A Problem and A Generalization

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ABSTRACT— ***In the short work, we generalize a problem in [1].This problem can be used in approximation theory and applied mathematical sciences. Finally, we solve a problem on a null set in a measure space, in a simple way.***

Keywords— Metric Space, Compact set, Closed set, Measure space, Null set

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1. INTRODUCTION

Maurice Fréchet introduced metric spaces in 1906[2].A metri**c** space is a set where a notion of distance (called a metric) between elements of the set is defined.

The metric space which most closely corresponds to our intuitive understanding of space is the 3-dimensional Euclidean space. In fact, the notion of "metric" is a generalization of the Euclidean metric arising from the four long-known properties of the Euclidean distance. The Euclidean metric defines the distance between two points as the length of the straight line segment connecting them. Other metric spaces occur for example in elliptic geometry and hyperbolic geometry, where distance on a sphere measured by angle is a metric, and the hyperboloid model of hyperbolic geometry is used by special relativity as a metric space of velocities. For more detail see [3-7].

A metric space also induces topological properties like open and closed sets which leads to the study of even more abstract topological spaces.

In this paper, we have discussed on the the minimum distance between a closed set and a compact set in a metric space which are distinct. Finally, we prove an assertion in a measure space.

2. A GENERALIZATION

In the section, first we have a problem which is in [1]. Next, a generalization of it, is proved.
Let us to present the problem in special case.
Problem2.1. Let $A \text{ and } B$ be compact and closed subsets of $\mathbb{R}$, respectively. Further, suppose that $A \text{ and } B$ be

distinct. Then there is a positive number δ such that $|a-b| \geq \delta$ for all $a \in A \text{ and } b \in B.$

$$|a-b| < \delta$$

Remark2.2. Notice that the condition "distinctness" in the above problem is necessary.
Now, we are ready to prove the generalization of the previous problem.

Generalization2.3. Let A and B be compact and closed subsets of a metric space (X, d), respectively. Further, suppose that A and B be distinct. Then there is a positive number δ such that $d(a,b) \geq \delta$ for all $a \in A$ and $b \in B$.

Proof. By contrary, Let for each $\delta > 0$, there are $a \in A$ *and* $b \in B$ so that

$$d(a,b) < \delta$$

Let n be an arbitrary positive integer. Set $\delta = \frac{1}{n}$. Then we can obtain two sequences (a_n) and (b_n) of elements in A *and* B, respectively so that

$$d(a_n, b_n) < \frac{1}{n}$$

For sufficiently large n we have

$$d(a_n, b_n) \to 0$$

Since A is compact, so (a_n) has a convergent subsequence. Without loss of generality one may consider (a_n) as this subsequence. Hence

$$a_n \to a^* \qquad (n \to \infty)$$

For some $a^* \in A$. On the other hand we have

$$d(b_n, a^*) \leq d(b_n, a_n) + d(a_n, a^*)$$

If $n \to \infty$, then the right side of the last inequality approaches to zero. Therefore

$$b_n \to a^*$$

As $n \to \infty$. Since B is closed, so $a^* \in B$ which is impossible; Since A *and* B are distinct. □

Remark2.4. According to the generalization2.3, $d(A,B) = \delta$, where

$$d(A,B) = inf\{d(a,b): a \in A, b \in B\}$$

3. A PROBLEM ON A NULL SET

In the section, We present a problem which appeared in a measure space; This problem is as follow:
Problem3.1. Let X be a nonempty set and $\mu: P(X) \to [0,\infty)$ an outer measure. Suppose that (A_n) be a sequence of subsets in $P(X)$ such that $\sum_n \mu(A_n) < \infty$. Consider the set $E = \{x \in X: x \text{ belong to infinite many of } A_k^{\prime}s\}$. Then $\mu(E)$=0.

Proof. To show that $\mu(E)$=0,We prove that E is countable. By definition of E, For each $x \in E$, there is $n_x \in \mathbb{N}$ so that $x \in \bigcap_{k=n_x}^{\infty} A_k$. Define the relation $\sim$ on E as follow:

$$x \sim y \Leftrightarrow n_x = n_y$$

It is easy to verify that $\sim$ is an equivalence relation on E.Set $N_E := \{n_x : x \in E\}$. Clearly, $N_E \subset \mathbb{N}$. Now, Consider the function $f: E \to N_E$ defined by $f([x])= n_x$, where $[x]$ denotes the equivalence class of x. Since the equivalence classes partite E, so f is well-defined. Also, evidently f is onto. Let $n_x = n_y$. This implies that $x \sim y$, i. e., $x \in [y]$. On the other hand, $y \in [x]$ and so we must have $[x] = [y]$. This means that f is also one to one.Therefore, E is countable. □

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