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A New Analytical of the Homotopy Analysis Method for Solving Singular Two-Point BVPs

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ABSTRACT---- ***In this paper, we will introduce a new adjustment of the homotopy analysis method (HAM) to obtain approximate or accurate solutions to problems BVPs. The results showed that this proposed amendment is very effective compared to the basic homotopy analysis method (HAM), where in most cases solutions are accurate in the first iteration. Some examples will be given to illustrate the suggested method.***

Keywords---- homotopy analysis method; Singular two-point boundary value problems

1. INTRODUCTION

In this work, we consider the singular two-point boundary value problems (BVPs) of the type

$$\frac{1}{\rho(x)}u'' + \frac{1}{\sigma(x)}u' + \frac{1}{\beta(x)}u(x) = z(x) \qquad , \qquad 0 < x \le 1 \tag{1}$$

subject to the boundary conditions

$$u(0) = a \ \text{ and } u(1) = b \tag{2}$$

where a, b are constants, and $\rho(x), \sigma(x), \beta(x), z(x)$ are continuous functions. The equation (1)-(2) appeared in different fields , including the mechanical inhibits, in chemical reactions and many applied maths disciplines [13]. Obtaining approximate or accurate solutions to this type of problem is of great importance because of the wide application of this equation in scientific research. Given the importance of this type of equation, it has become a point of view for many researchers. the above problems solved by homotopy-perturbation method (HPM) [8] . Junfeng Lu [9] used variational iteration method (VIM) to search for approximate solutions of singular two-point BVPs. Approximate solutions have also been obtained in researchs[5],[7],[16].

Homotopy analysis method (HAM) [14] has been used to solve many singular initial value problems with diverse variations and some modificationss of HAM have published to facilitatee and accurate the calculations and accelerates the rapid convergenceof the series solution and reduce the size of work [2-4],[6],[11],[12],[15,16]. In this work, we present anew approach of homotopy analysis method (HAM) to obtain approximate solutions of the singular two-point BVPs (1)-(2).

2. BASIC IDEAS OF STANDARD HAM

To illustrate the basic idea of the standard HAM and to achieve our goal of improving the method , we will assume the following non-linear differential equation $\aleph[u(x)] = z(x)$ (3)

Where $\aleph$ is a nonlinear operator.
construct the following zero-order deformation equations:

$$(1-p)L[\emptyset(x;p) - u_0(x)] = ph\mathcal{H}(x)\{\aleph[\emptyset(x;p] - z(x)\} \tag{4}$$

Where $p \in [0,1]$ denotes the so-called embedded parameter, L is an auxiliary linear operator with the property $L[g] = 0$, when $g = 0$, $h \neq 0$ is an auxiliary parameter, $\mathcal{H}(x)$ denotes a non-zero auxiliary function.

It is obvious that when, $p = 0$ and $p = 1$, equation (4) becomes:

$$\emptyset(x;0) = u_0(x), \qquad \emptyset(x;1) = u(x)$$

respectively. Thus as p increases from 0 to 1, the solution $\emptyset(x;p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$.

Having the freedom to choosen $u_0(x)$, L, h, $\mathcal{H}(x)$, we can assume that all of them can be properly chosen so that the solution $\emptyset(x;p)$ of equation (4) exists for $p \in [0,1]$.

Expanding $\emptyset(x;p)$ in Taylor series, one has:

$$\emptyset(x;p) = \sum_{r=0}^{+\infty} u_r(x) p^r, \tag{5}$$

where

$$u_r(x) = \frac{1}{r!} \frac{\partial^r \emptyset(x;p)}{\partial p^r} |_{p=0} \tag{6}$$

Assume that h, $\mathcal{H}(x)$, $u_0(x)$, L are so properly chosen such that the series (5) converges at $p = 1$ and

$$u(x) = \emptyset(x;1) = u_0(x) + \sum_{r=1}^{+\infty} u_r(x) \tag{7}$$

Defining the vector $u_s(x) = \{u_0(x), u_1(x), u_2(x), \dots, u_s(x)\}$, differentiating equation (4) r times with respect to p and then setting $p = 0$ and finally dividing them by $r!$ we have the so-called r^{th} order deformation equation:

$$L[u_r(x) - X_r\, u_{r-1}(x)] = h\mathcal{H}(x) R_r(\vec{u}_{r-1}(x)), \tag{8}$$

where,

$$R_r(\vec{u}_{r-1}(x)) = \frac{1}{(r-1)!} \frac{\partial^{r-1}(\aleph[\emptyset(x;p)] - z(x))}{\partial p^{r-1}} |_{p=0} \tag{9}$$

and,

$$X_r = \begin{cases} 0 & r \leq 1 \\ 1 & otherwise \end{cases} \tag{10}$$

It should be emphasized that $u_r(x)$ for $r \geq 1$ is governed by the linear equation (8) with the linear initial conditions that come from the original problem, which can be easily solved by symbolic computation softwares such as Mathematica, Maple and Matlab.

Basic idea behind the new approach of HAM (naHAM)

The new technique of HAM is established. We consider the equation of type in (1) , In the modified homotopy analysis method. We will express the nonhomogeneous part $z(x)$ in (1) in Taylor series with respect to p,

$$z(x) \to \mathcal{Z}(x;p) = \sum_{r=0}^{\infty} z_r\, p^r \tag{11}$$

Where

$$z_r = \sum_{s=0}^{1} \frac{1}{(2r+s)!} \left[\frac{d^{(2r+s)} z(x)}{dx^{(2r+s)}}\right] x^{(2r+s)}$$

Now, according to (11), the new zeroth-order deformation equation given by the Taylor series expansion is

$$(1-p)L[\emptyset(x;p) - u_0(x)] = ph\mathcal{H}(x)\{\aleph[\emptyset(x;p] - \mathcal{Z}(x;p)\} \tag{12}$$

It is obvious that when, $p = 0$ and $p = 1$, equation (12) becomes:

$$\emptyset(x;0) = u_0(x), \qquad \emptyset(x;1) = u(x)$$

Respectively. Thus as p increases from 0 to 1, the solution $\emptyset(x;p)$ varies from the initial boundary approximation $u_0(x)$ to the solution $u(x)$.

and the r^{th}-order deformation equation is

$$L[u_r(x) - X_r u_{r-1}(x)] = h\mathcal{R}_r(\overrightarrow{u_{r-1}}(x)) \qquad (13)$$

where,

$$R_r(\vec{u}_{r-1}(x)) = \frac{1}{(r-1)!}\frac{\partial^{r-1}(\aleph[\emptyset(x;p)] - Z(x;p))}{\partial p^{r-1}}|_{p=0}$$

And X_r as define by (10). Applying L^{-1} to (13) we get

$$u_r(x) = X_r u_{r-1}(x) + hL^{-1}[\mathcal{R}_r(\overrightarrow{u_{r-1}}(x))] \ .$$

3. APPLICATIONS

Example 1: consider the singular two-point BVPs [8]

$$\left(1 - \frac{x}{2}\right)u'' + \frac{3}{2}\left(\frac{1}{x} - 1\right)u' + \left(\frac{x}{2} - 1\right)u = z(x) \qquad , \qquad 0 < x \le 1 \qquad (14)$$

Subject to the boundary conditions

$$u'(0) = 0 \ \text{ and } \ u(1) = 0 \qquad (15)$$

having the exact solution as $u(x) = x^2 - x^3$ and $z(x) = 5 - \frac{29x}{2} + \frac{13x^2}{2} + \frac{3x^3}{2} - \frac{x^4}{2}$.

HAM solution. To solve (14) by means of the standard HAM, we choose the initial approximation

$$u_0(x) = 0$$

And the linear operator L will be take the form $= \frac{d^2}{dx^2} + \frac{3}{2x}\frac{d}{dx}$.

with the property $L\left[-\frac{2c_1}{\sqrt{x}} + c_2\right] = 0$, where $c_i (i = 1,2)$ are constants of integration. According to equations (8) and (9) we obtain,

$$u_r(x) = X_r u_{r-1}(x) + h\,L^{-1}\left[\left(1 - \frac{x}{2}\right)u''_{r-1} + \frac{3}{2}\left(\frac{1}{x} - 1\right)u'_{r-1} + \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial p^{r-1}}\left[\left(\frac{x}{2} - 1\right)u_{r-1} - z(x)\right]\Big|_{p=0}\right]$$

Then we obtain

$$u_1(x) = h(-x^2 + \frac{29x^3}{21} - \frac{13x^4}{36} - \frac{3x^5}{55} + \frac{x^6}{78})$$
$$u_2(x) = h(-x^2 + \frac{29x^3}{21} - \frac{13x^4}{36} - \frac{3x^5}{55} + \frac{x^6}{78}) + h(-hx^2 + \frac{37hx^3}{21} - \frac{37hx^4}{42} + \frac{8hx^5}{231} + \frac{6947hx^6}{108108} - \frac{43hx^8}{72930} + \frac{hx^9}{13338})$$
$$\vdots$$

Then, the series solution expression by HAM can be written as:

$$u(x,h) \cong U_R(x,h) = \sum_{i=0}^{R} u_i\,(x,h) \qquad (16)$$

The equation (16) is the approximate solution set for the problem (1–2) in terms of the convergence parameter h. In order to obtain the valid area of the h curves, we will take the the 8^{th}-order HAM at different values of x are drawn in Figure (1) . We will select the line segment parallel to the horizontal axis as a valid area for h that provides a simple way to adjust the proximity zone.

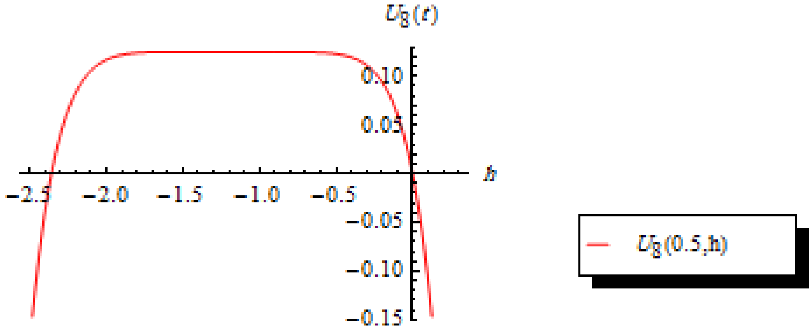


Figure (1): h-curve for HAM approximation solutions $u_8(x)$ Of problem (14-15).

Hence, the series solution of (16) at $h = -1$ is

$$u(x) = x^2 - x^3 + \frac{16x^8}{2431} - \frac{3904x^9}{340119} + \frac{3266464x^{10}}{1688511825} + \frac{83292714574x^{11}}{24350029028325} + \cdots \cong \; x^2 - x^3$$

naHAM solution: we expand the homotopy $Z(x;p)$ in powers of the embedding parameter p as (11)

Then we get

$$z_0(x) = 5 - \frac{29x}{2}, \quad z_1(x) = \frac{13x^2}{2} + \frac{3x^3}{2}, \quad z_2(x) = -\frac{x^4}{2}, \quad z_3(x) = 0 \text{ for all } i \geq 3$$

We choose the initial boundary approximation

$$u_0(x) = 0$$

And the linear operator L will be take the form $= \frac{d^2}{dx^2} + \frac{3}{2x}\frac{d}{dx}$.

According to equation (13) we obtain:

$$u_r(x) = X_r u_{r-1}(x) + h\, L^{-1}\left[\left(1 - \frac{x}{2}\right)u''_{r-1} + \frac{3}{2}\left(\frac{1}{x} - 1\right)u'_{r-1} + \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial p^{r-1}}\left[\left(\frac{x}{2} - 1\right) u_{r-1} - z_{r-1}(x)\right]\right]\Bigg|_{p=0}$$

Then we obtain

$$u_1(x) = h(-x^2 + x^3)$$

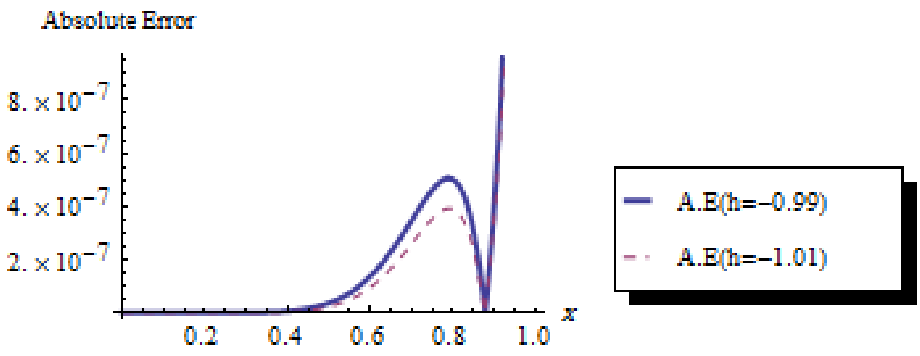
$$u_2(x) = h(-x^2 + x^3) + h(-hx^2 + \frac{8x^3}{21} + \frac{29hx^3}{21} - \frac{13x^4}{36} - \frac{13hx^4}{36} - \frac{3x^5}{55} - \frac{3hx^5}{55} + \frac{x^6}{78} + \frac{hx^6}{78})$$

$$\vdots$$

Choosing the auxiliary parameter. $h = -1$, we get

$u_1(x) = x^2 - x^3$ and $u_r(x) = 0$ for $r \geq 2$

Thus we obtain an exact solution at level $r = 1$, when $h = -1$. Also, can be obtained approximate solutions when using different value of h.figure (2) show the absolute error at $h = -0.99$ and $h = -1.01$.



Figure(2) : absolute errorof $u_8(x)$ of naHAM of problem (14-15) at different values of x and h.

Example 2: We supposed the singular two-point BVP[10]

$$u'' + \frac{1}{x}u' + u - 4 + 9x - x^2 + x^3 = 0 \qquad , \qquad 0 < x \leq 1 \qquad (17)$$

subject to the boundary conditions

$$u(0) = 0 \text{ and } u(1) = 0 \qquad (18)$$

having the exact solution as $u(x) = x^2 - x^3$.

HAM solution. To solve (17) by means of the standard HAM, we choose the initial approximation

$$u_0(x) = 0$$

The linear operator L will be take the form $L = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$.
with the property $L[c_1 \ln(x) + c_2] = 0$, where $c_i (i = 1,2)$ are constants of integration. According to equations (8) and (9) we get,

$$u_r(x) = X_r u_{r-1}(x) + h\, L^{-1}\left[u''_{r-1} + \frac{1}{x}u'_{r-1} + \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial p^{r-1}}[\, u_{r-1} - 4 + 9x - x^2 + x^3]\Big|_{p=0}\right]$$

Then we obtain

$$u_1(x) = h(-x^2 + x^3 + \frac{x^4}{16} - \frac{x^5}{25})$$
$$u_2(x) = h(-x^2 + x^3 + \frac{x^4}{16} - \frac{x^5}{25}) + h(-hx^2 + hx^3 + \frac{hx^6}{576} - \frac{hx^7}{1225})$$
$$\vdots$$

Then, we can write the serial solution by HAM as follows

$$u(x,h) \cong U_R(x,h) = \sum_{i=0}^{R} u_i\,(x,h) \qquad (19)$$

The equation (19) is the approximate solution set for the problem (17-18) in terms of the convergence parameter h. In order to obtain the valid area of the h curves, we will take the the 8th-order HAM at different values of x are drawn in Figure (3) . We will select the line segment parallel to the horizontal axis as a valid area for h that provides a simple way to adjust the proximity zone.

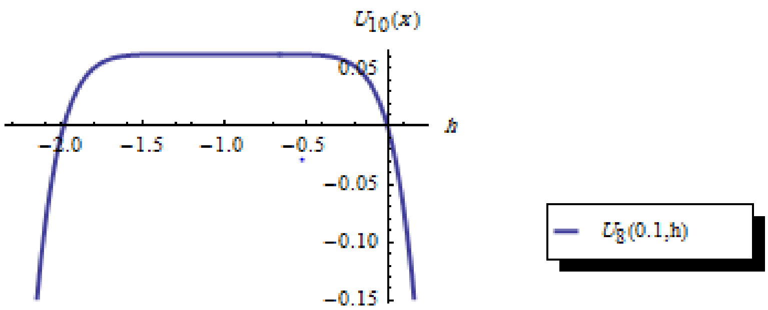


Figure (3): h-curve for HAM approximation solutions $u_8(x)$ Of problem (17-18).

Hence, the series solution of (19) at h = -1 is

$$u(x) = x^2 - x^3 - \frac{x^4}{8} + \frac{2x^5}{25} + \frac{x^6}{288} - \frac{2x^7}{1225} - \frac{x^8}{18432} + \frac{2x^9}{99225} + \cdots \cong x^2 - x^3$$

naHAM solution: we expand the homotopy $Z(x;p)$ in powers of the embedding parameter p as (11)

Then we get
$z_0(x) = 4 - 9x$, $z_1(x) = x^2 - x^3$, $z_2(x) = 0$ for all $i \geq 2$
We choose the initial boundary approximation

$$u_0(x) = 0$$

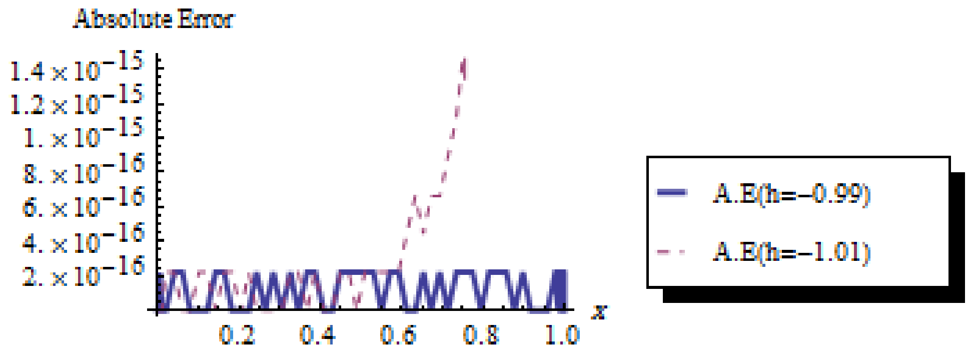
And the linear operator L will be take the form $= \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$.
According to equation (13) we obtain:

$$u_r(x) = X_r u_{r-1}(x) + h\, L^{-1}\left[u''_{r-1} + \frac{1}{x}u'_{r-1} + \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial p^{r-1}}[\,u_{r-1} - z_{r-1}(x)]\Big|_{p=0}\right]$$

Then we obtain

$$u_1(x) = h(-x^2 + x^3)$$
$$u_2(x) = h(-x^2 + x^3) + h(-hx^2 + hx^3 - \frac{x^4}{16} - \frac{hx^4}{16} + \frac{x^5}{25} + \frac{hx^5}{25})$$
$$\vdots$$

Choosing the auxiliary parameter. $h = -1$, we get
$u_1(x) = x^2 - x^3$ and $u_r(x) = 0$ for $r \geq 2$
Thus we obtain an exact solution at level $r = 1$, when$h = -1$. Also, can be obtained approximate solutions when using different value of h.figure (4) show the absolute error at $h = -0.99$ and $h = -1.01$.



Figure(4) : absolute errorof $u_8(x)$ of naHAM of problem (17-18) at different values of x and h.

Example 3: Consider the singular two-point BVP [9],

$$u'' + \frac{1}{x}u' + u - \frac{5}{4} - \frac{x^2}{16} = 0 \qquad , \qquad 0 < x \leq 1 \qquad (20)$$

subject to the boundary conditions

$$u(0) = 1 \text{ and } u(1) = \frac{17}{16} \qquad (21)$$

having the exact solution as $u(x) = 1 + \frac{x^2}{16}$.

HAM solution. To solve (20) by means of the standard HAM, we choose the initial approximation

$$u_0(x) = 1$$

the linear operator L will be take the form $= \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$.
with the property $L[c_1 \ln(x) + c_2] = 0$, where $c_i(i = 1,2)$ are constants of integration. Based on the equations (8) and (9) we get,

$$u_r(x) = X_r u_{r-1}(x) + h\, L^{-1}\left[u''_{r-1} + \frac{1}{x}u'_{r-1} + \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial p^{r-1}}\left[\,u_{r-1} - \frac{5}{4} - \frac{x^2}{16}\right]\Big|_{p=0}\right]$$

Then we have

$u_1(x) = h(-\frac{x^2}{16} - \frac{x^4}{256})$
$u_2(x) = h(-\frac{x^2}{16} - \frac{x^4}{256}) + h(-\frac{hx^2}{16} - \frac{hx^4}{128} - \frac{hx^6}{9216})$
$\vdots$

Then, we can write the serial solution by HAM as follows

$$u(x,h) \cong U_R(x,h) = \sum_{i=0}^{R} u_i\,(x,h) \qquad (22)$$

The equation (22) is the approximate solution set for the problem (20-21) in terms of the convergence parameter h. In order to obtain the valid area of the h curves, we will take the the 8th-order HAM at different values of x are drawn in Figure (5) . We will select the line segment parallel to the horizontal axis as a valid area for h that provides a simple way to adjust the proximity zone.

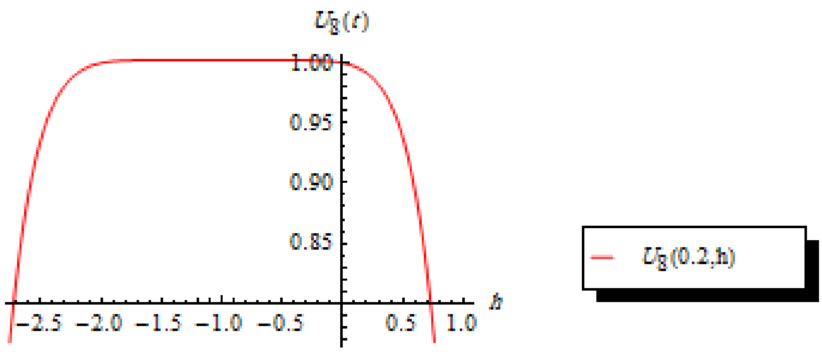


Figure (5): h-curve for HAM approximation solutions
$u_8(x)$ Of problem (20-21).

Hence, the series solution of (22) at $h = -1$ is $u(x) = 1 + \frac{x^2}{16} - \frac{x^{18}}{138078474102374400} + \cdots \cong 1 + \frac{x^2}{16}$

naHAM solution: we expand the homotopy $Z(x;p)$ in powers of the embedding parameter p as (11)

Then we get

$g_0(\mathrm{x}) = \frac{5}{4},\ g_1(\mathrm{x}) = \frac{x^2}{16},\ g_2(\mathrm{x}) = 0$ for all $i \geq 2$

We choose the initial boundary approximation

$u_0(x) = 1$

And the linear operator L will be take the form $= \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$.

According to equation (13) we obtain:

$$u_r(x) = X_r u_{r-1}(x) + h\,L^{-1}\left[u''_{r-1} + \frac{1}{x}u'_{r-1} + \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial p^{r-1}}[\,u_{r-1} - z_{r-1}(x)]\Big|_{p=0}\right]$$

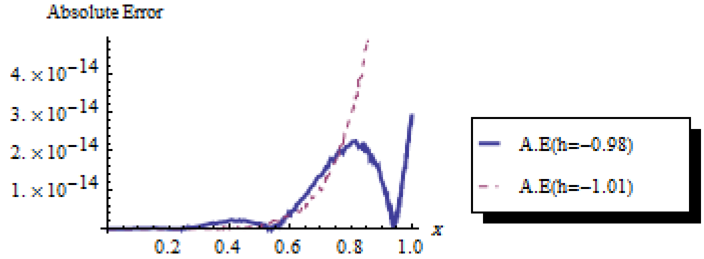
Then we obtain

$u_1(x) = -\frac{hx^2}{16}$
$u_2(x) = -\frac{hx^2}{16} + h(-\frac{hx^2}{16} - \frac{x^4}{256} - \frac{hx^4}{256})$
$\vdots$

Choosing the auxiliary parameter. $h = -1$, we get $u_0(x) = 1$ and

$u_1(x) = \frac{x^2}{16},\ u_r(x) = 0$ for $r \geq 2$

Thus we obtain an exact solution at level $r = 1$, when$h = -1$. Also, can be obtained approximate solutions when using different value of h.figure (6) show the absolute error at $h = -0.98$ and $h = -1.01$.



Figure(6) : absolute errorof $u_8(x)$ of naHAM of problem (20-21) at different values of x and h.

4. CONCLUSION

In this paper, we introduced a new modification of the homotopy analysis method for solving linear and non-linear singular two-point boundary value problems (BVPs). the results in the examples shown and demonstrated that the proposed method is better and gives the exact solution in only some iterations. Finally, we conclude that the suggested method is better than the standard homotopy analysis method.

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