FUZZY EQUIVALENCE RELATION, FUZZY CONGRUNCE RELATION AND FUZZY NORMAL SUBGROUPS ON GROUP G OVER t-NORMS

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ABSTRACT. In this study, by using t-norms, fuzzy equivalence relation, fuzzy congrunce relation on group G, fuzzy relation of subgroup H of group G, fuzzy normal subgroups of fuzzy subgroups, direct product of fuzzy subgroups(normal fuzzy subgroups) are introduced and some the their properties will be discussed. Next by using group homomorphisms, the image and pree image of them will be investigated.

1. INTRODUCTION

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right. In abstract algebra, a normal subgroup is a subgroup that is invariant under conjugation by members of the group of which it is a part. In other words, a subgroup H of a group G is normal in G if and only if $gH = Hg$ for all g in G. For centuries probability theory and error calculus have been the only models to treat imprecision and uncertainty. However recently a lot of new models have been introduced for handling incomplete information. The fact that crisp relations fail in interpreting real life phenomenon was first expressed by Poincare [25] in 1902 . Half a century later, Menger [21] addressed the issue raised by Poincare and proposed his Probabilistic relations. According to Menger in order to be in harmony with real life continuum, we should sacrifice transitivity and classical definition of relations should be changed and a probability of being related should be allocated to every pair of points belonging to the universe under consideration. Even after this development there remained a silence regarding re-building a rigorous theory of relations with different probabilities associated with them. Undoubtedly the notion of fuzzy set theory initiated by Zadeh [42] in 1965 in a seminal paper, plays the central role for further development. This notion tries to show that an object corresponds more or less to the particular category we want to assimilate it to; that was how the idea of defining the membership of an element to a set not on the Aristotelian pair $\{0, 1\}$ any more but on the continuous interval $[0, 1]$ was born. The notion of a fuzzy set is completely non-statistical in nature and the concept of fuzzy set provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables. In fact the idea of describing all shades of reality was for long the obsession of some logicians [22, 38]. During last four decades the fuzzy set

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theory has rapidly developed into an area which scientifically as well as from the application point of view, is recognized as a very valuable contribution to the existing knowledge [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 23, 24, 43, 44, 41]. After the emergence of fuzzy set theory in 1965 [42], the simple task of looking at relations as fuzzy sets on the universe $X \times X$ was accomplished in a celebrated paper by Zadeh [40], he introduced the concept of fuzzy relation, defined the notion of equivalence, and gave the concept of fuzzy ordering. Fuzzy relations have broad utility. Compared with crisp relations, they have greater expressive power. They are considered as softer models for expressing the strength of links between elements. Starting in early seventies, fuzzy relations have been defined, investigated, and applied in many different ways e.g., in fuzzy modeling, fuzzy diagnosis, and fuzzy control. They also have applications in fields such as Artificial Intelligence, Psychology, Medicine, Economics, and Sociology. In 1971 the study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld [37]. In fact many basic properties in group theory are found to be carried over to fuzzy groups. In 1979 Anthony and Sherwood [2] redefined a fuzzy subgroup of a group using the concept of triangular norm (t-norm, for short). The notion of fuzzy congruence on a group was introduced by Kroki [18] and the concept of fuzzy quotient group was studied by some authors [3, 17, 20, 39]. The author by using norms, investigated some properties of fuzzy algebra [26, 27, 28, 29, 30, 31, 32, 33, 34,35, 36]. The purpose of this study is to deal with the algebraic structure of fuzzy equivalence relation, fuzzy congrunce relation on group G, fuzzy relation of subgroup H of group G, fuzzy normal subgroups of fuzzy subgroups, direct product of fuzzy subgrops(normal fuzzy subgroups) with respet to t-norms. In Section 2, we summarize some basic conceps which will be used throughout the paper. In Section 3, by using t-norms, we introduse fuzzy equivalence relation and fuzzy congrunce relation on group G and we investigate the intersection of them. Also we define fuzzy relation of subgroup H of group G under t-norms and we show that it will be fuzzy equivalence relation on group G under t-norms. Later, we define and investigate some properties as intersection and composition of fuzzy subgroups of normal subgroups of group G by using t-norms. Also we define normal fuzzy subgroups of group G by using t-norms and investigate them. Next we define fuzzy normal subgroups of fuzzy subgroups under t-norms and we prove that the intersection of them is also fuzzy normal subgroup with respect to t-norms. In Section 4, we introduce the direct product of fuzzy subgroups (normal fuzzy subgroups) and we prove that it will be fuzzy subgroup(normal fuzzy subgroup) with respect to t-norms. In Section 5, we prove that image and pre image of fuzzy subgroups (normal fuzzy subgroups) with respect to t-norms is also fuzzy subgroups (normal fuzzy subgroups) under group homomorphisms.

2. PRELIMINARY

Definition 2.1. (See [9, 40]) A fuzzy subset of G, we mean a function $\mu : G \to [0,1]$. The set of all fuzzy subsets of G is called the $[0,1]$-power set of G and is denoted $[0,1]^G$. By a fuzzy relation on G we mean a function $\mu : G \times G \to [0,1]$. Denote by $FR(G)$, the set of all fuzzy relations on G.

Definition 2.2. (See [9, 40]) Let $\mu_1, \mu_2 \in FR(G)$ and $x, y \in G$. We define
(1) $\mu_1 \subseteq \mu_2$ iff $\mu_1(x,y) \leq \mu_2(x,y)$,
(2) $\mu_1 = \mu_2$ iff $\mu_1(x,y) = \mu_2(x,y)$.

Definition 2.3. (See [1]) A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),
for all $x,y,z \in [0,1]$.

We say that T is idempotent if for all $x \in [0,1]$,$T(x,x) = x$.

Example 2.4. (1) Standard intersection T-norm $T_m(x,y) = \min\{x,y\}$.
(2) Bounded sum T-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product T-norm $T_p(x,y) = xy$.
(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x,y \in [0,1]$.

Lemma 2.5. *(See [1]) Let T be a t-norm. Then*

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z)),$$

for all $x,y,w,z \in [0,1]$.

Definition 2.6. (See 36]) Let μ be a fuzzy subset of a group G. Then μ is called a fuzzy subgroup of G under a t-norm T (T-fuzzy subgroup) iff for all $x,y \in G$
(1) $\mu(xy) \geq T(\mu(x),\mu(y))$
(2) $\mu(x^{-1}) \geq \mu(x)$.
Denote by $TF(G)$, the set of all T-fuzzy subgroup of G.

Example 2.7. Let $G = \{1, i, -1, -i\}$ be a group with respect to multiplication. Define fuzzy subset $\mu : G \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.9 & \text{if } x = 1 \\ 0.8 & \text{if } x = -1 \\ 0.6 & \text{if } x = \pm i \end{cases}$$

If $T(x,y) = T_m(x,y) = min\{x,y\}$ for all $x,y \in G$, then $\mu \in TF(G)$.

Definition 2.8. (See [36]) Let f be a mapping from $\frac{G_1}{H_1}$ into $\frac{G_2}{H_2}$, $\mu \in [0,1]^{\frac{G_1}{H_1}}$ and $\nu \in [0,1]^{\frac{G_2}{H_2}}$. Define $f(\mu) \in [0,1]^{\frac{G_2}{H_2}}$ as

$$f(\mu)(g_2H_2) = \begin{cases} \sup\{\mu(g_1H_1) \mid g_1H_1 \in \frac{G_1}{H_1}, f(g_1H_1) = g_2H_2\} & \text{if } f^{-1}(g_2H_2) \neq \emptyset \\ 0 & \text{if } f^{-1}(g_2H_2) = \emptyset \end{cases}$$

for all $g_2H_2 \in G_2/H_2$.

Also define $f^{-1}(\nu) \in [0,1]^{\frac{G_1}{H_1}}$ by $f^{-1}(\nu)(g_1H_1) = \nu(f(g_1H_1))$ for all $g_1H_1 \in G_1/H_1$.

Definition 2.9. (See [36]) Given two groups G and H, a group homomorphism is a map $f : G \to H$ such that $f(xy) = f(x)f(y)$ for all $x, y \in G$.

Example 2.10. The map $exp : (\mathbb{R}, +) \to (\mathbb{R}, .)$ with $exp(x) = e^x$ is a group homomorphism.

Definition 2.11. (See [36]) Let H be normal subgroup of G. The group homomorphism $\pi : G \to \dfrac{G}{H}$ with $\pi(g) = gH$ is called the natural or canonical map or projection.

Remark 2.12. (See [36]) We have the following terminology: epimorphism=surjective homomorphism.

3. FUZZY EQUIVALENCE RELATION, FUZZY CONGRUNCE RELATION, FUZZY QUOTIENT SUBGROUPS AND NORMAL FUZZY QUOTIENT SUBGROUPS WITH RESPECT TO A t-NORM

Throughout this Section G be an arbitrary group. In this study we define some new special T-fuzzy equivalence relations and derive some simple consequences. Then using those relations we define suitable T-fuzzy quotient subgroup of $\frac{G}{H}$ differently.

Definition 3.1. A fuzzy relation $\mu : G \times G \to [0,1]$ on a group G is a T-fuzzy equivalence relation on G if the following conditions are satisfied:
(1) $\mu(x, x) = 1$.
(2) $\mu(x, y) = \mu(y, x)$.
(3) $\mu(x, z) \geq T(\mu(x, y), \mu(y, z))$,
for all $x, y, z \in G$.

Example 3.2. Let $G = (\mathbb{Z}, +)$ be a group of integers numbers. Define $\mu : \mathbb{Z} \times \mathbb{Z} \to [0,1]$ by

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0.55 & \text{otherwise.} \end{cases}$$

If $T(x, y) = T_m(x, y) = \min\{x, y\}$ for all $x, y \in [0,1]$, Then μ is a T-fuzzy equivalence relation on G.

Definition 3.3. μ_1 and μ_2 be two T-fuzzy equivalence relations on G. We define

$$(\mu_1 \cap \mu_2)(x, y) = T(\mu_1(x, y), \mu_2(x, y))$$

for all $x, y \in G$.

Remark 3.4. μ_1 and μ_2 and μ_3 be three T-fuzzy equivalence relations on G. Then from properties T3 and T4 of Definition 2.3 we get that $\mu_1 \cap \mu_2 = \mu_2 \cap \mu_1$ and $\mu_1 \cap \mu_2 \cap \mu_3 = (\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3)$.

Proposition 3.5. *If μ_1 and μ_2 be two T-fuzzy equivalence relations on G, then so is $\mu_1 \cap \mu_2$.*

Proof. Let $x, y, z \in G$.
(1) $(\mu_1 \cap \mu_2)(x,x) = T(\mu_1(x,x), \mu_2(x,x)) = T(1,1) = 1$.
(2) $(\mu_1 \cap \mu_2)(x,y) = T(\mu_1(x,y), \mu_2(x,y)) = T(\mu_1(y,x), \mu_2(y,x)) = (\mu_1 \cap \mu_2)(y,x)$.
(3)

$$\begin{aligned}(\mu_1 \cap \mu_2)(x,y) &= T(\mu_1(x,y), \mu_2(x,y)) \\ &\geq T(T(\mu_1(x,z), \mu_1(z,y)), T(\mu_2(x,z), \mu_2(z,y)) \\ &= T(T(\mu_1(x,z), \mu_2(x,z)), T(\mu_1(z,y), \mu_2(z,y))) \quad \text{(by Lemma 2.5)} \\ &= T((\mu_1 \cap \mu_2)(x,z), (\mu_1 \cap \mu_2)(z,y)).\end{aligned}$$

Thus $\mu_1 \cap \mu_2$ will be T-fuzzy equivalence relations on G. □

Corollary 3.6. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\}$ be T-fuzzy equivalence relations on G, then so is $\mu = \cap_{i \in I_n} \mu_i$.*

Definition 3.7. A fuzzy relation $\mu : G \times G \to [0,1]$ on a group G is a T-fuzzy congruence relation on G if the following conditions are satisfied:
(1) $\mu(x,x) = 1$.
(2) $\mu(x,y) = \mu(y,x)$.
(3) $\mu(x,z) \geq T(\mu(x,y), \mu(y,z))$.
(4) $\mu(xz, yt) \geq T(\mu(x,y), \mu(z,t))$,
for all $x, y, z, t \in G$.

Example 3.8. Let $G = (\mathbb{R}, +)$ be a group of real numbers. Define $\mu : \mathbb{R} \times \mathbb{R} \to [0,1]$ by

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0.35 & \text{otherwise.} \end{cases}$$

If $T(x,y) = T_m(x,y) = \min\{x,y\}$ for all $x, y \in [0,1]$, Then μ is a T-fuzzy congruence relation on G.

Proposition 3.9. *If μ_1 and μ_2 be two T-fuzzy congruence relations on G, then so is $\mu_1 \cap \mu_2$.*

Proof. Let $x, y, z, t \in G$. Then
(1) $(\mu_1 \cap \mu_2)(x,x) = T(\mu_1(x,x), \mu_2(x,x)) = T(1,1) = 1$.
(2) $(\mu_1 \cap \mu_2)(x,y) = T(\mu_1(x,y), \mu_2(x,y)) = T(\mu_1(y,x), \mu_2(y,x)) = (\mu_1 \cap \mu_2)(y,x)$.
(3)

$$\begin{aligned}(\mu_1 \cap \mu_2)(x,y) &= T(\mu_1(x,y), \mu_2(x,y)) \\ &\geq T(T(\mu_1(x,z), \mu_1(z,y)), T(\mu_2(x,z), \mu_2(z,y)) \\ &= T(T(\mu_1(x,z), \mu_2(x,z)), T(\mu_1(z,y), \mu_2(z,y))) \quad \text{(by Lemma 2.5)} \\ &= T((\mu_1 \cap \mu_2)(x,z), (\mu_1 \cap \mu_2)(z,y)).\end{aligned}$$

(4)

$$\begin{aligned}(\mu_1 \cap \mu_2)(xz,yt) &= T(\mu_1(xz,yt), \mu_2(xz,yt)) \\ &\geq T(T(\mu_1(x,y), \mu_1(z,t)), T(\mu_2(x,y), \mu_2(z,t))) \\ &= T(T(\mu_1(x,y), \mu_2(x,y)), T(\mu_1(z,t), \mu_2(z,t))) \quad \text{(by Lemma 2.5)} \\ &= T((\mu_1 \cap \mu_2)(x,y), (\mu_1 \cap \mu_2)(z,t)).\end{aligned}$$

Then $\mu_1 \cap \mu_2$ is T-fuzzy congruence relation on G. □

Corollary 3.10. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\}$ be T-fuzzy congruence relations on G, then so is $\mu = \cap_{i \in I_n} \mu_i$.*

We define some special fuzzy relations and give some its results.

Definition 3.11. Let H be a subgroup of G and $\mu_H \in TF(H)$. A T-fuzzy relation $\theta : G \times G \to [0, 1]$ can be defined by

$$\theta(x, y) = \begin{cases} T(\mu_H(x), \mu_H(y)) & \text{if } x \neq y \\ \mu_H(e) = 1 & \text{if } x = y \end{cases}$$

Corollary 3.12. *$\theta(x^{-1}, y^{-1}) \geq \theta(x, y)$ for all $x, y \in G$.*

Proof. Since $\mu_H \in TF(H)$ so

$$\theta(x^{-1}, y^{-1}) = T(\mu_H(x^{-1}), \mu_H(y^{-1})) \geq T(\mu_H(x), \mu_H(y)) = \theta(x, y).$$

□

Proposition 3.13. *θ is a T-fuzzy equivalence relation on G.*

Proof. Let $x, y, z \in G$, then
(1) $\theta(x, x) = 1$.
(2)

$$\theta(x, y) = T(\mu_H(x), \mu_H(y)) = T(\mu_H(y), \mu_H(x)) = \theta(y, x).$$

(3)

$$\begin{aligned} \theta(x, y) &= T(\theta(x, y), 1) = T(\theta(x, y), \theta(z, z)) \\ &= T(T(\mu_H(x), \mu_H(y)), T(\mu_H(z), \mu_H(z))) \\ &= T(T(\mu_H(x), \mu_H(z)), T(\mu_H(y), \mu_H(z))) \quad \text{(by Lemma 2.5)} \\ &= T(\theta(x, z), \theta(z, y)). \end{aligned}$$

Then θ is a T-fuzzy equivalence relation on G. □

Proposition 3.14. *The T-fuzzy relation θ defined on G is T-fuzzy congruence relation.*

Proof. Let $x, y, z, t \in G$ and $\mu_H \in TF(H)$. Then
(1) $\theta(x, x) = 1$.
(2)

$$\theta(x, y) = T(\mu_H(x), \mu_H(y)) = T(\mu_H(y), \mu_H(x)) = \theta(y, x).$$

(3)

$$\begin{aligned} \theta(x, y) &= T(\theta(x, y), 1) = T(\theta(x, y), \theta(z, z)) \\ &= T(T(\mu_H(x), \mu_H(y)), T(\mu_H(z), \mu_H(z))) \\ &= T(T(\mu_H(x), \mu_H(z)), T(\mu_H(y), \mu_H(z))) \quad \text{(by Lemma 2.5)} \\ &= T(\theta(x, z), \theta(z, y)). \end{aligned}$$

(4)

$$\begin{aligned} \theta(xz, yt) &= T(\mu_H(xz), \mu_H(yt)) \\ &\geq T(T(\mu_H(x), \mu_H(z)), T(\mu_H(y), \mu_H(t))) \\ &= T(T(\mu_H(x), \mu_H(y)), T(\mu_H(z), \mu_H(t))) \quad \text{(by Lemma 2.5)} \end{aligned}$$

$$= T(\theta(x,y),\theta(z,t)).$$

Thus θ will be T-fuzzy congruence relation on G. □

Definition 3.15. Let G be a group and H be a normal subgroup of G. Then $\mu_{\frac{G}{H}} : \frac{G}{H} \to [0,1]$ can be defined by $\mu_{\frac{G}{H}}(xH) = \theta(x,h)$ for all $x \in G$ and $h \in H$.

Now we show some algebraic properties of μ.

Lemma 3.16. *Let $\mu_H \in TF(H)$ and T be idempotent t-norm. Then $\mu_{\frac{G}{H}} \in TF(\frac{G}{H})$.*

Proof. Let $xH, yH \in \frac{G}{H}$ and $\mu_H \in TF(H)$. Then

$$\mu_{\frac{G}{H}}(xHyH) = \mu_{\frac{G}{H}}(xyH) = \theta(xy,h)$$
$$= T(\mu_H(xy),\mu_H(h)) \geq T(T(\mu_H(x),\mu_H(y)),\mu_H(h))$$
$$= T(T(\mu_H(x),\mu_H(y)),T(\mu_H(h),\mu_H(h)))$$
$$= T(T(\mu_H(x),\mu_H(h)),T(\mu_H(y),\mu_H(h))) \quad \text{(by Lemma 2.5)}$$
$$= T(\theta(x,h),\theta(y,h)) = T(\mu_{\frac{G}{H}}(xH),\mu_{\frac{G}{H}}(yH)).$$

Also

$$\mu_{\frac{G}{H}}(xH)^{-1} = \mu_{\frac{G}{H}}(x^{-1}H) = \theta(x^{-1},h)$$
$$= T(\mu_H(x^{-1}),\mu_H(h)) \geq T(\mu_H(x),\mu_H(h))$$
$$= \theta(x,h) = \mu_{\frac{G}{H}}(xH).$$

Therefore $\mu_{\frac{G}{H}} \in TF(\frac{G}{H})$. □

Proposition 3.17. *If T be idempotent -norm, then for all $xH \in \frac{G}{H}$, and $n \geq 1$,*
(1) $\mu_{\frac{G}{H}}(H) \geq \mu_{\frac{G}{H}}(xH)$;
(2) $\mu_{\frac{G}{H}}(xH)^n \geq \mu_{\frac{G}{H}}(xH)$;
(3) $\mu_{\frac{G}{H}}(xH) = \mu_{\frac{G}{H}}(xH)^{-1}$.

Proof. Let $xH \in \frac{G}{H}$ and $n \geq 1$. From Lemma 3.16 we have that $\mu_{\frac{G}{H}} \in TF(\frac{G}{H})$.
(1)

$$\mu_{\frac{G}{H}}(H) = \mu_{\frac{G}{H}}(xx^{-1}H) = \mu_{\frac{G}{H}}(xHx^{-1}H)$$
$$\geq T(\mu_{\frac{G}{H}}(xH),\mu_{\frac{G}{H}}(x^{-1}H)) \geq T(\mu_{\frac{G}{H}}(xH),\mu_{\frac{G}{H}}(xH))$$
$$= \mu_{\frac{G}{H}}(xH).$$

(2) $\mu_{\frac{G}{H}}(xH)^n = \mu_{\frac{G}{H}}(\underbrace{xHxH...xH}_{n}) \geq T(\underbrace{\mu_{\frac{G}{H}}(xH),\mu_{\frac{G}{H}}(xH),...,\mu_{\frac{G}{H}}(xH)}_{n}) = \mu_{\frac{G}{H}}(xH)$.

(3) $\mu_{\frac{G}{H}}(xH) = \mu_{\frac{G}{H}}((x^{-1}H))^{-1} \geq \mu_{\frac{G}{H}}(x^{-1}H) \geq \mu_{\frac{G}{H}}(xH)$. So $\mu_{\frac{G}{H}}(xH) = \mu_{\frac{G}{H}}(x^{-1}H)$. □

Proposition 3.18. *Let $\mu_{\frac{G}{H}}$ be a fuzzy subset of a finite group $\frac{G}{H}$ and T be idempotent t-norm. If $\mu_{\frac{G}{H}}$ satisfies condition (1) of Definition 2.6, then $\mu_{\frac{G}{H}} \in TF(\frac{G}{H})$.*

Proof. Let $xH \in \frac{G}{H}, x \notin H$. Since $\frac{G}{H}$ is finite, xH has finite order, say $n > 1$. So $(xH)^n = H$ and $x^{-1}H = x^{n-1}H$. Now by using (1) repeatedly, we have that

$$\mu_{\frac{G}{H}}(x^{-1}H) = \mu_{\frac{G}{H}}(x^{n-1}H) = \mu_{\frac{G}{H}}(x^{n-2}xH)$$
$$\geq T(\mu_{\frac{G}{H}}(x^{n-2}H), \mu_{\frac{G}{H}}(xH)) \geq T(\underbrace{\mu_{\frac{G}{H}}(xH), \mu_{\frac{G}{H}}(xH), ..., \mu_{\frac{G}{H}}(xH)}_{n})$$
$$= \mu_{\frac{G}{H}}(xH).$$

□

Proposition 3.19. *Let $\mu_{\frac{G}{H}}, \nu_{\frac{G}{H}} \in TF(\frac{G}{H})$. Then $\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}} \in TF(\frac{G}{H})$.*

Proof. Let $xH, yH \in \frac{G}{H}$. Then

$$(\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(xHyH) = T(\mu_{\frac{G}{H}}(xHyH), \nu_{\frac{G}{H}}(xHyH))$$
$$\geq T(T(\mu_{\frac{G}{H}}(xH), \mu_{\frac{G}{H}}(yH)), T(\nu_{\frac{G}{H}}(xH), \nu_{\frac{G}{H}}(yH)))$$
$$= T(T(\mu_{\frac{G}{H}}(xH), \nu_{\frac{G}{H}}(xH)), T(\mu_{\frac{G}{H}}(yH), \nu_{\frac{G}{H}}(yH))) \quad \text{(by Lemma 2.5)}$$
$$= T((\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(xH), (\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(yH)).$$

Also

$$(\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(xH)^{-1} = (\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(x^{-1}H)$$
$$= T(\mu_{\frac{G}{H}}(x^{-1}H), \nu_{\frac{G}{H}}(x^{-1}H)) \geq T(\mu_{\frac{G}{H}}(xH), \nu_{\frac{G}{H}}(xH))$$
$$= (\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(xH).$$

Thus $\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}} \in TF(\frac{G}{H})$. □

Corollary 3.20. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq TF(\frac{G}{H})$, Then $\mu = \cap_{i \in I_n} \mu_i \in TF(\frac{G}{H})$.*

Proposition 3.21. *Let $\mu_{\frac{G}{H}} \in TF(\frac{G}{H})$ and $xH \in \frac{G}{H}$. If T be idempotent t-norm, then $\mu_{\frac{G}{H}}(xHyH) = \mu_{\frac{G}{H}}(yH)$ for all $yH \in \frac{G}{H}$ if and only if $\mu_{\frac{G}{H}}(xH) = \mu_{\frac{G}{H}}(H)$.*

Proof. Suppose that $\mu_{\frac{G}{H}}(xHyH) = \mu_{\frac{G}{H}}(yH)$ for all $yH \in \frac{G}{H}$. Then by letting $y = H$, we get that $\mu_{\frac{G}{H}}(xH) = \mu_{\frac{G}{H}}(H)$.
Conversely, suppose that $\mu_{\frac{G}{H}}(xH) = \mu_{\frac{G}{H}}(H)$. By Proposition 3.17 we have that $\mu_{\frac{G}{H}}(xH) \geq \mu_{\frac{G}{H}}(xHyH), \mu_{\frac{G}{H}}(yH)$. Now we have

$$\mu_{\frac{G}{H}}(xHyH) \geq T(\mu_{\frac{G}{H}}(xH), \mu_{\frac{G}{H}}(yH)) \geq T(\mu_{\frac{G}{H}}(yH), \mu_{\frac{G}{H}}(yH))$$
$$= \mu_{\frac{G}{H}}(yH) = \mu_{\frac{G}{H}}(x^{-1}xyH) = \mu_{\frac{G}{H}}(x^{-1}HxHyH)$$
$$\geq T(\mu_{\frac{G}{H}}(x^{-1}H), \mu_{\frac{G}{H}}(xHyH)) \geq T(\mu_{\frac{G}{H}}(xH), \mu_{\frac{G}{H}}(xHyH))$$
$$\geq T(\mu_{\frac{G}{H}}(xHyH), \mu_{\frac{G}{H}}(xHyH)) = \mu_{\frac{G}{H}}(xHyH).$$

Then $\mu_{\frac{G}{H}}(xH) = \mu_{\frac{G}{H}}(H)$. □

Definition 3.22. Let $\frac{G}{H}$ be a quotient subgroup and let $\mu_{\frac{G}{H}}, \nu_{\frac{G}{H}} : G/H \to [0,1]$ be two fuzzy sets in $\frac{G}{H}$. The composition $\mu_{\frac{G}{H}}\ O\ \nu_{\frac{G}{H}} : \frac{G}{H} \to [0,1]$ is defined by

$$(\mu_{\frac{G}{H}}\ O\ \nu_{\frac{G}{H}})(xH) = \begin{cases} \sup_{aHbH=xH} T(\mu_{\frac{G}{H}}(aH), \nu_{\frac{G}{H}}(bH)) & \text{if } aHbH = xH \\ 0 & \text{if } aHbH \neq xH \end{cases}$$

Proposition 3.23. *Let $\mu^{-1}_{\frac{G}{H}}$ be the inverse of $\mu_{\frac{G}{H}}$ such that $\mu^{-1}_{G/H}(xH) = \mu_{\frac{G}{H}}(x^{-1}H)$. Then $\mu_{\frac{G}{H}} \in TF(\frac{G}{H})$ if and only if $\mu_{\frac{G}{H}}$ satisfies the following conditions:*
(1) $\mu_{\frac{G}{H}}\ O\ \mu_{\frac{G}{H}} \subseteq \mu_{\frac{G}{H}}$;
(2) $\mu^{-1}_{\frac{G}{H}} = \mu_{\frac{G}{H}}$.

Proof. Let $xH, yH, zH \in \frac{G}{H}$ such that $xH = yHzH$. If $\mu_{\frac{G}{H}} \in TF(\frac{G}{H})$, then

$$\mu_{\frac{G}{H}}(xH) = \mu_{\frac{G}{H}}(yHzH) \geq T(\mu_{\frac{G}{H}}(yH), \mu_{\frac{G}{H}}(zH)) = (\mu_{\frac{G}{H}}\ O\ \mu_{\frac{G}{H}})(xH)$$

so $\mu_{\frac{G}{H}}\ O\ \mu_{\frac{G}{H}} \subseteq \mu_{\frac{G}{H}}$. Also by Proposition 3.17 we have $\mu^{-1}_{\frac{G}{H}}(xH) = \mu_{\frac{G}{H}}(x^{-1}H) = \mu_{\frac{G}{H}}(xH)$.
Conversely, let $xH \in \frac{G}{H}$. Then

$$\mu_{\frac{G}{H}}(yHzH) = \mu_{\frac{G}{H}}(xH) \geq (\mu_{\frac{G}{H}}\ O\ \mu_{\frac{G}{H}})(xH)$$
$$= \sup_{xH=yHzH} T(\mu_{\frac{G}{H}}(yH), \mu_{\frac{G}{H}}(zH)) \geq T(\mu_{\frac{G}{H}}(yH), \mu_{\frac{G}{H}}(zH)).$$

□

Definition 3.24. We say that $\mu \in TF(G)$ is a normal T-fuzzy subgroup of G if for all $x, y \in G$, $\mu(xyx^{-1}) = \mu(y)$. Also we denote by $NTF(G)$ the set of all normal T-fuzzy subgroups of G.

Corollary 3.25. *If $\mu_H \in NTF(H)$, then $\mu_{\frac{G}{H}} \in NTF(\frac{G}{H})$.*

Proof. Let $xH, yH \in \frac{G}{H}$ and $\mu_H \in NTF(H)$. Then

$$\mu_{\frac{G}{H}}(xHyH(xH)^{-1}) = \mu_{\frac{G}{H}}(xHyHx^{-1}H) = \mu_{\frac{G}{H}}(xyx^{-1}H)$$
$$= \theta(xyx^{-1}, h) = T(\mu_H(xyx^{-1}), \mu_H(h))$$
$$= T(\mu_H(y), \mu_H(h)) = \theta(y, h) = \mu_{\frac{G}{H}}(yH).$$

□

Proposition 3.26. *Let $\mu_{\frac{G}{H}}, \nu_{\frac{G}{H}} \in NTF(\frac{G}{H})$. Then $\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}} \in NTF(\frac{G}{H})$.*

Proof. Let $xH, yH \in \frac{G}{H}$. Then

$$(\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(xHyH(xH)^{-1}) = (\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(xHyHx^{-1}H)$$
$$= T(\mu_{\frac{G}{H}}(xHyHx^{-1}H), \nu_{\frac{G}{H}}(xHyHx^{-1}H)) = T(\mu_{\frac{G}{H}}(yH), \nu_{\frac{G}{H}}(yH))$$
$$= (\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(yH).$$

□

Corollary 3.27. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq NTF(\frac{G}{H})$, Then $\mu = \cap_{i \in I_n} \mu_i \in NTF(\frac{G}{H})$.*

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Definition 3.28. Let $\mu_{\frac{G}{H}}, \nu_{\frac{G}{H}} \in TF(\frac{G}{H})$ and $\mu_{\frac{G}{H}} \subseteq \nu_{\frac{G}{H}}$. Then $\mu_{\frac{G}{H}}$ is called a normal subgroup of the subgroup $\nu_{\frac{G}{H}}$, written $\mu_{\frac{G}{H}} \trianglerighteq \nu_{\frac{G}{H}}$, if for all $xH, yH \in \frac{G}{H}$ we have that

$$\mu_{\frac{G}{H}}(xHyH(xH)^{-1}) \geq T(\mu_{\frac{G}{H}}(yH), \nu_{\frac{G}{H}}(xH)).$$

Lemma 3.29. *Let T be idempotent t-norm. If $\mu_{\frac{G}{H}} \in NTF(\frac{G}{H})$ and $\nu_{\frac{G}{H}} \in TF(\frac{G}{H})$, then $\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}} \trianglerighteq \nu_{\frac{G}{H}}$.*

Proof. From Proposition 3.19 we have $(\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}}) \in TF(\frac{G}{H})$. Now for all $xH, yH \in \frac{G}{H}$ we have

$$\begin{aligned}
&(\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(xHyH(xH)^{-1}) \\
&= T(\mu_{\frac{G}{H}}(xHyH(xH)^{-1}), \nu_{\frac{G}{H}}(xHyH(xH)^{-1})) \\
&= T(\mu_{\frac{G}{H}}(yH), \nu_{\frac{G}{H}}(xHyH(xH)^{-1})) \\
&\geq T(\mu_{\frac{G}{H}}(yH), T(T(\nu_{\frac{G}{H}}(xH), \nu_{\frac{G}{H}}(yH)), \nu_{\frac{G}{H}}(xH))) \\
&= T(\mu_{\frac{G}{H}}(yH), T(\nu_{\frac{G}{H}}(yH), \nu_{\frac{G}{H}}(xH))) \\
&= T(T(\mu_{\frac{G}{H}}(yH), \nu_{\frac{G}{H}}(yH)), \nu_{\frac{G}{H}}(xH)) \\
&= T((\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(yH), \nu_{\frac{G}{H}}(xH)).
\end{aligned}$$

Hence $\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}} \trianglerighteq \nu_{\frac{G}{H}}$. □

Proposition 3.30. *Let T be idempotent and $\mu_{\frac{G}{H}}, \mu_{\frac{G}{H}}, \xi_{\frac{G}{H}} \in TF(\frac{G}{H})$. If $\mu_{\frac{G}{H}}, \nu_{\frac{G}{H}} \trianglerighteq \xi_{\frac{G}{H}}$, then $\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}} \trianglerighteq \xi_{\frac{G}{H}}$.*

Proof. Clearly, $\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}} \in TF(\frac{G}{H})$ and $\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}} \subseteq \xi_{\frac{G}{H}}$. If $xH, yH \in \frac{G}{H}$, then

$$\begin{aligned}
&(\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(xHyH(xH)^{-1}) \\
&= T(\mu_{\frac{G}{H}}(xHyH(xH)^{-1}), \nu_{\frac{G}{H}}(xHyH(xH)^{-1})) \\
&\geq T(T(\mu_{\frac{G}{H}}(yH), \xi_{\frac{G}{H}}(xH)), T(\nu_{\frac{G}{H}}(yH), \xi_{\frac{G}{H}}(xH))) \\
&= T(T(\mu_{\frac{G}{H}}(yH), \nu_{\frac{G}{H}}(yH)), T(\xi_{\frac{G}{H}}(xH), \xi_{\frac{G}{H}}(xH))) \quad \text{(by Lemma 2.5)} \\
&= T(T(\mu_{\frac{G}{H}}(yH), \nu_{\frac{G}{H}}(yH)), \xi_{\frac{G}{H}}(xH)) \\
&= T((\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}})(yH), \xi_{\frac{G}{H}}(xH)).
\end{aligned}$$

Therefore $\mu_{\frac{G}{H}} \cap \nu_{\frac{G}{H}} \trianglerighteq \xi_{\frac{G}{H}}$. □

Corollary 3.31. *Let $I_n = \{1, 2, ..., n\}$ and $\{\mu_i \mid i \in I_n\} \subseteq TF(\frac{G}{H})$ such that $\{\mu_i \mid i \in I_n\} \trianglerighteq \xi$. Then $\mu = \cap_{i \in I_n} \mu_i \trianglerighteq \xi$.*

4. Direct product of T-fuzzy quotient subgroups and normal T-fuzzy quotient subgroups

Definition 4.1. Let $\mu_{\frac{G_1}{H_1}}$ and $\nu_{\frac{G_2}{H_2}}$ be T-fuzzy subgroups of the groups $\frac{G_1}{H_1}$ and $\frac{G_2}{H_2}$, respectively. The direct product of $\mu_{\frac{G_1}{H_1}}$ and $\nu_{\frac{G_2}{H_2}}$, denoted by $\mu_{\frac{G_1}{H_1}} \times \nu_{\frac{G_2}{H_2}}$, is the function defined by setting for all xH_1 in $\frac{G_1}{H_1}$ and yH_2 in $\frac{G_2}{H_2}$ we have that

$$(\mu_{\frac{G_1}{H_1}} \times \nu_{\frac{G_2}{H_2}})(xH_1, yH_2) = T(\mu_{\frac{G_1}{H_1}}(xH_1), \nu_{\frac{G_2}{H_2}}(yH_2)).$$

Proposition 4.2. *Let $\mu_i \in TF(\frac{G_i}{H_i})$ for $i = 1, 2$. Then $(\mu_1 \times \mu_2) \in TF(\frac{G_1}{H_1} \times \frac{G_2}{H_2})$.*

Proof. Let $(a_1H_1, b_1H_2), (a_2H_1, b_2H_2) \in \frac{G_1}{H_1} \times \frac{G_2}{H_2}$. Then

$$\begin{aligned}
&(\mu_1 \times \mu_2)((a_1H_1, b_1H_2)(a_2H_1, b_2H_2)) \\
&= (\mu_1 \times \mu_2)(a_1H_1a_2H_1, b_1H_2b_2H_2) \\
&= T(\mu_1(a_1H_1a_2H_1), \mu_2(b_1H_2b_2H_2)) \\
&\geq T(T(\mu_1(a_1H_1), \mu_1(a_2H_1)), T(\mu_2(b_1H_2), \mu_2(b_2H_2))) \\
&= T(T(\mu_1(a_1H_1), \mu_2(b_1H_2), T(\mu_1(a_2H_1), \mu_2(b_2H_2)) \quad \text{(by Lemma 2.5)} \\
&= T((\mu_1 \times \mu_2)(a_1H_1, b_1H_2), (\mu_1 \times \mu_2)(a_2H_1, b_2H_2)).
\end{aligned}$$

Also

$$\begin{aligned}
&(\mu_1 \times \mu_2)(a_1H_1, b_1H_2)^{-1} = (\mu_1 \times \mu_2)((a_1H_1)^{-1}, (b_1H_2)^{-1}) \\
&= T(\mu_1(a_1H_1)^{-1}, \mu_2(b_1H_2)^{-1}) \geq T(\mu_1(a_1H_1), \mu_2(b_1H_2)) \\
&= (\mu_1 \times \mu_2)(a_1H_1, b_1H_2).
\end{aligned}$$

Thus $(\mu_1 \times \mu_2) \in TF(\frac{G_1}{H_1} \times \frac{G_2}{H_2})$. □

Corollary 4.3. *Let $\mu_{\frac{G_1}{H_1}} \in TF(\frac{G_1}{H_1})$ and $\nu_{\frac{G_2}{H_2}} \in TF(\frac{G_2}{H_2})$. Then $\mu_{\frac{G_1}{H_1}} \times 1_{\frac{G_2}{H_2}}, 1_{\frac{G_1}{H_1}} \times \nu_{\frac{G_2}{H_2}} \in TF(\frac{G_1}{H_1} \times \frac{G_2}{H_2})$.*

Corollary 4.4. *Let $\mu_i \in TF(\frac{G_i}{H_i})$ for $i = 1, 2, ..., n$. Then*

$$\mu_1 \times \mu_2 \times ... \times \mu_n \in TF(\frac{G_1}{H_1} \times \frac{G_2}{H_2} \times ... \times \frac{G_n}{H_n}).$$

Proposition 4.5. *Let $\mu_i \in NTF(\frac{G_i}{H_i})$ for $i = 1, 2$. Then $\mu_1 \times \mu_2 \in NTF(\frac{G_1}{H_1} \times \frac{G_2}{H_2})$.*

Proof. From Proposition 4.2 we have that $(\mu_1 \times \mu_2) \in TF(\frac{G_1}{H_1} \times \frac{G_2}{H_2})$. Let $(a_1H_1, b_1H_2), (a_2H_1, b_2H_2) \in \frac{G_1}{H_1} \times \frac{G_2}{H_2}$. Then

$$\begin{aligned}
&(\mu_1 \times \mu_2)((a_1H_1, b_1H_2)(a_2H_1, b_2H_2)(a_1H_1, b_1H_2)^{-1} \\
&= (\mu_1 \times \mu_2)((a_1H_1, b_1H_2)(a_2H_1, b_2H_2)((a_1H_1)^{-1}, (b_1H_2)^{-1}) \\
&= (\mu_1 \times \mu_2)(a_1H_1a_2H_1(a_1H_1)^{-1}, b_1H_2b_2H_2(b_1H_2^{-1}) \\
&= T(\mu_1(a_1H_1a_2H_1(a_1H_1)^{-1}), \mu_2(b_1H_2b_2H_2(b_1H_2)^{-1})) \\
&= T(\mu_1(a_2H_1), \mu_2(b_2H_2)) \\
&= (\mu_1 \times \mu_2)(a_2H_1, b_2H_2).
\end{aligned}$$

□

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Corollary 4.6. *Let $\mu_i \in NTF(\frac{G_i}{H_i})$ for $i = 1, 2, ..., n$. Then*

$$\mu_1 \times \mu_2 \times ... \times \mu_n \in NTF(\frac{G_1}{H_1} \times \frac{G_2}{H_2} \times ... \times \frac{G_n}{H_n}).$$

Proposition 4.7. *Let $\mu_i, \nu_i \in TF(\frac{G_i}{H_i})$ and $\mu_i \subseteq \nu_i$ for $i = 1, 2$. If $\mu_i \trianglerighteq \nu_i$, then $\mu_1 \times \mu_2 \trianglerighteq \nu_1 \times \nu_2$.*

Proof. Let $(a_1H_1, b_1H_2), (a_2H_1, b_2H_2) \in \frac{G_1}{H_1} \times \frac{G_2}{H_2}$. Then

$$\begin{aligned}
&(\mu_1 \times \mu_2)((a_1H_1, b_1H_2)(a_2H_1, b_2H_2)(a_1H_1, b_1H_2)^{-1}) \\
&= (\mu_1 \times \mu_2)((a_1H_1, b_1H_2)(a_2H_1, b_2H_2)((a_1H_1)^{-1}, (b_1H_2)^{-1})) \\
&= (\mu_1 \times \mu_2)(a_1H_1a_2H_1(a_1H_1)^{-1}, b_1H_2b_2H_2(b_1H_2)^{-1}) \\
&= T(\mu_1(a_1H_1a_2H_1(a_1H_1)^{-1}), \mu_2(b_1H_2b_2H_2(b_1H_2)^{-1})) \\
&\geq T(T(\mu_1(a_2H_1), \nu_1(a_1H_1)), T(\mu_2(b_2H_2), \nu_2(b_1H_2))) \\
&= T(T(\mu_1(a_2H_1), \mu_2(b_2H_2)), T(\nu_1(a_1H_1), \nu_2(b_1H_2))) \quad \text{(by Lemma 2.5)} \\
&= T((\mu_1 \times \mu_2)(a_2H_1, b_2H_2), (\nu_1 \times \nu_2)(a_1H_1, b_1H_2)).
\end{aligned}$$

□

Corollary 4.8. *Let $\mu_i, \nu_i \in TF(\frac{G_i}{H_i})$ and $\mu_i \subseteq \nu_i$ for $i = 1, 2, ..., n$. If $\mu_i \trianglerighteq \nu_i$, then $\mu_1 \times \mu_2 \times ... \times \mu_n \trianglerighteq \nu_1 \times \nu_2 \times ... \times \nu_n$.*

5. GROUP HOMOMORPHISMS AND FUZZY SUBGROUPS UNDER t-NORMS

Lemma 5.1. *Let f be a homomorphism of $\frac{G_1}{H_1}$ into $\frac{G_2}{H_2}$ and $\mu \in TF(\frac{G_1}{H_1})$. Then $f(\mu) \in TF(\frac{G_2}{H_2})$.*

Proof. Let $uH_2, vH_2 \in \frac{G_2}{H_2}$ and $xH_1, yH_1 \in \frac{G_1}{H_1}$. If $uH_2 \notin f(\frac{G_1}{H_1})$ or $vH_2 \notin f(\frac{G_1}{H_1})$, then

$$f(\mu)(uH_2) = f(\mu)(vH_2) = 0 \leq f(\mu)(uH_2vH_2).$$

Suppose $uH_2 = f(xH_1)$ and $vH_2 = f(yH_1)$ then

$$\begin{aligned}
&f(\mu)(uH_2vH_2) \\
&= \sup\{\mu(xH_1yH_1) \mid uH_2 = f(xH_1), vH_2 = f(yH_1)\} \\
&\geq \sup\{T(\mu(xH_1), \mu(yH_1)) \mid uH_2 = f(xH_1), vH_2 = f(yH_1)\} \\
&= T(\sup\{\mu(xH_1) \mid uH_2 = f(xH_1)\}, \sup\{\mu(yH_1) \mid vH_2 = f(yH_1)\}) \\
&= T(f(\mu)(uH_2), f(\mu)(vH_2)).
\end{aligned}$$

Also since $\mu \in TF(\frac{G_1}{H_1})$ we have

$$\begin{aligned}
f(\mu)((uH_2)^{-1}) &= f(\mu)((u^{-1}H_2)) \\
&= \sup\{\mu(xH_1) \mid u^{-1}H_2 = f(xH_1)\} \\
&\geq \sup\{\mu(x^{-1}H_1) \mid uH_2 = f(x^{-1}H_1)\} \\
&= f(\mu)(uH_2).
\end{aligned}$$

□

Lemma 5.2. *Let f be a homomorphism of $\frac{G_1}{H_1}$ into $\frac{G_2}{H_2}$ and $\nu \in TF(\frac{G_2}{H_2})$. Then $f^{-1}(\nu) \in TF(\frac{G_1}{H_1})$.*

Proof. Let $xH_1, yH_1 \in \frac{G_1}{H_1}$. Then

$$f^{-1}(\nu)(xH_1yH_1) = \nu(f(xH_1yH_1)) = \nu(f(xH_1)f(yH_1)$$
$$\geq T(\nu(f(xH_1)), \nu(f(yH_1))) = T(f^{-1}(\nu)(xH_1), f^{-1}(\nu)(yH_1)).$$

Also
$f^{-1}(\nu)(xH_1)^{-1} = \nu(f(xH_1)^{-1}) = \nu(f^{-1}(xH_1)) \geq \nu(f(xH_1)) = f^{-1}(\nu)(xH_1)$. □

Proposition 5.3. *Let $\mu \in NTF(\frac{G_1}{H_1})$ and f is an epimorphism of $\frac{G_1}{H_1}$ onto $\frac{G_2}{H_2}$. Then $f(\mu) \in NTF(\frac{G_2}{H_2})$.*

Proof. From Lemma 5.1 we have $f(\mu) \in TF(\frac{G_2}{H_2})$. Let $xH_2, yH_2 \in \frac{G_2}{H_2}$. Since f is a surjection, $f(uH_1) = xH_2$ for some $uH_1 \in \frac{G_1}{H_1}$. Then

$$f(\mu)(xH_2yH_2(xH_2)^{-1})$$
$$= \sup\{\mu(wH_1) \mid wH_1 \in \frac{G_1}{H_1}, f(wH_1) = xH_2yH_2(xH_2)^{-1}\}$$
$$= \sup\{\mu(u^{-1}H_1wH_1uH_1) \mid wH_1 \in \frac{G_1}{H_1}, f(u^{-1}H_1wH_1uH_1) = yH_2\}$$
$$= f(\mu)(yH_2).$$

□

Proposition 5.4. *Let $\frac{G_2}{H_2}$ be a group and $\nu \in NTF(\frac{G_1}{H_1})$. Suppose that f is a homomorphism of $\frac{G_1}{H_1}$ into $\frac{G_2}{H_2}$. Then $f^{-1}(\nu) \in NTF(\frac{G_1}{H_1})$.*

Proof. By Lemma 5.2 we obtain $f^{-1}(\nu) \in TF(\frac{G_1}{H_1})$. Now for any $xH_1, yH_1 \in \frac{G_1}{H_1}$, we have

$$f^{-1}(\nu)(xH_1yH_1(xH_1)^{-1})$$
$$= \nu(f(xH_1yH_1(xH_1)^{-1}))$$
$$= \nu(f(xH_1)f(yH_1)f(xH_1)^{-1})$$
$$= \nu(f(xH_1)f(yH_1)f^{-1}(xH_1))$$
$$= \nu(f(yH_1)) = f^{-1}(\nu)(yH_1).$$

Hence $f^{-1}(\nu) \in NTF(\frac{G_1}{H_1})$. □

Proposition 5.5. *Let $\mu, \nu \in TF(\frac{G_1}{H_1})$ and $\mu \trianglerighteq \nu$. Let $\frac{G_2}{H_2}$ be a group and f a homomorphism from $\frac{G_1}{H_1}$ into $\frac{G_2}{H_2}$. Then $f(\mu) \trianglerighteq f(\nu)$.*

Proof. By Lemma 5.1 we have $f(\mu), f(\nu) \in TF(\frac{G_2}{H_2})$. Let $xH_2, yH_2 \in \frac{G_2}{H_2}$ and $uH_1, vH_1 \in \frac{G_1}{H_1}$. Then

$$f(\mu)(xH_2yH_2(xH_2)^{-1})$$
$$= \sup\{\mu(zH_1) \mid zH_1 \in \frac{G_1}{H_1}, f(zH_1) = xH_2yH_2(xH_2)^{-1}\}$$
$$= \sup\{\mu(uH_1vH_1(uH_1)^{-1}) \mid uH_1, vH_1 \in \frac{G_1}{H_1}, f(uH_1) = xH_2, f(vH_1) = yH_2\}$$
$$\geq \sup\{T(\mu(vH_1), \nu(uH_1)) \mid f(uH_1) = xH_2, f(vH_1) = yH_2\}$$
$$= T(\sup\{\mu(vH_1) \mid yH_2 = f(vH_1)\}, \sup\{\nu(uH_1) \mid xH_2 = f(uH_1)\})$$
$$= T(f(\mu)(yH_2), f(\nu)(xH_2)).$$

Hence $f(\mu) \trianglerighteq f(\nu)$. □

Proposition 5.6. *Let $\frac{G_2}{H_2}$ be a group. Let $\mu, \nu \in TF(\frac{G_2}{H_2})$ and $\mu \trianglerighteq \nu$. If f be a homomorphism from $\frac{G_1}{H_1}$ into $\frac{G_2}{H_2}$, then $f^{-1}(\mu) \trianglerighteq f^{-1}(\nu)$.*

Proof. From Lemma 5.2 we have $f^{-1}(\mu), f^{-1}(\nu) \in TF(\frac{G_1}{H_1})$. Let $xH_1, yH_1 \in \frac{G_1}{H_1}$. Now

$$f^{-1}(\mu)(xH_1yH_1(xH_1)^{-1}) = \mu(f(xH_1yH_1(xH_1)^{-1}))$$
$$= \mu(f(xH_1)f(yH_1)f^{-1}(xH_1)) \geq T(\mu(f(yH_1)), \nu(f(xH_1)))$$
$$= T(f^{-1}(\mu)(yH_1), f^{-1}(\nu)(xH_1)).$$

Hence $f^{-1}(\mu) \trianglerighteq f^{-1}(\nu)$. □

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