\mu \subseteq \uparrow (F)$. Now $\lambda \cap \mu \subseteq \uparrow (E) \cap \uparrow (F) = \bigcup_{e \in E, \ f \in F} (\uparrow e \bigcap \uparrow f)$ so that, from Theorem 4.6, $\lambda \cap \mu$ is strongly compact.

(ii) Since $X$ is finitary and the only open set containing $X$, then $X$ is strongly compact. Let $x, y \in X$, since $' \leq '$is reflexive, $\forall x \in X$, $\uparrow x$ is finitary and since $' \leq '$transitive, then $\uparrow x$ is upper set. Hence $\uparrow x \cap \uparrow y$ is strongly compact.

**Note 4.4.** We refere that Theorem 4.7 is a generalization of the corresponding result in Proposition 1.11.(Proposition 5.5.1 [5]).

**Definition 4.5.** Let $(X, \leq, \delta)$ be a topological continuous information system. $X$ is called compactly complete iff $X$ is compact and let $\forall x, y \in X$, $ub(\{x, y\})$ is compact.

**Definition 4.6.** Let $(X, \leq, \delta)$ be a topological continuous information system. We say that $\delta$ has the property $F$ iff every compact open set in $(X, \delta)$ is finitary.

**Example 4.1** From Proposition 1.5 (Proposition 4.6.8 [5]).In $(X, \leq, \delta_S)$ where $X$ is a domain. and $\delta_S$ is Scott-topology induced by $' \leq '$ , $\delta_S$ has the property $F$.

**Theorem 4.8.** Let $(X, \leq, \delta)$ be a continuous information system, If

(1) $X$ is compactly complete; and

(2) $X$ is finitary and the intersection of two finitary upper sets is compact. Then

(i) If $\delta$ has the property $F$, $(1) \Rightarrow (2)$, and

(ii) If $' \leq '$ is reflexive, and each member of $\delta$ is upper set, then $(2) \Rightarrow (1)$.

**Proof** (i) Since $\delta$ has the property $F$, then $X$ is finitary. Let $U$ and $V$ be two finitary upper sets. Then there are finite sets $E$ and $M$ s.t. $U \subseteq \uparrow (E) \subseteq \uparrow (U) \subseteq U$ and $V \subseteq \uparrow (M) \subseteq \uparrow (V) \subseteq V$. So, So $U \subseteq \uparrow (E)$ and $V \subseteq \uparrow (M)$. Now $U \cap V \subseteq \uparrow (E) \cap \uparrow (M) = \bigcup_{e \in E, \ m \in M} (\uparrow e \bigcap \uparrow m)$ so that $U \cap V$ is compact.because a finite union of comact subset is compact.

(ii) Since $X$ is finitary , then $X$ is strongly compact. From Theorem 4.5, $X$ is compact. since $' \leq '$is reflexive, then $\forall x \in X$, $x \in \uparrow x$ so that $\uparrow x$ is finitary. Furthermore $\forall x \in X$, $\uparrow x$ is upper set. Hence $\forall x, y \in X$, $\uparrow x \cap \uparrow y$ is compact.

## 5- Conclusion

We have done the research tasks such that the correspoding results of completeness properties in R. Heckmann theorems [5], are strengthen by our results.Also, we can give open problem to the researchers if can get our results

if a binary relation $`\leq'$ be a binary relation set on $X \neq \phi$ have one only condition on Definition 1.2.

No conflict of interest

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