On a New Modification of Homotopy Analysis Method for Solving Nonlinear Nonhomogeneous Differential Equations

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Abstract
In this paper, new powerful modification of homotopy analysis technique (NMHAM) was submitted to create an approximate solution of nonhomogeneous nonlinear ordinary and partial differential equations. The NMHAM is a combination of the new technique of homotopy analysis method(NHAM) [4] and the new technique of homotopy analysis method(nHAM) [7].Three illustrative examples are employed to illustrate the accuracy and computational proficiency of this approach. The outcomes uncover that the NMHAM is more accurate than the NHAM and nHAM.

Keywords: Modified homotopy analysis method, Taylor Series, Nonhomogeneous differential equations.

1. Introduction

In recent years, many engineers and scientists in various sciences like Mathematics, Biology,Physics, and particularly in branches of engineering like Fluid mechanics, Numerical calculations in Aerospace and Electronics are faced with nonlinear phenomena and many nonlinear problems. Since solving nonlinear problems plays a crucial role in various fields of engineering and science, Scientists are interested in obtaining techniques for solving nonlinear problems and have performed extensive researchers to achieve nonlinear problem solving techniques. As solving nonlinear problems are generally difficult and achieving their exact solutions are hard, various approximate methods have been developed to solve them.
The homotopy analysis technique (HAM), proposed by Liao [14], is a powerful technique to solve non-linear problems. In recent years, this method has been effectively applied to numerous problems in science and engineering [15-27]. All of these successful applications verified the validity, effectiveness and flexibility of the HAM. Recently, some modifications of HAM have published to facilitate and accurate the calculations and accelerate the rapid convergence of the series solution and reduce the size of work [1-13]. It is the aim of this paper to submit a new powerful modification of the HAM. The NMHAM is a combination of the two modifications of homotopy analysis technique (NHAM) [4] and the nHAM [7]. The NMHAM demonstrates an accurate solution if compared with the NHAM and nHAM, and therefore it has been shown that to be computationally efficient in applied fields. The obtained results suggest that this newly improvement technique introduces a powerful improvement for solving nonlinear problems.

2. The New Technique of Homotopy Analysis Method (NHAM)

Consider the following nonlinear differential equation

$$\mathbb{N}[y(x,t)] = f(x,t) \tag{2.1}$$

Where $\mathbb{N}$ is a nonlinear operator , (x,t) means independent variables, $y(x,t)$ is an unknown function, and $f(x,t)$ is a non-homogeneous terms.

The non-homogeneous terms $f(x,t)$ in (2.1) can be expressed in Taylor series based on a kind of continuous homotopy mapping with respect to $\mathbb{q}$, where $\mathbb{q} \in [0,1]$ is an embedding parameter, $f(x,t) \to \mu(x,t;\mathbb{q})$ as [4]

$$\mu(x,t;\mathbb{q}) = \sum_{r=0}^{\infty} f_r{}^s(x,t)\mathbb{q}^r = f_0{}^s(x,t)\mathbb{q}^0 + f_1{}^s(x,t)\mathbb{q}^1 + \cdots + f_n{}^s(x,t)\mathbb{q}^n + \cdots \tag{2.2}$$

Where

$$f_r{}^s(x,t) = \frac{1}{rs!}\left[\frac{d^{rs}}{dt^{rs}}f(x,t)\right]_{t=0} t^{rs} + \frac{1}{(rs+1)!}\left[\frac{d^{(rs+1)}}{dt^{(rs+1)}}f(x,t)\right]_{t=0} t^{(rs+1)} + \cdots + \frac{1}{(rs+s-1)!}\left[\frac{d^{(rs+s-1)}}{dt^{(rs+s-1)}}f(x,t)\right]_{t=0} t^{(rs+s-1)} \tag{2.3}$$

We note that $f_r{}^s(x,t)$ depend on the order of the differential equation s.For example,

$$s = 1 \Longrightarrow f_r{}^1 = \frac{1}{r!}\left[\frac{d^r}{dt^r}f(x,t)\right]_{t=0} t^r,$$

$$s = 2 \Longrightarrow f_r{}^2 = \frac{1}{2r!}\left[\frac{d^{2r}}{dt^{2r}}f(x,t)\right]_{t=0} t^{2r} + \frac{1}{(2r+1)!}\left[\frac{d^{(2r+1)}}{dt^{(2r+1)}}f(x,t)\right]_{t=0} t^{(2r+1)},$$

$$s = 4 \Longrightarrow f_r{}^2 = \frac{1}{4r!}\left[\frac{d^{4r}}{dt^{4r}}f(x,t)\right]_{t=0} t^{4r} + \frac{1}{(4r+1)!}\left[\frac{d^{(4r+1)}}{dt^{(4r+1)}}f(x,t)\right]_{t=0} t^{(4r+1)} + \frac{1}{(4r+2)!}\left[\frac{d^{(4r+2)}}{dt^{(4r+2)}}f(x,t)\right]_{t=0} t^{(4r+2)} + \frac{1}{(4r+3)!}\left[\frac{d^{(4r+3)}}{dt^{(4r+3)}}f(x,t)\right]_{t=0} t^{(4r+3)}$$

Give us a chance to develop the supposed zeroth deformation equation as follows

$$(1-\mathbb{q})\mathrm{L}[\delta(x,t;\mathbb{q}) - y_0(x,t)] = \mathbb{q}\mathbb{h}\mathcal{H}(x,t)\mathbb{N}[\delta(x,t;\mathbb{q}) - \mu(x,t;\mathbb{q})] \tag{2.4}$$

Where $\mathbb{q} \in [0,1]$ is an embedding parameter, $\mathbb{h} \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $y_0(x,t)$ is the initial guesses of $y(x,t)$, $\delta(x,t;\mathbb{q})$ is an unknown functions , and $\mathcal{H}(x,t)$ denotes a nonzero auxiliary function. It is evident that when $\mathbb{q} = 0$ and $\mathbb{q} = 1$ becomes

$$\delta(x,t;0) = y_0(x,t) \ , \ \delta(x,t;1) = y(x,t) \tag{2.5}$$

respectively. In this way as $\mathbb{q}$ increments from 0 to 1, the solution $\delta(x,t;\mathbb{q})$ varies from the initial guess $y_0(x,t)$ to the solution $y(x,t)$. Having the freedom to select $y_0(x,t)$, L , $\mathbb{h}$,and $\mathcal{H}(x,t)$, we can expect that every one of them can be chosen with the goal that the solution $\delta(x,t;\mathbb{q})$ of (2.4) exists for $\mathbb{q} \in [0,1]$.

Expanding $\delta(x,t;\mathbb{q})$ in Taylor series, we have

$$\delta(x,t;\mathbb{q}) = y_0(x,t) + \sum_{r=1}^{+\infty} y_r(x,t)\mathbb{q}^r, \tag{2.6}$$

Where

$$y_r(x,t) = \frac{1}{r!}\frac{\partial^r \delta(x,t;\mathbb{q})}{\partial \mathbb{q}^r}\bigg|_{\mathbb{q}=0}. \tag{2.7}$$

Next, recall that $\mathbb{h}$, $\mathcal{H}(x,t), y_0(x,t)$,and L are select with the end goal that the series (2.6) converges at $\mathbb{q}$=1 and that

$$y(x,t) = \delta(x,t;1) = y_0(x,t) + \sum_{r=1}^{+\infty} y_r(x,t) \tag{2.8}$$

Let $y_e(x,t) = \{y_0(x,t), y_1(x,t), y_2(x,t), \dots y_e(x,t)\}$. (2.9)

Differentiating equation (2.4) for r times with respect to $\mathbb{q}$ and afterward setting $\mathbb{q}$=0 and lastly dividing the resulting equation by $r!$, we have the so-called rth order deformation equation as follows:

$$L[y_r(x,t) - X_r y_{r-1}(x,t)] = \mathbb{h}\mathcal{H}(x,t)\mathcal{R}_r(\overrightarrow{y_{r-1}}(x,t)), \tag{2.10}$$

where

$$\mathcal{R}_r\left(\overrightarrow{y_{r-1}}(x,t)\right) = \frac{1}{(r-1)!}\frac{\partial^{r-1}(\mathbb{N}[\delta(x,t,\mathbb{q})] - \mu(x,t;\mathbb{q}))}{\partial \mathbb{q}^{r-1}}\bigg|_{\mathbb{q}=0} \tag{2.11}$$

And $X_r = \begin{cases} 0, & r \le 1, \\ 1, & r > 1. \end{cases}$

It ought to be underscored that $y_r(x,t)$ for $r \ge 1$ is administered by the equation (2.10) with the boundary conditions that come from the original problem.

The homogenous part of equation (2.1) can be written as [7]

$$\begin{gathered} Ly(x,t) + \mathrm{A}y(x,t) + \mathrm{B}y(x,t) = 0, \\ y(x,0) = g_0(x), \\ \frac{\partial y(x,t)}{\partial t}\bigg|_{t=0} = g_1(x), \\ \vdots \\ \frac{\partial^{k-1} y(x,t)}{\partial^{k-1}}\bigg|_{t=0} = g_{k-1}(x). \end{gathered} \tag{2.12}$$

WhereL= $\partial^k/\partial t^k$, $k = 1, 2, \dots$ is the highest partial derivative with respect to t , A is a linear term , and B is a nonlinear term.

Hence, the equation (2.1) will be take the form:

$$\begin{gathered} Ly(x,t) + \mathrm{A}y(x,t) + \mathrm{B}y(x,t) = \mu(x,t;\mathbb{q}), \\ y(x,0) = g_0(x), \\ \left.\frac{\partial y\,(x,t)}{\partial t}\right|_{t=0} = g_1(x), \\ \vdots \\ \left.\frac{\partial^{k-1} y(x,t)}{\partial^{k-1}}\right|_{t=0} = g_{k-1}(x). \end{gathered} \tag{2.13}$$

And the so-called zero-order deformation equation (2.4) becomes

$$(1-\mathbb{q})L[\delta(x,t;\mathbb{q}) - y_0(x,t)] = \mathbb{q}\mathbb{h}\mathcal{H}(x,t)(Ly(x,t) + Ay(x,t) + By(x,t) - \mu(x,t;\mathbb{q})) \tag{2.14}$$

Hence, the r^{th} order deformation equation will becomes:

$$L[y_r(x,t) - X_r y_{r-1}(x,t)] = \mathbb{h}\mathcal{H}(x,t)\,(Ly_{r-1}(x,t) + Ay_{r-1}(x,t) + B(\overrightarrow{y_{r-1}}(x,t)) - {f_{r-1}}^s(x,t)) \tag{2.15}$$

Therefore

$$\begin{aligned} y_r(x,t) = X_r y_{r-1}(x,t) + \mathbb{h}L^{-1}[&\mathcal{H}(x,t)(Ly_{r-1}(x,t) + Ay_{r-1}(x,t) \\ &+B\left(\overrightarrow{y\lambda_{r-1}}(x,t)\right) - {f_{r-1}}^s(x,t))] \end{aligned} \tag{2.16}$$

Such that

$$L^{-1}(.) = \int\int\cdots\int(.)\underbrace{dtdt\cdots dt}_{ktimes} + c_1 t^{k-1} + c_2 t^{k-1} + \cdots + c_k\,. \tag{2.17}$$

Where $c_1, c_2, \dots, c_k$ are constants.

To solve (2.13) by means of HAM , we select the following initial approximation

$$y_0(x,t) = g_0(x) + g_1(x)t + g_2(x)\tfrac{t^2}{2!} + \cdots + g_{k-1}(x)\tfrac{t^{k-1}}{(k-1)!}\,. \tag{2.18}$$

Let $\mathcal{H}(x,t) = 1$, by means of (2.17) and (2.18); then (2.16) becomes

$$y_r(x,t) = X_r y_{r-1}(x,t) + \mathbb{h}\int_0^t\int_0^t\dots\int_0^t\Big(\frac{\partial^k y_{r-1}(x,\tau)}{\partial\tau^k} + Ay_{r-1}(x,\tau) + B\left(\overrightarrow{y_{r-1}}(x,\tau)\right) - {f_{r-1}}^s(x,\tau)\Big)\underbrace{d\tau d\tau\dots d\tau}_{ktimes}. \tag{2.19}$$

Now we have

$$y_r(x,t) = X_r y_{r-1}(x,t) + \hbar \int_0^t \int_0^t \dots \int_0^t \frac{\partial^k y_{r-1}(x,\tau)}{\partial \tau^k} \underbrace{d\tau d\tau \dots d\tau}_{ktimes} + \hbar \int_0^t \int_0^t \dots \int_0^t A y_{r-1}(x,\tau) + B\left(\overrightarrow{y_{r-1}}(x,\tau)\right) - {f_{r-1}}^s(x,\tau)) \underbrace{d\tau d\tau \dots d\tau}_{ktimes}$$

$$= X_r y_{r-1}(x,t) + \hbar y_{r-1}(x,t) - \hbar(y_{r-1}(x,0) + t\frac{\partial y_{r-1}(x,0)}{\partial t} + \dots + \frac{t^{k-1}}{(k-1)!}\frac{\partial^{k-1} y_{r-1}(x,0)}{\partial t^{k-1}})$$

$$+ \hbar \int_0^t \int_0^t \dots \int_0^t (A y_{r-1}(x,\tau) + B\left(\overrightarrow{y_{r-1}}(x,\tau)\right) - {f_{r-1}}^s(x,\tau)) \underbrace{d\tau d\tau \dots d\tau}_{ktimes} \qquad (2.20)$$

For $r = 1$, $X_r = 0$,and

$$y_0(x,0) + t\frac{\partial y_0(x,0)}{\partial t} + \frac{t^2}{2!}\frac{\partial^2 y_0(x,0)}{\partial t^2} + \dots + \frac{t^{k-1}}{(k-1)!}\frac{\partial^{k-1} y_0(x,0)}{\partial t^{k-1}}$$

$$= g_0(x) + g_1(x)t + g_2(x)\frac{t^2}{2!} + \dots + g_{k-1}(x)\frac{t^{k-1}}{(k-1)!} \qquad (2.21)$$

$$= y_0(x,t)$$

Substituting this equality into (2.20), we obtain

$$y_1(x,t) = \hbar \int_0^t \int_0^t \dots \int_0^t A y_0(x,\tau) + B\left(\overrightarrow{y_0}(x,\tau)\right) - {f_0}^s(x,\tau)) \underbrace{d\tau d\tau \dots d\tau}_{ktimes} \qquad (2.22)$$

For $r > 1$, $X_r = 1$, and

$$y_r(x,0) = 0\,, \frac{\partial y_r(x,0)}{\partial t} = 0\,, \frac{\partial^2 y_r(x,0)}{\partial t^2} = 0\,, \cdots, \frac{\partial^{(k-1)} y_r(x,0)}{\partial t^{(k-1)}} = 0\,. \qquad (2.23)$$

Substituting this equality into (2.20), we obtain

$$y_r(x,t) = (1+\hbar) y_{r-1}(x,t) + \hbar \int_0^t \int_0^t \dots \int_0^t (A y_{r-1}(x,\tau) + B\left(\overrightarrow{y_{r-1}}(x,\tau)\right) - {f_{r-1}}^s(x,\tau)) \underbrace{d\tau d\tau \dots d\tau}_{ktimes} \qquad (2.24)$$

The NHAM is powerful when $k = 1$, and the solution of NHAM can be written as the following series:

$$y(x,t;\hbar) = Y_r(x,t;\hbar) = \sum_{i=0}^{r} y_i(x,t;\hbar) \qquad (2.25)$$

But when $k \geq 2$, there are too many additional terms where harder and more timeconsuming computations are performed.so, the closed form solution needs more numbers of iteration.

3. The New Modified Homotopy Analysis Method (NMHAM)

When $k \geq 2$, we rewrite (2.1) as in the following system:

$$y_t = y1$$
$$y1_t = y2$$
$$\vdots \tag{3.1}$$
$$y\{k-1\}_t = -Ay(x,t) - By(x,t) + f(x,t)$$

Set the initial approximation

$$y_0(x,t) = g_0(x)$$
$$y1_0(x,t) = g_1(x)$$
$$\vdots \tag{3.2}$$
$$y\{k-1\}_0(x,t) = g_{k-1}(x)$$

We note that the order of differential equation (3.1) is the first order ($s = 1$) Since all the equations of the system (3.1) of the first order then The non-homogeneous terms $f(x,t)$ in the last equation of (3.1) can be expressed in Taylor series based on a kind of continuous homotopy mapping with respect to $\mathbb{q}$,$f(x,t) \to \mu(x,t;\mathbb{q})$ as the following

$$\mu(x,t;\mathbb{q}) = \sum_{r=0}^{\infty} f_r{}^1(x,t)\mathbb{q}^r = f_0{}^1(x,t)\mathbb{q}^0 + f_1{}^1(x,t)\mathbb{q}^1 + \cdots + f_n{}^1(x,t)\mathbb{q}^n + \cdots \tag{3.3}$$

Where

$$f_r{}^1 = \frac{1}{r!}\left[\frac{d^r}{dt^r}f(x,t)\right]_{t=0} t^r \tag{3.4}$$

Using the iteration formulas (2.22) and (2.24) as follows:

$$y_1(x,t) = \mathbb{h}\int_0^t \big(-y1_0(x,\tau)\big)d\tau$$
$$y1_1(x,t) = \mathbb{h}\int_0^t \big(-y2_0(x,\tau)\big)d\tau$$
$$\vdots \tag{3.5}$$
$$y\{k-1\}_1(x,t) = \mathbb{h}\int_0^t \Big(Ay_0(x,\tau) + B\big(y_0(x,\tau)\big) - f_0{}^1(x,\tau)\Big)d\tau$$

For $r > 1$, $X_r = 1$, and

$$y_r(x,0) = 0 , y1_r(x,0) = 0 , y2_r(x,0) = 0, \cdots, y\{k-1\}_r(x,0) = 0 . \tag{3.6}$$

Substituting in (2.20),we obtain

$$y_r(x,t) = (1+\hbar)y_{r-1}(x,t) + \hbar\int_0^t\left(-y1_{r-1}(x,\tau)\right)d\tau \,,$$

$$y1_r(x,t) = (1+\hbar)y1_{r-1}(x,t) + \hbar\int_0^t(-y2_{r-1}(x,\tau))d\tau \,,$$

$$\vdots \tag{3.7}$$

$$y\{k-1\}_r(x,t) = (1+\hbar)y\{k-1\}_{r-1}(x,t) + \hbar\int_0^t(A\,y_{r-1}(x,\tau) + \;B\left(y_{r-1}(x,\tau)\right) - {f_{r-1}}^1(x,\tau))d\tau$$

4. Applications

4.1 Example 1 Consider a Duffings equation [4]

$$\frac{d^2y}{dt^2} + 3y - 2y^3 = f(t) \tag{4.1}$$

Equation (4.1) with the initial condition

$$\text{y(0)=0} \;\; ,y'(0)\text{=1} \,, \tag{4.2}$$

and $f(t) = \cos(t)\sin(2t)$ has the exact solution

$$y(t) = \sin(t) \tag{4.3}$$

The problem (4.1)-(4.2) solved by (NHAM)[4].

In order to solve (4.1)–(4.2) by the proposed approach (NMHAM) we construct the following system:

$$y_t(t) = v(t),$$

$$v_t(t) = -3y + 2y^3 + \mu(t;\mathbb{q}) \tag{4.4}$$

with the following initial conditions :

$$y_0(t) = 0\,, v_0(t) = 1 \tag{4.5}$$

We expand the homotopy $\mu(t;\mathbb{q})$ in powers of the parameter $\mathbb{q}$ with s=1:

$$\mu(t;\mathbb{q}) = \sum_{r=0}^{\infty} {f_r}^1\,(t)\mathbb{q}^r \; = {f_0}^1(t)\mathbb{q}^0 + {f_1}^1(t)\mathbb{q}^1 + \cdots + {f_n}^1(t)\mathbb{q}^n + \cdots \tag{4.6}$$

Where

$${f_r}^1(t) = \frac{1}{r!}\left[\frac{d^r}{dt^r}f(t)\right]_{t=0} t^r \tag{4.7}$$

such that

$${f_0}^1(t) = 0 \quad ,{f_1}^1(t) = 2t \quad ,{f_2}^1(t) = 0 \quad ,{f_3}^1(t) = -\frac{7}{3}t^3 \,, \; ....$$

and the following linear operators:

$$Ly(t) = \frac{\partial y(t)}{\partial t} \;, Lv(t) = \frac{\partial v(t)}{\partial t} \tag{4.8}$$

$$Ay_{r-1}(t) = 3y_{r-1}(t) - f^1{}_{r-1}(t)$$

$$By_{r-1}(t) = -2\sum_{i=0}^{r-1} y_{r-1-i} \sum_{j=0}^{i} y_j y_{i-j} \tag{4.9}$$

we obtain

$$y_1(t) = \mathbb{h}\int_0^t \big(-v_0(\tau)\big)d\tau$$

$$v_1(t) = \mathbb{h}\int_0^t (3y_0(\tau) - 2{y_0}^3(\tau) - {f_0}^1(\tau))d\tau \tag{4.10}$$

Now, for $r \geq 2$, we get

$$y_r(t) = (1+\mathbb{h})y_{r-1}(t) + \mathbb{h}\int_0^t (-v_{r-1}(\tau))\,d\tau\,,$$

$$v_r(t) = (1+\mathbb{h})v_{r-1}(t) + \mathbb{h}\int_0^t (Ay_{r-1}(\tau) + By_{r-1}(\tau))\,d\tau \tag{4.11}$$

And the following results are obtained:

$$y_1(t) = -\mathbb{h}t$$

$$v_1(t) = 0$$

$$y_2(t) = -\mathbb{h}(1+\mathbb{h})t$$

$$v_2(t) = \mathbb{h}(-t^2 - \frac{3\,\mathbb{h}t^2}{2})$$

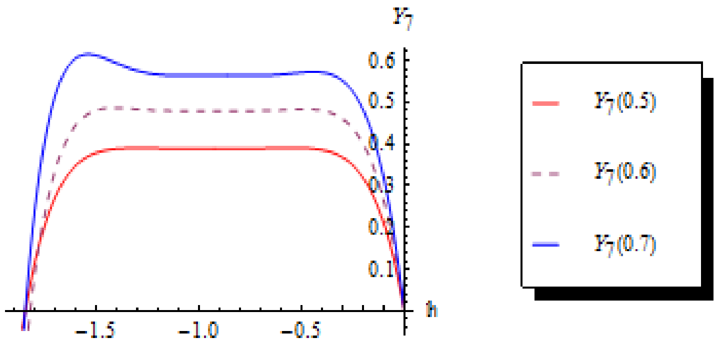
$$y_3(t) = -\mathbb{h}(1+\mathbb{h})^2 t + \mathbb{h}(\frac{\mathbb{h}t^3}{3} + \frac{\mathbb{h}^2 t^3}{2})$$

$\vdots$

Then, the series solution of the NMHAM is:

$$y(t,\mathbb{h}) \cong Y_R(t,\mathbb{h}) = \sum_{i=0}^{R} y_i\,(t,\mathbb{h}) \tag{4.12}$$

Equation (4.12) is an approximation solutions for the problem (4.1)-(4.2) depending on the parameter $\mathbb{h}$. To determine the valid region of , the $\mathbb{h}$-curves given by the 7th-order NMHAM at different values of t are drawn in figure (1).



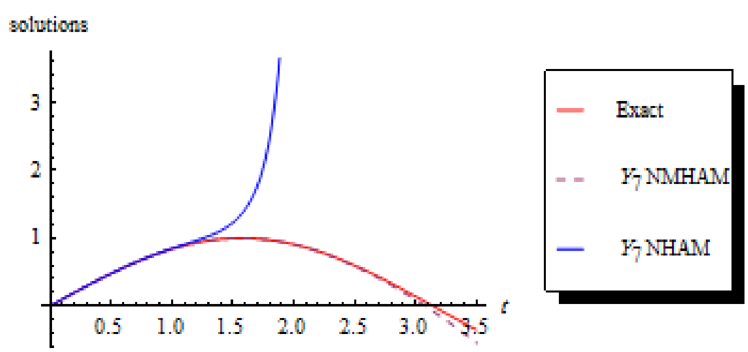
Figure(1) ħ-curve for NMHAM approximation solutions$Y_7(t)$ of problem (4.1) –(4.2)at different values of t .

Figure(2) show the comparison between Y_7 of NMHAM and Y_7 of NHAM at$0 \leq t \leq 3.5$ with the exact solution(4.3) .Figure (3) comparison between Y_6 , $Y_4$of NMHAM and Y_7 of NHAM with the exact solution (4.3) at $0 \leq t \leq 3$ which indicates that the speed of convergence of NMHAM is faster in comparison of NHAM.

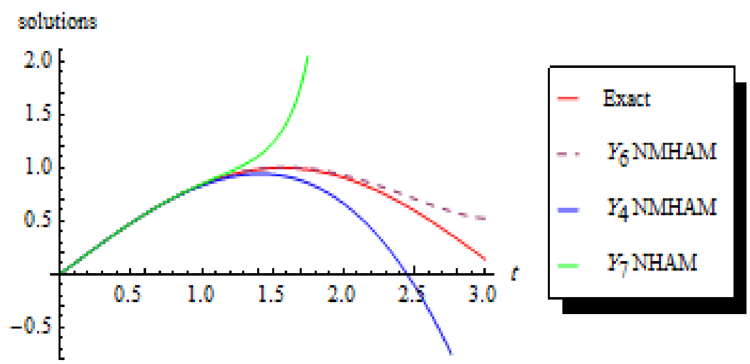
The absolute error of the 7th order approximate solution of NMHAM compared with 7th order approximate solution of NHAM are calculated by the formula

Absolute Error $(A.E) = |Y_{exact} - Y_{approx}|$

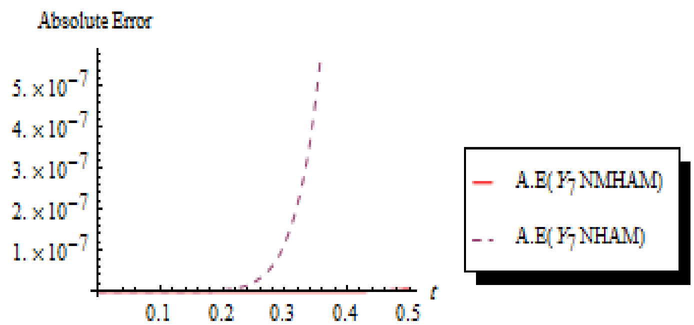
Figures (4) show that the series solution Y_7 obtained by NMHAM at $0 \leq t \leq 0.5$ is more accuratefrom the series solution Y_7obtained byNHAM.Figures (5) show that the series solution Y_7 obtained by NMHAM is more accurate from the series solution Y_7obtained by NHAM at larger t $(0.5 \leq t \leq 1)$. Figures (6) show that the series solution Y_6 obtained by NMHAM at $0 \leq t \leq 0.5$ is moreand faster converge from the series solution Y_7obtained by NHAM. Figures (7) show that the series solution Y_6 obtained by NMHAM is moreand faster converge from the series solution Y_7obtained by NHAM at larger t $(0.5 \leq t \leq 1)$.



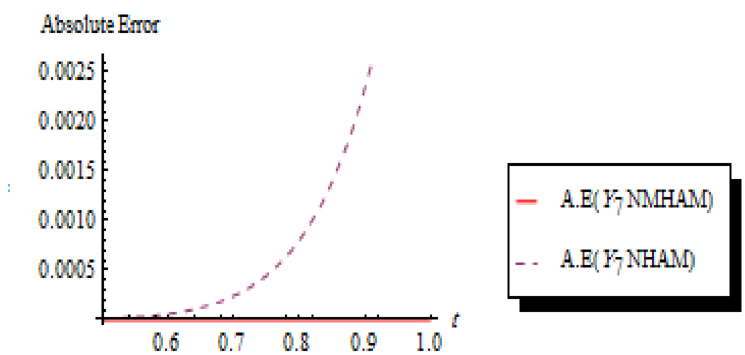
Figure(2): Comparison of the 7th order approximations of NMHAM and NHAM at $0 \leq t \leq 3.5, ħ = -1$ with the exact solution(4.3).



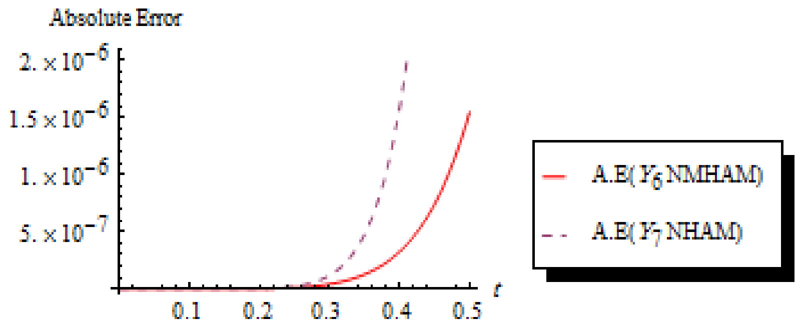
Figure(3):Comparison of Y_6 , $Y_4$of NMHAM and $Y_7$of NHAM with the exact solution (4.3) at $0 \leq t \leq 3, \mathbb{h} = -1$.



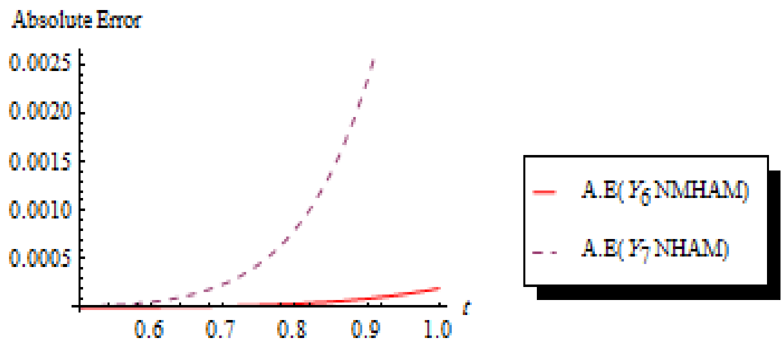
Figure(4):Absolute errors of $Y_7$of NMHAM and Y_7 of NHAMat$0 \leq t \leq 0.5, \mathbb{h} = -1$.



Figure(5):Absolute errors of $Y_7$of NMHAM andY_7 of NHAMat$0.5 \leq t \leq 1, \mathbb{h} = -1$.



Figure(6):Absolute errors of Y_6 of NMHAM and Y_7 of NHAM at $0 \leq t \leq 0.5, \hbar = -1$



Figure(7):Absolute errors of Y_6 of NMHAM and Y_7 of NHAM at $0.5 \leq t \leq 1, \hbar = -1$

Table (1) show the comparison of Y_7 of NMHAM ,NHAM and nHAM ,with the exact solution(4.3) at different values of t .Table (2) show the comparison between the absolute errors of Y_7 of NMHAM ,NHAM and nHAM at different values of t. Tables (1) and (2) indicate that the series solution obtained by NMHAM is more accurate from the series solution obtained by NHAM and the series solution obtained by nHAMis divergent for all t except $t = 0$ and the absolute error monotonously increases very quickly.

Table1: Comparison of the 7th order approximations of NMHAM , 7th order approximations of NHAM and 7th order approximations of nHAM with the exact solution.

t	*exact*	*NMHAM*	*NHAM*	*nHAM*
0	0	0	0	0
0.2	0.198669	0.198669	0.198669	−2.376868
0.4	0.389418	0.389418	0.389419	−4.709543
0.6	0.564642	0.564642	0.564703	−6.959303
0.8	0.717356	0.717356	0.718162	−9.10208
1	0.841471	0.841468	0.847508	−11.144744
1.2	0.932039	0.932025	0.963396	−13.152658

1.4	0.985449	0.985393	1.111087	−15.294508
1.6	0.999574	0.999389	1.424605	−17.912939
1.8	0.973848	0.973317	2.459322	−21.629384
2	0.909297	0.907937	7.466350	−27.489478

Table2: Comparisonof the Absolute errors of 7th order approximations of NMHAM , 7th order approximations of NHAM and 7th order approximations of nHAM with the exact solution

t	***A.E NMHAM***	***A.E NHAM***	***A.E nHAM***
0	0	0	0
0.2	1.410399×10^{-12}	3.083948×10^{-9}	2.575537
0.4	7.213487×10^{-10}	1.575220×10^{-6}	5.098961
0.6	2.768075×10^{-8}	0.000060	7.523945
0.8	3.677249×10^{-7}	0.000805	9.819431
1	2.730839×10^{-6}	0.006037	11.986216
1.2	1.403454×10^{-5}	0.031357	14.084697
1.4	5.593443×10^{-5}	0.125637	16.279957
1.6	1.850367×10^{-4}	0.425032	18.912513
1.8	5.308537×10^{-4}	1.485475	22.603231
2	1.360919×10^{-3}	6.557053	28.398776

4.2 Example 2 Consider a nonlinear ordinary differential equation[28]

$$y_{ttt} - t y_{tt} + t^2 y^2 = f(t) \quad (4.13)$$

Subject the initial condition

$$y(0) = 0, \; y_t(0) = 1 \;\;, y_{tt}(0) = 1 \quad (4.14)$$

The exact solution when $f(t) = t\sin(t) - \cos(t) + t^2 \sin(t)^2$ is

$$y(t) = \sin(t)\,. \quad (4.15)$$

4.2.1. NHAM solution: To solve (4.13-4.14) by means of the NHAM, expanding the homotopy $\mu(x,t;\mathbb{q})$ in powers of the parameter $\mathbb{q}$ with $s = 3$:

$$\mu(t;\mathbb{q}) = \sum_{r=0}^{\infty} f_r{}^3(t)\mathbb{q}^r = f_0{}^3(t)\mathbb{q}^0 + f_1{}^3(t)\mathbb{q}^1 + \cdots + f_n{}^3(t)\mathbb{q}^n + \cdots, \quad (4.16)$$

where

$$f_r{}^3(t) = \frac{1}{3r!}[\frac{d^{3r}}{dt^{3r}}f(t)]_{t=0}\, t^{3r} + \frac{1}{(3r+1)!}[\frac{d^{(3r+1)}}{dt^{(3r+1)}}f(t)]_{t=0}\, t^{(3r+1)}\, \frac{1}{(3r+2)!}[\frac{d^{(3r+2)}}{dt^{(3r+2)}}f(t)]_{t=0}\, t^{(3r+2)} \tag{4.17}$$

such that

$$f_0{}^3(t) = -1 + \frac{3t^2}{2} \quad , f_1{}^3(t) = \frac{19t^4}{24} \quad , f_2{}^3(t) = -\frac{233t^6}{720} + \frac{1783t^8}{40320} \quad , f_3{}^3(t) = -\frac{11509t^{10}}{3628800}\ , ... \tag{4.18}$$

Let we choose the initial approximation

$$y_0(t) = t, \tag{4.19}$$

And the linear operator

$$L[\delta(t;\mathbb{q})] = \frac{\partial^3\delta(t;\mathbb{q})}{\partial t^3} \tag{4.20}$$

with the property $L[c] = 0$, (4.21)

where$c$ is a constant of integration.
The nonlinear operator to the problem (4.13-4.14) under NHAM define as

$$\mathbb{N}[\delta(t;\mathbb{q})] = \frac{\partial^3\delta(t;\mathbb{q})}{\partial t^3} - t\frac{\partial^2\delta(t;\mathbb{q})}{\partial t^2} + t^2\delta^2(t;\mathbb{q}) - \mu(t;\mathbb{q}), \tag{4.22}$$

According to (2.4), the zero order deformation equation with the initial approximation (4.19) and linear operator (4.20) with (4.21) will be:

$$(1-\mathbb{q})\mathrm{L}[\delta(t;\mathbb{q}) - y_0(t)] = \mathbb{q}\mathbb{h}\mathbb{N}[\delta(t;\mathbb{q}) - \mu(t;\mathbb{q})], \tag{4.23}$$

and the rth order deformation equation as follows:

$$L[y_r(t) - X_r y_{r-1}(t)] = \mathbb{h}\mathcal{R}_r(\overrightarrow{y_{r-1}}(t)), \tag{4.24}$$

with the initial conditions $y_r(x,0) = 0$, $y_{r_t}(x,0) = 1$ and $y_{r_{tt}}(x,0) = 0$
Where

$$\mathcal{R}_r\left(\overrightarrow{y_{r-1}}(t)\right) = y_{\mathrm{ttt}\,\mathrm{r}-1}(t) - t\, y_{tt_{r-1}}(t) + t^2\sum_{i=0}^{r-1} y_i y_{r-1-i} - f^3{}_{r-1}(t) \tag{4.25}$$

Now, the solution of (4.24)for $r \geq 1$ becomes

$$y_r(t) = X_r y_{r-1}(t) + \mathbb{h}L^{-1}\mathcal{R}_r\left(\overrightarrow{y_{r-1}}(t)\right) \tag{4.26}$$

And the following results are obtained:

$$y_1(t) = \frac{1}{840}\mathbb{h}t^3(140 - 21t^2 + 4t^4)$$

$$y_2(t) = \frac{1}{840}\mathbb{h}t^3(140 - 21t^2 + 4t^4) + \mathbb{h}(-\frac{19t^7}{5040} + \mathbb{h}(\frac{t^3}{6} - \frac{t^5}{24} + \frac{t^7}{140} + \frac{t^9}{3780} - \frac{t^{11}}{19800} + \frac{t^{13}}{180180}))$$

⋮

Then, the series solution of the NHAM is:

$$y(t,\mathbb{h}) \cong Y_R(t,\mathbb{h}) = \sum_{i=0}^{R} y_i\,(t,\mathbb{h}) \tag{4.27}$$

Equation (4.27) is an approximation solutions for the problem (4.13)-(4.14) depending on the parameters $\mathbb{h}$.

4.2.2. NMHAM solution: in order to solve (4.13))-(4.14)by the proposed approach (NMHAM) we construct a system of differential equations as follows :

$$y_t(t) = v(t)$$

$$v_t(t) = z(t) \tag{4.28}$$

$$z_t(t) = t\,z(t) - t^2 y^2(t) + \mu(t;\mathbb{q})$$

with the following initial conditions:

$$y_0(t) = 0$$

$$v_0(t) = 1 \tag{4.29}$$

$$z_0(t) = 0$$

Expanding the homotopy $\mu(t;\mathbb{q})$ in powers of the parameter $\mathbb{q}$ with $s = 1$:

$$\mu(t;\mathbb{q}) = \sum_{r=0}^{\infty} f_r{}^1\,(t)\mathbb{q}^r = f_0{}^1(t)\mathbb{q}^0 + f_1{}^1(t)\mathbb{q}^1 + \cdots + f_n{}^1(t)\mathbb{q}^n + \cdots, \tag{4.30}$$

Where

$$f_r{}^1(t) = \frac{1}{r!}\Big[\frac{d^r}{dt^r} f(t)\Big]_{t=0}\, t^r \tag{4.31}$$

such that

$$f_0{}^1(t) = -1, \quad f_1{}^1(t) = 0, \qquad f_2{}^1(t) = \frac{3t^2}{2}, \cdots \tag{4.32}$$

And the following linear operators:

$$Ly(t) = \frac{\partial y\,(t)}{\partial t},\ Lv(t) = \frac{\partial v\,(t)}{\partial t}\quad ,Lz(t) = \frac{\partial z\,(t)}{\partial t} \tag{4.33}$$

$$Ay_{r-1}(t) = -f^1{}_{r-1}(t)$$

$$B y_{r-1}(x,t) = -t\, z_{r-1}(t) + t^2 \sum_{i=0}^{r-1} y_i y_{r-1-i} \tag{4.34}$$

We obtain

$$y_1(t) = \mathbb{h} \int_0^t \big(-v_0(\tau)\big) d\tau$$

$$v_1(t) = \mathbb{h} \int_0^t \big(-z_0(\tau)\big) d\tau$$

$$z_1(t) = \mathbb{h} \int_0^t (-t z_0(\tau) + t^2 {y_0}^2(\tau) - {f_0}^1(\tau)) d\tau \tag{4.35}$$

Now, for $r \geq 2$, we get

$$y_r(t) = (1+\mathbb{h}) y_{r-1}(t) + \mathbb{h} \int_0^t \big(-v_{r-1}(\tau)\big) d\tau$$

$$v_r(t) = (1+\mathbb{h}) v_{r-1}(t) + \mathbb{h} \int_0^t \big(-z_{r-1}(\tau)\big) d\tau$$

$$z_r(t) = (1+\mathbb{h}) z_{r-1}(t) + \mathbb{h} \int_0^t \Big(A y_{r-1}(\tau) + B\big(y_{r-1}(\tau)\big)\Big) d\tau \tag{4.36}$$

And the following results are obtained:

$$y_1(t) = -\mathbb{h} t,\ v_1(t) = 0,\ z_1(t) = \mathbb{h} t$$

$$y_2(t) = -\mathbb{h}(1+\mathbb{h}) t,\ v_2(t) = -\frac{1}{2}\mathbb{h}^2 t^2,\ z_2(t) = \mathbb{h}(1+\mathbb{h}) t - \frac{\mathbb{h}^2 t^3}{3},$$

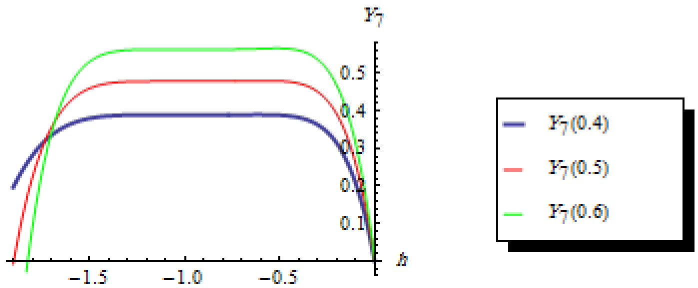
$$y_3(t) = -\mathbb{h}(1+\mathbb{h})^2 t + \frac{\mathbb{h}^3 t^3}{6},$$

$\vdots$

Then, the series solution of the NMHAM is:

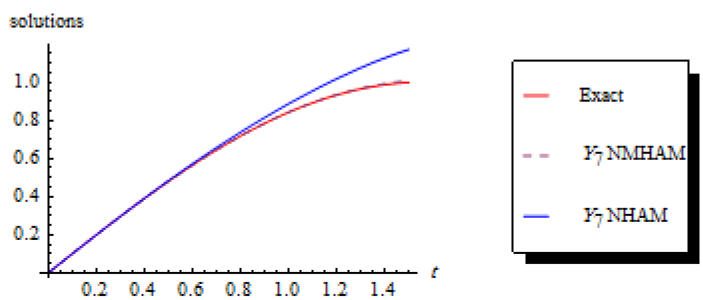
$$y(t,\mathbb{h}) \cong Y_R(t,\mathbb{h}) = \sum_{i=0}^{R} y_i\,(t,\mathbb{h}) \tag{4.37}$$

Equation (4.37) is an approximation solutions for the problem (4.13)-(4.14) depending on the parameter $\mathbb{h}$. To determine the valid region of $\mathbb{h}$, the $\mathbb{h}$-curves given by the 7^{th} order of NMHAM at different values of t are drawn in figure (8).

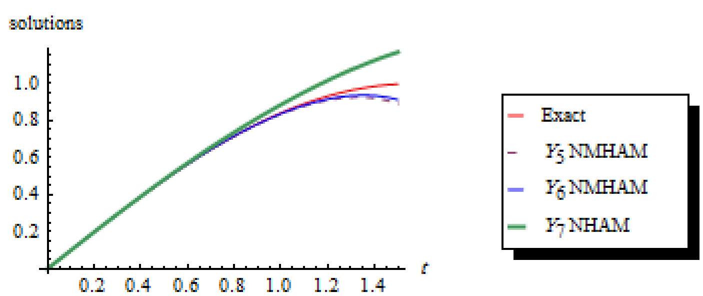


Figure(8): ℏ-curve for NMHAM approximation solutionsY_7 of problem (4.13)-(4.14) at different values of t .

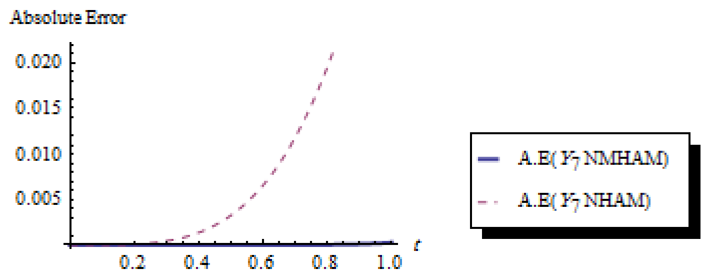
Figure(9) shows the comparison between Y_7 of NMHAM and Y_7 of NHAM at $0 \le t \le 2$, with the exact solution(4.15) which indicates that the series solution Y_7 obtained by MNHAMis more accurate from the series solution Y_7 obtained by NHAM.Figure (10) shows the comparison of Y_6 , $Y_5$of NMHAM and Y_7 of NHAM with the exact solution (4.15) at $0 \le t \le 1.5$ which indicates that the speed of convergence of NMHAM is faster and more convergence in comparison of NHAM.Figures (11) shows that the series solution Y_7 obtained by NMHAM at $0 \le t \le 1$ is more accurate from the series solutionY_7 obtained by NHAMFigures (12) shows that the series solutionY_6 obtained by NMHAM is more and faster convergence from the series solutionY_7 obtained by NHAM at $0 \le t \le 1$.



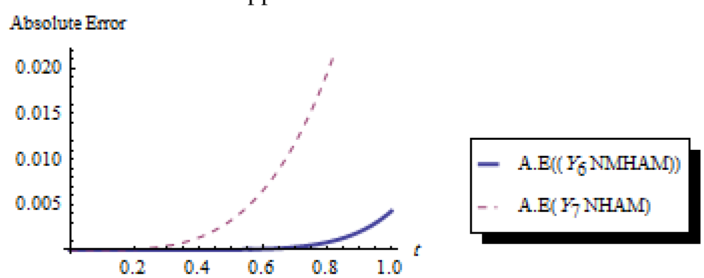
Figure(9): Comparison of the 7th order approximations of NMHAM and NHAM at $0 \le t \le 1.5$, $\hbar = -1$with the exact solution(4.15).



Figure(10):Comparison of Y_6 and Y_5 of NMHAM and Y_7 of NHAM with the exact solution (4.15) at $0 \le t \le 1.5$ $\hbar = -1$.



Figure(11): The absolute errors of the 7th order approximations of NMHAM and NHAM at $0 \leq t \leq 1$, $\hbar = -1$.



Figure(12):The absolute errors of the 6th order approximations of NMHAM and the 7th order approximations of NHAM at $0 \leq t \leq 1, \hbar = -1$.

Table (3) show the comparison of Y_7 of NMHAM ,NHAM and nHAM ,with the exact solution (4.15) at different values of of t .Table (4) shows the comparison of the absolute errors of Y_7 of NMHAM ,NHAM and nHAM at different values of of t. Tables (3) and (4) indicate that the series solution obtained by NMHAM is more accurate from the series solution obtained by NHAM and the series solution obtained by nHAM is divergent.

Table3: Comparison of the 7th order approximations of NMHAM, NHAM and nHAM with the exact solution (4.15).

t	***Exact***	***NMHAM***	***NHAM***	***nHAM***
0	0	0	0	0
0.1	0.099833	0.099833	0.099839	−350.966
0.2	0.198669	0.198669	0.198762	−701.933
0.3	0.295520	0.295520	0.295974	−1052.902
0.4	0.389418	0.389418	0.390805	−1403.874
0.5	0.479426	0.479426	0.482704	−1754.851
0.6	0.564642	0.564646	0.571229	−2105.831
0.7	0.644218	0.644230	0.656040	−2456.816
0.8	0.717356	0.717397	0.736880	−2807.808
0.9	0.783327	0.783442	0.813554	−3158.805
1	0.841471	0.841760	0.885889	−3509.807

Table4: Comparison of the absolute errors of 7th order approximations of NMHAM , NHAM and nHAM with the exact solution(4.15).

t	*A.E NMHAM*	*A.E NHAM*	*A.E nHAM*
0	0	0	0
0.1	3.275435×10^{-13}	6.012234×10^{-6}	351.066
0.2	1.671137×10^{-10}	9.274402×10^{-5}	702.132
0.3	6.386566×10^{-9}	4.535345×10^{-4}	1053.198
0.4	8.435251×10^{-8}	0.0013867	1053.198
0.5	6.217155×10^{-7}	0.003279	1755.329
0.6	3.165286×10^{-6}	0.006587	2106.396
0.7	1.247286×10^{-5}	0.011823	2457.461
0.8	4.071103×10^{-5}	0.019524	2808.526
0.9	1.149799×10^{-4}	0.030227	3159.588
1	2.894769×10^{-4}	0.044418	3510.648

4.3 Example 3 Consider non-linear Klein-Gordon equation as follows [1]

$$y_{tt} - y_{xx} + y^2 = f(x,t) \quad (4.38)$$

Subject the initial condition

$$y(x,0) = 0 \quad , \quad y_t(x,0) = 0 \quad (4.39)$$

The exact solution when $f(x,t) = 2x^2 - 2t^2 + x^4t^4$ is

$$y(x,t) = x^2t^2 \quad (4.40)$$

4.3.1. NHAM solution: To solve (4.38-4.39) by means of the NHAM, expanding the homotopy $\mu(x,t;q)$ in powers of the parameter q with $s = 2$:

$$\mu(x,t;q) = \sum_{r=0}^{\infty} f_r{}^2 (x,t) q^r = f_0{}^2(x,t)q^0 + f_1{}^2(x,t)q^1 + \cdots + f_n{}^2(x,t)q^n + \cdots, \quad (4.41)$$

Where

$$f_r{}^2(x,t) = \frac{1}{2r!}[\frac{d^{2r}}{dt^{2r}} f(x,t)]_{t=0}\, t^{2r} + \frac{1}{(2r+1)!}[\frac{d^{(2r+1)}}{dt^{(2r+1)}} f(x,t)]_{t=0}\, t^{(2r+1)} \quad (4.42)$$

such that

$$f_0{}^2(x,t) = 2x^2 \quad , f_1{}^2(x,t) = -2t^2 \quad , f_2{}^2(x,t) = t^4x^4 \quad , f_3{}^2(x,t) = 0 , ... \quad (4.43)$$

Let we choose the initial approximation

$$y_0(x,t) = 0, \quad (4.44)$$

And the linear operator

$L[\delta(x,t;q)] = \frac{\partial^2\delta(x,t;q)}{\partial t^2}$ (4.45)

with the property$L[c] = 0$, (4.46)

where$c$ is a constant of integration.

The problem (4.38-4.39) under NHAM suggests to define a nonlinear operator as

$$\mathbb{N}[\delta(x,t;q)] = \frac{\partial^2\delta(x,t;q)}{\partial t^2} - \frac{\partial^2\delta(x,t;q)}{\partial x^2} + \delta^2(x,t;q) - \mu(x,t;q), \quad (4.47)$$

According to (2.4), the zero order deformation equation with the initial approximation (4.44) and linear operator (4.45) with (4.46) will be:

$$(1-q)\mathrm{L}[\delta(x,t;q) - y_0(x,t)\,] = q\mathbb{h}\mathbb{N}[\delta(x,t;q) - \mu(x,t;q)], \quad (4.48)$$

and the rth order deformation equation as follows:

$$L[y_r(x,t) - X_r y_{r-1}(x,t)] = \mathbb{h}\mathcal{R}_r(\overrightarrow{y_{r-1}}(x,t)), \quad (4.49)$$

with the initial conditions $y_r(x,0) = 0$ and $y_{r_t}(x,0) = 0$

Where

$$\mathcal{R}_r\left(\overrightarrow{y_{r-1}}(x,t)\right) = y_{\mathrm{tt}_{\mathrm{r}-1}}(x,t) - y_{xx_{r-1}}(x,t) + \sum_{i=0}^{r-1} y_i y_{r-1-i} - f^2{}_{r-1}(x,t) \quad (4.50)$$

Now, the solution of (4.49)for $r \geq 1$ becomes

$$y_r(x,t) = X_r y_{r-1}(x,t) + \mathbb{h}L^{-1}\mathcal{R}_r\left(\overrightarrow{y_{r-1}}(x,t)\right) \quad (4.51)$$

We now successively obtain

$$y_1(x,t) = -\mathbb{h}t^2x^2$$

$$y_2(x,t) = -\mathbb{h}t^2x^2 + \frac{1}{6}\mathbb{h}((1+\mathbb{h})t^4 - 6\mathbb{h}t^2x^2$$

⋮

Then, the series solution of the NHAM is:

$$y(x,t,\mathbb{h}) \cong Y_R(x,t,\mathbb{h}) = \sum_{i=0}^{R} y_i\,(x,t,\mathbb{h}) \quad (4.52)$$

Equation (4.52) is a family of approximation solutions to the problem (4.38)-(4.39) in terms of the convergence parameters $\mathbb{h}$.

4.3.2. NMHAM solution: in order to solve (4.38))-(4.39)by the proposed approach (NMHAM) we construct the following system:

$$y_t(x,t) = v(x,t)\,,$$

$$v_t(x,t) = y_{xx} - y^2 + \mu(x,t;q) \quad (4.53)$$

with the following initial conditions

$$y_0(x,t) = 0 \,, v_0(x,t) = 0 \tag{4.54}$$

Expanding the homotopy $\mu(x,t;\mathbb{q})$ in powers of the parameter $\mathbb{q}$ with $s = 1$:

$$\mu(x,t;\mathbb{q}) = \sum_{r=0}^{\infty} f_r^{\ 1}(x,t)\mathbb{q}^r = f_0^{\ 1}(x,t)\mathbb{q}^0 + f_1^{\ 1}(x,t)\mathbb{q}^1 + \cdots + f_n^{\ 1}(x,t)\mathbb{q}^n + \cdots, \tag{4.55}$$

Where

$$f_r^{\ 1}(x,t) = \frac{1}{r!}\left[\frac{d^r}{dt^r} f(x,t)\right]_{t=0} t^r \tag{4.56}$$

such that

$$f_0^{\ 1}(x,t) = \frac{5}{4} \quad , f_1^{\ 1}(x,t) = 0 \quad , f_2^{\ 1}(x,t) = \frac{t^2}{16} \quad , f_3^{\ 1}(x,t) = 0\,, \ \ldots.$$

and the following linear operators:

$$Ly(x,t) = \frac{\partial y(x,t)}{\partial t} \,, Lv(x,t) = \frac{\partial v(x,t)}{\partial t} \tag{4.57}$$

$$Ay_{r-1}(t) = -y_{xx\,r-1}(x,t) - f^{1}{}_{r-1}(x,t)$$

$$By_{r-1}(x,t) = \sum_{i=0}^{r-1} y_i y_{r-1-i} \tag{4.58}$$

we obtain

$$y_1(x,t) = \mathbb{h}\int_0^t \big(-v_0(x,\tau)\big)d\tau$$

$$v_1(x,t) = \mathbb{h}\int_0^t (-y_{0_{xx}}(x,\tau) + y_0^{\ 2}(x,\tau) - f_0^{\ 1}(x,\tau))d\tau \tag{4.59}$$

Now, for $r \geq 2$, we get

$$y_r(x,t) = (1+\mathbb{h})y_{r-1}(x,t) + \mathbb{h}\int_0^t (-v_{r-1}(x,\tau))\, d\tau\,,$$

$$v_r(x,t) = (1+\mathbb{h})v_{r-1}(x,t) + \mathbb{h}\int_0^t (Ay_{r-1}(x,\tau) + By_{r-1}(x,\tau))\, d\tau \tag{4.60}$$

And the following results are obtained:

$$y_1(x,t) = 0, \ \ v_1(x,t) = -2\mathbb{h}tx^2$$

$$y_2(x,t) = \mathbb{h}^2t^2x^2, \ \ v_2(x,t) = -2\mathbb{h}(1+\mathbb{h})tx^2$$

$$y_3(x,t) = \mathbb{h}^2(1+\mathbb{h})t^2x^2 + \mathbb{h}(\mathbb{h}t^2x^2 + \mathbb{h}^2t^2x^2)$$

⋮

Then, the series solution of the NMHAM is:

$$y(x,t,\hbar) \cong Y_R(x,t,\hbar) = \sum_{i=0}^{R} y_i\,(x,t,\hbar) \quad (4.61)$$

Equation (4.61) is an approximation solutions for the problem (4.38)- (4.39) depending on the parameter $\hbar$.To determine the valid region of $\hbar$, the $\hbar$-curves given by the 5th-order NMHAM at different values of x and t are drawn in figure (13).

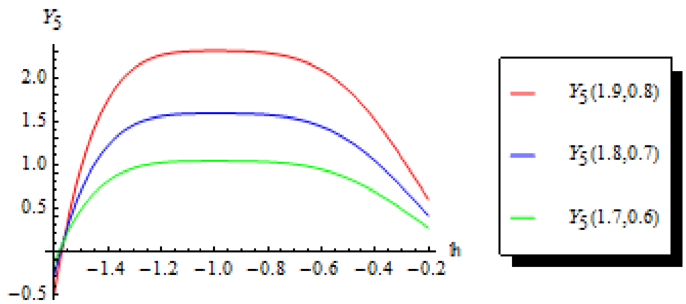
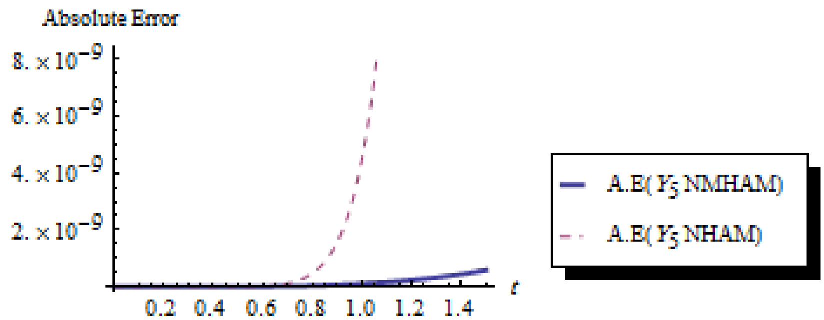
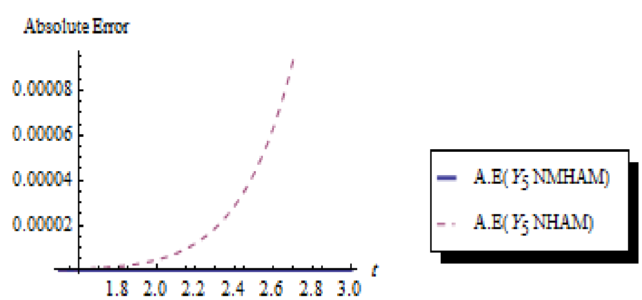


Figure (13): $\hbar$-curve for NMHAM approximation solutionsY_5 of problem (4.38)-(4.39) at different values of x and t.

Figures (14) show that the series solution Y_5 obtained by NMHAM at $0 \le t \le 1.5, x = 0.2$ is more accurate from the series solution Y_5obtained by NHAM.Figures (15) show that the series solution Y_5 obtained by NMHAM is more accurate from the series solution Y_5obtained by NHAM at larger $t(1.5 \le t \le 3, x = 0.2)$.



Figure(14):The absolute errors of $Y_5$of NMHAM and NHAM at $0 \le t \le 1.5$, $x = 0.2$, $\hbar = -0.99$.



Figure(15):The absolute errors of $Y_5$of NMHAM and NHAM at $1.5 \le t \le 3$, $x = 0.2$, $\hbar = -0.99$.

Table (5) show the comparison ofY_5 of NMHAM ,NHAM and nHAM ,with the exact solution (4.40).Table (6) shows the comparison between the absolute errors ofY_5 of NMHAM ,NHAM and nHAM. Tables (5) and (6) indicate that the series solution obtained by NMHAM is more accurate from the series solution obtained by NHAM and nHAM.

Table5: Comparison of the 5th order approximations of NMHAM , NHAM and nHAM at different values of $t$and $x$with the exact solution(4.40).

x	***t***	***exact***	***NMHAM***	***NHAM***	***nHAM***
0.2	0	0	0	0	0
	0.5	0.01	0.009999	0.010000	0.010002
	1	0.04	0.039999	0.040000	0.040339
	1.5	0.09	0.089999	0.090000	0.097930
	2	0.16	0.159999	0.160000	0.236556
	2.5	0.25	0.249999	0.250004	0.698986
0.6	0	0	0	0	0
	0.5	0.09	0.089999	0.090000	0.090078
	1	0.36	0.359999	0.360000	0.366891
	1.5	0.809999	0.809999	0.8100000	0.925110
	2	1.44	1.439999	1.440000	2.374766
	2.5	2.25	2.249999	2.250004	7.228376
1	0	0	0	0	0
	0.5	0.25	0.249999	0.249999	0.250549
	1	1	0.999999	0.999999	1.040476
	1.5	2.25	2.249999	2.249999	2.812751
	2	4	3.999999	3.999998	7.961905
	2.5	6.25	6.249999	6.249990	25.287156

Table 6:Comparison of the absolute errors of 5th order approximations of NMHAM , NHAM and nHAM with the exact solution (4.40).

x	***t***	***A.E NMHAM***	***A.E NHAM***	***A.E nHAM***
0.2	0	0	0	0
	0.5	7.291737×10^{-14}	4.391019×10^{-13}	1.949405×10^{-6}
	1	1.166678×10^{-12}	4.496417×10^{-10}	3.390476×10^{-4}
	1.5	5.906234×10^{-12}	2.592851×10^{-8}	0.007930
	2	1.866693×10^{-11}	4.604308×10^{-7}	0.076556
	2.5	4.557352×10^{-11}	4.288093×10^{-6}	0.448986
0.6	0	0	0	0
	0.5	7.294165×10^{-14}	3.717859×10^{-13}	7.754464×10^{-5}
	1	1.166733×10^{-12}	3.806401×10^{-10}	0.006891
	1.5	5.906164×10^{-12}	2.194879×10^{-8}	0.115110
	2	1.866729×10^{-11}	3.897552×10^{-7}	0.934766
	2.5	4.557421×10^{-11}	3.629856×10^{-6}	4.978376
1	0	0	0	0
	0.5	7.291389×10^{-14}	1.007444×10^{-12}	5.487351×10^{-4}
	1	1.166622×10^{-12}	1.031669×10^{-9}	0.040476

	1.5	5.907275×10^{-12}	5.949359×10^{-8}	0.562751
	2	1.866685×10^{-11}	1.056486×10^{-6}	3.961905
	2.5	4.557509×10^{-11}	9.839353×10^{-6}	19.037156

5. Conclusion

In this article, new powerful modification of homotopy analysis method (NMHAM) was proposed to create an approximate solution of nonhomogeneous nonlinear ordinary and partial differential equations. The main advantage of the NMHAM is that it requires less computational work compared with the NHAM and nHAM in finding approximate solutions for nonlinear nonhomogeneous differential equations. Illustrative examples show that the series solution obtained by NMHAM is more accurate from the series solution obtained byNHAM andnHAM. Therefore, depending on the results of this work, we can say that the NMHAM is more effective than NHAM and nHAM.

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