Asian Journal of Fuzzy and Applied Mathematics (ISSN: 2321 – 564X)
Volume 8– Issue 2, June 2020

A Mathematical Study on a Diseased Prey-Predator Model with Predator Harvesting

Srinivasarao Thota

Department of Applied Mathematics
School of Applied Natural Sciences
Adama Science and Technology University
Post Box No. 1888, Adama, Ethiopia
Email: srinivasarao.thota [AT] astu.edu.et

ABSTRACT---- ***In this paper, we present a mathematical model for a prey-predator system with infectious disease in the prey population. We assumed that there is harvesting from the predator and a defensive property against predation. This model is constituted by a system of nonlinear decoupled ordinary first order differential equations, which describe the interaction among the healthy prey, infected prey and predator. The existence, uniqueness and boundedness of the system solutions are investigated. Local stability of the system at equilibrium points is discussed.***

Keywords--- Pre-predator model, Equilibrium points, Stability Analysis, Non-linear differential equations

1. INTRODUCTION

The prey-predator species interaction has been studied by many researchers and engineers since Lotka-Volterra model see, for example, [1]. Similarly, the susceptible-infected-recovered population interaction is becoming an interesting research work since the pioneering work of Kermack and Mc Kendrick [2]. The dynamics of disease with in the ecological systems is playing an important role in eco-epidemiology research. In [3], Anderson and May were introduced the research on combination of these two systems, and the term "eco-epidemiology" is used first by Chattopadhyay and Arino in [4] for such type of models. Several scientists and engineers are studied the dynamics of prey-predator using various effects of variety of biological factors within the last decades, see, for example [5-8], and different types of mathematical models have been created in epidemiology using different types of incidence rates and disease, see, for example [9-13]. In this paper, we propose a mathematical model for a prey-predator system with infectious disease in the prey population. We assumed that there is harvesting from the predator and a defensive property against predation. The existence, uniqueness and boundedness of the system solutions are investigated. Local stability of the system at equilibrium points is discussed and the analytical results obtained in proposed model are justified using numerical simulations.

The paper is organized as follows. Section 2 presents the mathematical model formulation; Section 3 discusses the local stability of all possible equilibrium points of the system; and in Section 4, we discuss the conclusion of the proposed model.

2. MODEL FORMULATION

In the proposed model, we study a prey-predator system involving infected disease in prey. We assumed that there is harvesting from the predator and a defensive property against predation. In this model, the population density of prey is divided into two parts namely, the susceptible population density at time t given by $x(t)$ and the infected population density at time t denoted by $y(t)$, and the population density of predator at time t is given by $z(t)$.

2.1 Assumptions

According to the following hypotheses, the mathematical model of an eco-epidemic prey-predator model with harvesting in predator is formulated.

(a). The prey population grows logistically with intrinsic growth rate $r > 0$. It is assumed that the infected can't reproduce rather than that it competes with the susceptible individuals for food and space.

(b). The susceptible prey population becomes infected by contact with the infected prey according to the simple mass action kinetics with $\beta > 0$ as the rate of infection.

(c). The constants $e_1 \in (0,1)$ and $e_2 \in (0,1)$ are the conversion rates from susceptible and infected preys to predator respectively.

(d). The disease cases a death in the infected population that represented by diseased death rate $d_1 > 0$. While in the absence of prey, the predator decay exponentially with natural death rate $d_2 > 0$.

(e). We use the coefficients $\alpha > 0$ and $\gamma > 0$ for the competing coefficient of *y* over *x* and the computing coefficient of *z* over *y* respectively.

2.2 Mathematical Model

According to the above set of hypotheses the dynamics of a diseased prey-predator model with predator harvesting can be describe in the following set of first order nonlinear differential equations.

$$\left.\begin{aligned} \frac{dx}{dt} &= rx - \frac{\beta}{1+y}xy - \alpha xz \\ \frac{dy}{dt} &= -d_1 y + \frac{\beta}{1+y}xy - \gamma yz \\ \frac{dz}{dt} &= -d_2 z + e_1 \alpha xz + e_2 \gamma yz - ez \end{aligned}\right\} \qquad (1)$$

where $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$.

3. STABILITY ANALYSIS

In this section, we discuss about the equilibrium points of the model (1), existence and stability analysis of equilibrium points.

3.1 Equilibrium Points

The equilibrium points of the system are necessary for the purpose of studying the local stability nature of the eco-epidemic prey-predator model. The system (1), under investigation, has the following four equilibrium points.

(i). Fully washed state or extent state: $E_1 = (0,0,0)$

(ii). Infected species washed state: $E_2 = \left(\frac{d_2 + e}{\alpha e_1}, 0, \frac{r}{\alpha} \right)$

(iii). Predator washed state: $E_3 = \left(\frac{d_1}{\beta - r}, \frac{r}{\beta - r}, 0 \right)$. This equilibrium point exists when $\beta > r$.

(iv). Coexistence state: $E_4 = \left(x^*, y^*, z^* \right)$, where

$x^* = \frac{(e_2\gamma + d_2 + e)(\alpha d_1 + \gamma r) - (d_2 + e)\beta\gamma}{\alpha[(\alpha d_1 + \gamma r)e + \beta\gamma(e_2 - e_1)]}$, $y^* = \frac{\beta(d_2 + e) - (\alpha d_1 + \gamma r)}{\alpha[(\alpha d_1 + \gamma r)e + \beta\gamma(e_2 - e_1)]}$ and

$z^* = \frac{\alpha d_1 e_1 + e_2 \gamma r + (d_2 + e)(r - \beta)}{\alpha[(d_2 + e) + \gamma(e_2 - e_1)]}$.

This equilibrium point E_4 exists when $e_2 > e_1$, $(e_2\gamma + d_2 + e)(\alpha d_1 + \gamma r) > (d_2 + e)\beta\gamma$ and $\beta(d_2 + e) > (\alpha d_1 + \gamma r)$.

3.2 Existence and Stability Analysis of Equilibrium Points

The Jacobin matrix for the system (1) at equilibrium point *E* = (*x*, *y*, *z*) is given by

$$J_E = \begin{pmatrix} r - \frac{\beta y}{1+y} - \alpha z & -\frac{\beta x}{1+y} + \frac{\beta xy}{(1+y)^2} & -\alpha x \\ \frac{\beta y}{1+y} & -d_1 + \frac{\beta x}{1+y} - \frac{\beta xy}{(1+y)^2} - \gamma z & -\gamma y \\ \alpha e_1 z & e_2 \gamma z & \alpha e_1 x + e_2 \gamma y - d_2 - e \end{pmatrix} \qquad (2)$$

Based on the nature of Eigen values, the dynamical system (1) gets stable when all three Eigen values are negative in case of real roots or negative real parts in case of complex roots of the characteristic equation for the above Jacobin matrix (2), otherwise the dynamical system is unstable.

Theorem 1: The dynamical system (1) is unstable at the equilibrium points E_1 and E_2.

Proof: (i) The Eigen values of the dynamical system (1) at $E_1 = (0, 0, 0)$ are computed using the Jacobin matrix at E_1.

$$J_{E_1} = \begin{pmatrix} r & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 - e \end{pmatrix}.$$

Now the Eigen values are $r, -d_1$ and $-d_2 - e$. Therefore, the equilibrium point E_1 is saddle point.

(ii) Similarly, the Eigen values of the dynamical system (1) at $E_2 = \left(\frac{d_2 + e}{\alpha e_1}, 0, \frac{r}{\alpha} \right)$ are computed using the Jacobin matrix at E_2.

$$J_{E_2} = \begin{pmatrix} 0 & -\frac{\beta(d_2+e)}{\alpha e_1} & -\frac{d_2+e}{e_1} \\ 0 & -\frac{\alpha d_1 e_1 + e_1 \gamma r - \beta d_2 - \beta e}{\alpha e_1} & 0 \\ r e_1 & \frac{r e_2 \gamma}{\alpha} & 0 \end{pmatrix}.$$

The characteristic equation of J_{E_2} is

$$\lambda^3 + \left[\frac{\alpha d_1 e_1 + \gamma r e_2 - \beta(d_2+e)}{\alpha e_1} \right] \lambda^2 + [r(d_2+e)]\lambda + \left[\frac{r(d_2+e)(\alpha d_1 + \gamma r)e_1 - \beta(d_2+e)}{\alpha e_1} \right] = 0,$$

and the Eigen values are $\pm i \sqrt{(d_2+e)r}$ and $\frac{\beta(d_2+e) - (\alpha d_1 + \gamma r)e_1}{\alpha e_1}$. Since there is no real part exists in case of complex Eigen values, the dynamical system is unstable.

Theorem 2: The dynamical system (1) is locally asymptotically stable at $E_3 = \left(\frac{d_1}{\beta - r}, \frac{r}{\beta - r}, 0 \right)$ if $r > \beta$ and $\beta^2 > r(\beta + \tfrac{1}{4} d_1)$, $\beta > r$.

Proof: For this equilibrium point the corresponding Jacobin matrix is

$$J_{E_3} = \begin{pmatrix} 0 & -\dfrac{d_1(\beta - r)}{\beta} & -\dfrac{\alpha d_1}{\beta - r} \\ r & -\dfrac{rd_1}{\beta} & -\dfrac{\gamma r}{\beta - r} \\ 0 & 0 & \dfrac{\alpha d_1 e_1 + \gamma r e_2 + (d_2 + e)(r - \beta)}{\beta - r} \end{pmatrix}.$$

The corresponding Eigen values are $-\dfrac{d_1 r}{2\beta} \pm i \dfrac{\sqrt{(d_1 r)[\beta^2 - r(\beta + \frac{1}{4} d_1)]}}{\beta}$, $-\dfrac{[(\alpha d_1 e_1 + \gamma r e_2) + (r - \beta)(d_2 + e)]}{r - \beta}$.

Hence the system (1) is locally asymptotically stable when $r > \beta$ and $\beta^2 > r(\beta + \frac{1}{4} d_1)$, $\beta > r$.

Theorem 3: The equilibrium point $E_4 = (x^*, y^*, z^*)$ exists if $e_2 > e_1$, $\beta(d_2 + e) > (\alpha d_1 + \gamma r)$ and $(e_2\gamma + d_2 + e)(\alpha d_1 + \gamma r) > (d_2 + e)\beta\gamma$.

Proof: Let x^*, y^*, z^* be the positive solutions of the equations

$$\frac{dx^*}{dt} = rx^* - \frac{\beta}{1 + y^*} x^* y^* - \alpha x^* z^*$$
$$\frac{dy^*}{dt} = -d_1 y^* + \frac{\beta}{1 + y^*} x^* y^* - \gamma y^* z^*$$
$$\frac{dz^*}{dt} = -d_2 z^* + e_1 \alpha x^* z^* + e_2 \gamma y^* z^* - ez^*$$

Solving above equations for x^*, y^*, z^*, we obtain

$$x^* = \frac{(e_2\gamma + d_2 + e)(\alpha d_1 + \gamma r) - (d_2 + e)\beta\gamma}{\alpha[(\alpha d_1 + \gamma r)e + \beta\gamma(e_2 - e_1)]}, \quad y^* = \frac{\beta(d_2 + e) - (\alpha d_1 + \gamma r)}{\alpha[(\alpha d_1 + \gamma r)e + \beta\gamma(e_2 - e_1)]} \text{ and}$$
$$z^* = \frac{\alpha d_1 e_1 + e_2 \gamma r + (d_2 + e)(r - \beta)}{\alpha[(d_2 + e) + \gamma(e_2 - e_1)]}.$$

These would be positive when $e_2 > e_1$, $(e_2\gamma + d_2 + e)(\alpha d_1 + \gamma r) > (d_2 + e)\beta\gamma$ and $\beta(d_2 + e) > (\alpha d_1 + \gamma r)$. So, the interior equilibrium point E_4 for system (1) exists if $e_2 > e_1$, $(e_2\gamma + d_2 + e)(\alpha d_1 + \gamma r) > (d_2 + e)\beta\gamma$ and $\beta(d_2 + e) > (\alpha d_1 + \gamma r)$.

Theorem 4: The dynamical system (1) at the coexistent equilibrium point E_4 is locally asymptotically stable if $a_0 > 0, a_2 > 0$ and $a_0 a_1 - a_2 > 0$ otherwise is unstable.

Proof: The Jacobin matrix corresponding to the equilibrium point E_4 is

$$J_{E_4} = \begin{pmatrix} 0 & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & 0 \end{pmatrix}$$

where

$$H_{12} = -\frac{\left(e_2\gamma(\alpha d_1+\gamma r)+(d_2+e)(\alpha d_2+\gamma(r-\beta))\right)\left(e_1(\alpha d_1-\beta\gamma)+(\beta e_2+re_1)\gamma\right)}{\alpha\beta\left((d_2+e)+\gamma(e_2-e_1)\right)^2},$$

$$H_{13} = -\frac{\left(e_2\gamma(\alpha d_1+\gamma r)+(d_2+e)(\alpha d_2+\gamma(r-\beta))\right)}{\left(e_1(\alpha d_1-\beta\gamma)+(\beta e_2+re_1)\gamma\right)},$$

$$H_{21} = -\frac{\left((\alpha d_1+\gamma r)e_1-\beta(d_2+e)\right)}{\left((d_2+e)+\gamma(e_2-e_1)\right)},$$

$$H_{22} = \frac{1}{\alpha\beta\left((d_2+e)+\gamma(e_2-e_1)\right)}(e_1e_2\gamma(\alpha d_1+\gamma r)^2+(\alpha d_1d_2e_1+\alpha d_1e_1e)(\alpha d_1-\beta\gamma)$$
$$+(d_2+e)(2\alpha d_1e_1\gamma r-\alpha\beta d_1e_2\gamma+e_1\gamma^2r^2)-(e_1+e_2)(\beta d_2\gamma^2 r+\beta e\gamma^2 r)$$
$$+(d_2+2e)(-\alpha\beta d_1d_2+\beta^2d_2\gamma-\beta d_2\gamma r)-\alpha\beta d_1e^2+\beta^2\gamma e^2-\beta e^2\gamma r),$$

$$H_{23} = \frac{\left(\gamma(e_1(\alpha d_1+\gamma r)-\beta(d_2+e)\right)}{\left(e_1(\alpha d_1-\beta\gamma)+(\beta e_2+re_1)\gamma\right)},$$

$$H_{31} = \frac{\left(\alpha d_1e_1+e_2\gamma r+(d_2+e)(r-\beta)\right)e_1}{\left((d_2+e)+\gamma(e_2-e_1)\right)},$$

$$H_{32} = \frac{\left(\alpha d_1e_1+e_2\gamma r+(d_2+e)(r-\beta)\right)e_2\gamma}{\left((d_2+e)+\gamma(e_2-e_1)\right)\alpha}$$

The characteristic equation J_{E_4} is $\lambda^3+a_0\lambda^2+a_1\lambda+a_2=0$. Here $a_0=-H_{22}$, $a_1=-(H_{12}H_{21}+H_{13}H_{31}+H_{23}H_{32})$ and $a_2=-H_{31}H_{12}H_{23}-H_{13}H_{21}H_{32}+H_{13}H_{22}H_{31}$. Therefore, by Routh-Hurwitz criteria, the equilibrium point E_4 is locally asymptotically stable if $a_0>0, a_2>0$ and $a_0a_1-a_2>0$ otherwise is unstable.

4. CONCLUSIONS AND DISCUSSIONS

This paper elucidates an ecological model of a prey-predator system with infectious disease in the prey population. This model is constituted by a system of nonlinear decoupled ordinary first order differential equations, which describe the interaction among the healthy prey, infected prey and predator. By using perturbed method, we identify the local stability nature of the system at each possible equilibrium point and also the existence, uniqueness and boundedness of the system solutions are investigated.

(i). From Theorem 2, we can observe that the dynamical system (1) is locally asymptotically stable at the equilibrium point E_3 when $r>\beta$ and $\beta^2>r(\beta+\frac{1}{4}d_1)$, $\beta>r$.

(ii). One can also notice, from Theorem 4, that the dynamical system (1) at the coexistent equilibrium point E_4 = (x^*, y^*, z^*) is locally asymptotically stable if $a_0>0, a_2>0$ and $a_0a_1-a_2>0$ otherwise is unstable, where $a_1=-(H_{12}H_{21}+H_{13}H_{31}+H_{23}H_{32})$, $a_2=-H_{31}H_{12}H_{23}-H_{13}H_{21}H_{32}+H_{13}H_{22}H_{31}$, $a_0=-H_{22}$, and $H_{ij}, 1\le i, j\le 3$ are as given Theorem 4.

5. REFERENCES

[1]. M. R. May, Stability and Complexity in Model Ecosystems. Princeton, NJ: Princeton University, 1973.
[2]. Kermack WO, Mc Kendrick AG. Contributions to the mathematical theory of epidemics, part1. Proceedings of the Royal Society of London Series A1927; 115:700–721.
[3]. Anderson RM, May RM. The invasion, persistence and spread of infectious diseases within animal and plant communities. Philosophical Transactions of the Royal Society of London. Series B 1986; 314:533–570.
[4]. Chattopadhyay J, Arino O. A predator–prey model with disease in the prey. Nonlinear Analysis 1999; 36:747–766.

[5]. Z. Ma, F. Chen, C. Wu, and W. Chen, "Dynamic behaviors of a Lotka-Volterra predator-prey model incorporating a prey refuge and predator mutual interference," Applied Mathematics and Computation, vol. 219, no. 15, pp. 7945–7953, 2013.

[6]. P. J. Pal and P. K. Mandal, "Bifurcation analysis of a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and strong Allee effect," Mathematics and Computers in Simulation, vol. 97, pp. 123–146, 2014.

[7]. Raid Kamel Naji and Salam Jasim Majeed. The Dynamical Analysis of a Prey-Predator Model with a Refuge-Stage Structure Prey Population. International Journal of Differential Equations. Volume 2016, Article ID 2010464, 10 pages.

[8]. Ahmed Sami Abdulghafour and Raid Kamel Naji. The impact of refuge and harvesting on the dynamics of prey-predator system. Sci. Int. (Lahore), 30 (2), 315-323. 2018

[9]. D. Greenhalgh, Q. J. A. Khan, and F. I. Lewis, "Hopf bifurcation in two SIRS density dependent epidemic models," Mathematical and Computer Modelling, vol. 39, no. 11-12, pp. 1261–1283, 2004.

[10]. T. L. Zhang, J. L. Liu, and Z. D. Teng, "Stability of Hopf bifurcation of a delayed SIRS epidemic model with stage structure," Nonlinear Analysis: Real World Applications, vol. 11, no. 1, pp. 293–306, 2010.

[11]. Y. Z. Pei, S. Li, C. Li, and S. Chen, "The effect of constant and pulse vaccination on an SIR epidemic model with infectious period," Applied Mathematical Modelling, vol. 35, no. 8, pp. 3866–3878, 2011.

[12]. Raid Kamel Naji and Burhan Haqi Abdulateef. The dynamics of SICIR model with nonlinear incidence rate and saturated treatment function. Sci. Int. (Lahore), 29 (6), 1223-1236, 2017.

[13]. S. Thota: Prey-Predator Model for Awash National Park, Oromia, Ethiopia and Its Stability Analysis with Simulations, Journal of Science and Sustainable Development, 7 (2) (2019), 15-21.