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On Recent Modifications of Extended Weibull Families Distributions and Its Applications

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Abstract

The paper discusses an extended form of family of Weibull distributions and some important cases of this distributions. The family of Weibull distributions and its derivate models finds many applications. The author proposes a new six parameters extended Weibull model, which generalizes the family of weibull distributions. The applicability of the new models is well justified by means of two real data sets. Several methods are used to estimate the Weibull parameters which are (MLE) and (ME). Various properties of this new family are investigated and then exploited to derive several related results, especially characterizations, in probability. As a motivation, the statistical applications of the results based on health related data are included which investigates reliability properties of a flexible extended weibull family of distributions. The findings of this work will be useful for the practitioners in various fields of theoretical and applied sciences.

Keywords: Weibull distribution, reliability, generating and quantile functions, Hazard rate function, Parameter estimation, Estimation Methods.

1. Introduction

The two-parameter Weibull is the most popular distribution for modeling lifetime data. It is most widely used in reliability engineering. The Weibull distribution in probability and statistics was first published in 1939, and it was named after Swedish Mathematician E. H. Waloddi Weibull (1887-1979). The distribution was the first to model the breaking strength of materials. An alternative approach is based on the application of the Weibull distribution to the interevent times of an earthquake sequence asithas been performed by many authors [22], [21]. The first standard parameterization of probability density function of a Weibull life distribution model with two parameters is as equation (1). The Weibull distribution is commonly used in industry to model failure data. It has probability density function (pdf) is given by the following equation:

$$f(x;\lambda,k)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^{k}},\quad x\geq 0\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In the important case of the alternative parameterizations, the shape parameter k is the same as above and the scale parameter is $b=(1/\lambda)^{k}$, the probability density function is:

$$f(x;\lambda,k)=b\,k\,x^{k-1}e^{-b\,x^{k}},\quad x\geq 0\ldots\ldots\ldots\ldots\ldots\ldots(1)'$$

The most general expression of the Weibull probability density function with three-parameter expression is given by (2):

$$f(x;\lambda,k,\mu)=\frac{k}{\lambda}\left(\frac{x-\mu}{\lambda}\right)^{k-1}e^{-\left(\frac{x-\mu}{\lambda}\right)^{k}}\ ,\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Where $k\succ 0,\lambda\succ 0,-\infty\prec\mu\prec+\infty$. The parameters k is the shape parameter, also known as the Weibull slope , λ is the scale parameter and μ is the location parameter. The special cases represent $k=1$ exponential distribution. The Rayleigh is actually a special case of the Weibull distribution with, $\mu=0$
A quite important wind speed distributions.which finds applications in . $k=2$
case and most often found in literature is for value of $k\sim 3$. The one-parameter Weibull probability density function is obtained by again setting $\mu=0$ and assuming $\lambda=1$ assumed constant value, then:

$$f(x;k)=kx^{k-1}e^{-x^{k}}\ ,\ldots\ldots\ldots\ldots\ldots\ldots(2)'$$

where the only unknown parameter is the shape parameter k . The Weibull, and related models have been used in many applications, and for solving a variety of problems from many disciplines. The plot at the left shows the Weibull distribution for various valuables of the parameters k , $\mu = 0$ and λ .

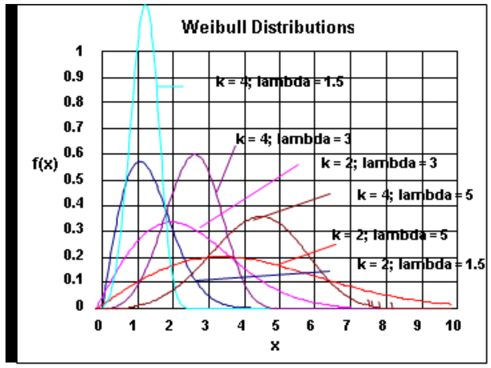

Figure 1: Graph of $f(x; \lambda, k, \mu)$ for k , $\mu = 0$ and for various value of λ

The plot shows that as the value of c increases for a given value of k the shape of the distribution gets wider. Because of this, λ is called the scale parameter; it has dimensions of velocity. The plot also shows that as k increases from 2 to 4 for a given value of λ, the maximum in the probability density function (pdf) increases. Because of this, k is called the shape parameter; it is dimensionless. Since many extensions of the Weibull distribution have been proposed to enhance its capability to fit diverse lifetime data [13].

Proposition: The W follows a Weibull distribution with cumulative distribution function $F(w; \lambda, k) = 1 - e^{-\left(\frac{w}{\lambda}\right)^k}$, where $w \geq 0, \lambda \succ 0, k \succ 0$. then X=$(W/\lambda)^k$ follows an exponential distribution .

Proof: $W \sim Weibull$ **,** $w \geq 0, \lambda \succ 0, k \succ 0$ **,** then $F(w; \lambda, k) = 1 - e^{-\left(\frac{w}{\lambda}\right)^k}$, by derivation of cumulative distribution function $f(w, \lambda, k) = \frac{k}{\lambda^k} w^{k-1} e^{-(\frac{w}{\lambda})k}$ **,** $w \geq 0, \lambda \succ 0, k \succ 0$ **.** For $X = (W/\lambda)^k$ **,** $W \sim Weibull(\lambda, k)$**,** by transformation $W = \lambda(x)^{\frac{1}{k}}$ **,** $|J| = \left| \frac{\lambda}{k} x^{\frac{1}{k}-1} \right|$**,** by ordered the formula $f_X(x) = \frac{k}{\lambda^k} (\lambda x^{\frac{1}{k}})^{k-1} e^{-(\frac{\lambda x^{\frac{1}{k}}}{\lambda})^k} \frac{\lambda}{k} x^{\frac{1}{k}-1} = e^{-x}$, Then $X \sim \exp(1)$.

The Four-parameter Weibull distribution with cumulative distribution function was proposed by kies (1958) and its survival function is given by [7]

$$F(x; \lambda, k, a, b) = 1 - \exp\left\{ -\lambda \left[\frac{(x-a)}{(b-x)} \right]^k \right\};.....................(3)$$

where $0 \prec a \leq x \leq b \prec +\infty,\ \lambda \succ 0, k \succ 0$.

A four parameter was proposed by Jeong J. H which is given as [6]

$$F(x; \alpha, \delta, \lambda, k) = 1 - \exp\left[-\frac{\lambda^{1-k}}{k} \left\{ \left(\frac{x}{\delta} \right)^{\alpha} + \lambda \right\}^{k} - \frac{\lambda}{k} \right],$$

where $x \succ 0, \lambda \succ 0, \alpha \succ 0, \beta \succ 0, k \in IR$.

The Five-parameter Weibull distribution with cumulative distribution function was proposed by Phani (1987), extension of this model are due to kies (1958) and its survival function is given by Phani. [16]:

$$F(x; \lambda, k, a, b) = 1 - \exp\left\{ -\lambda \frac{(x-a)^{\alpha_1}}{(b-x)^{\alpha_2}} \right\};\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where $0 \prec a \leq x \prec b \prec +\infty,\ \lambda \succ 0, \alpha_1, \alpha_2 \succ 0$.

2. Generalized Extended Weibull Families Distributions

During last 40 years, various extension of the extended Weibull families and their applications have been attempted, but one the important model was by Slymen and Lachenbruch (1984). They proposed a modified Weibull model with survival formula named Slymen-Lachenbruch Modified Weibull Distribution**.** [18].

In 1993, Mudholkar and Srivastava proposed a modification to the standard Weibull model through the introduction of an additional parameters named in general by Exponentiated Weibull Distribution [11]. The distribution has been studied extensively by Mudholkar and Hutson (1996)[9]. The so called generalized Weibull model was derived by Mudholkar and Kollia (1994) named by Generalized Weibull Family [10] and from the basic two-parameter Weibull distribution by appending an additional parameter [12].

It has been pointed by Gurvich et al. (1997) that several of the modifications of the Weibull distribution named Generalized Weibull Distribution of Gurvich et al. [5] .

Another generalization of Weibull was introduced by Xie et al. (2002) named Weibull Extension Model [24] and a detailed statistical analysis was given in Tang et al. (2003)[19] .

The beta-Weibull was first proposed by Famoye et al. (2005) by coupling the beta density and the Weibull distribution function such that the distribution function of the new distribution named Beta-Weibull Distribution [4] .

Nadarajah and Kotz (2005) proposed a generalization of the standard Weibull model with four parameters named Generalized Weibull-Gompertz Distribution.[14]**.**

Cooray (2006) has constructed a generalized Weibull family called the odd Weibull family based on the idea of evaluating the distribution named the odd weibull distribution .[3].

Nikulin and Haghighi (2006) introduced a family of distributions with survival function named Generalized Power Weibull Family[15].

Bebbington et al. (2007a) introduced a distribution which is quite simple and yet very flexible to model reliability data, this distribution named flexible weibull distribution **.**[2].

The Weibull-geometric distribution, abbreviated as WG, was proposed and studied by Barreto-Souza et al. (2010) named the weibull-geometric distribution**.**[1].

Silva et al. (2010) proposed a new distribution called the beta modified Weibull based on the new construction named the Beta Modified Weibull Distribution**.** [17] .

3. New Extended Model Weibull Family Distributions

In the last few years, new classes of distributions were proposed by extending the Weibull distribution. The new model is named as new extended Weibull distribution. A review of some of these models includes the exponentiated Weibull distribution. A new way of introducing a model function $\varphi(x)$ to expand a family of extension of the exponential and Weibull. The one extended model weibull family distribution with cumulative distribution function was proposed with survival function is given by:

$$F(x;\alpha,\delta,\lambda,k,\eta)=1-\exp\left\{-\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\},$$ where

$x\succ 0,\lambda\succ 0,\alpha\succ 0,\beta\succ 0,r,k\in IR$

This case represent with general formula $F(x;\varphi)=1-\exp\{-\varphi(x)\}$, where $\varphi(x)=\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]$. A new extended weibull distribution is proposed has cumulative distribution function is given by the following equation:

$$F(x;\varphi,\rho,p)=1-\exp\left\{-\rho\left[\frac{\varphi(x)}{1-\varphi(x)}\right]^{p}\right\},\ \text{where }\ \varphi(x)\succ 0,\rho,p\in IR\ .$$

4. Model and Parameter Estimation of Five-parameters Weibull

This section introduces the assumed model for product life and also describes the test method.

4.1. The Model Assumptions

The one of new extended model weibull model named five-parameters weibull distributions has density function of the form:

$$f(x;\alpha,\delta,\lambda,k,\eta)=-\frac{\alpha}{\delta}\left(\frac{x-\eta}{\delta}\right)^{\alpha-1}\exp\left\{-\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\};$$

$$x\succ 0,\eta,\lambda\succ 0,\alpha\succ 0,\beta\succ 0,k\in IR$$

where k , δ , parameter slope as theknown λ, also sthe shape parameter are α is the scale parameter and μ is the location parameter of the distribution. The cumulative distribution function is:

$$F(x;\alpha,\delta,\lambda,k,\eta)=1-\exp\left\{-\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\},\ x\succ 0,\eta,\lambda\succ 0,\alpha\succ 0,\beta\succ 0,k\in IR.$$

4.2. Estimation of Parameters with Maximum Likelihood Estimation Method

The maximum likelihood is one of the most important and widely used methods in statistics, and is one of popular methods for computing shape and scale factors because its ability to deal with large number of data. The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize the of the sample data. Since the lifetimes $x_1,x_2,...,x_n$ of n items are independent and identically distributed random variable with simple $X_1,X_2,...,X_n$, then their likelihood function is given by;

$$f(x_1,x_2,...,x_n;\alpha,\delta,\lambda,k,\eta)=f(x_1;\alpha,\delta,\lambda,k,\eta)f(x_2;\alpha,\delta,\lambda,k,\eta)...f(x_n;\alpha,\delta,\lambda,k,\eta)$$

$$=\left(-\frac{\alpha}{\delta}\right)^{n}\prod_{i=1}^{n}\left(\frac{x_i-\eta}{\delta}\right)^{\alpha-1}\exp\left\{-\sum_{i=1}^{n}\left[\left(\frac{x_i-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\};$$

The method of maximum likelihood estimates λ,α,β,k by finding a value of θ that maximizes $\ln(f(x_1,x_2,...,x_n;\alpha,\delta,\lambda,k,\eta))$. In the maximum likelihood method, $\alpha,\delta,\lambda,k,\eta$ are obtained by solving maximize the likelihood equation;

$$\ln\left(f\left(x_1,x_2,\ldots,x_n;\alpha,\delta,\lambda,k,\eta\right)\right)=\ln\left(\left(-\frac{\alpha}{\delta}\right)^n\prod_{i=1}^{n}\left(\frac{x_i-\eta}{\hat{\delta}}\right)^{\alpha-1}\exp\left\{-\sum_{i=1}^{n}\left[\left(\frac{x_i-\eta}{\delta}\right)^{\alpha}+\lambda^k\right]\right\}\right)$$

$$=n\ln\left(-\frac{\alpha}{\delta}\right)+\sum_{i=1}^{n}\left(\frac{x_i-\eta}{\hat{\delta}}\right)^{\alpha-1}-\sum_{i=1}^{n}\left[\left(\frac{x_i-\eta}{\delta}\right)^{\alpha}+\lambda^k\right]$$

An MLE estimate is the same regardless of whether we maximize the likelihood or the log-likelihood function, since log is a monotonically increasing function. The maximum likelihood equivalently differentiating partially to parameters $\alpha,\delta,\lambda,k,\eta$ in turn and equating to zero, we obtain the estimating equations. One ends up with a nonlinear equation in $\hat{\lambda},\hat{\alpha},\hat{\beta},\hat{k},\hat{\eta}$ this cannot be solved in closed form, there are basically two methods and they are called root finding methods, they are based on the calculus theorem that says that when a function is continuous, and changes signs on an interval, it is zero on that interval. For this particular problem there already coded in MATLAB, a MLE method called weibullfit, that also provides a confidence interval. For general optimization, the function in Matlab is fmin for multiple variables, and fmins you could also look at how to use optimize in Splus. Maximizing log likelihood, with and without constraints, can be an unsolvable problem in closed form, then we have to use iterative procedures.

4.2.2. Estimation of Parameters with Moment Estimation Method

In this method (ME), λ,α,β,k are obtained by solving the following equations composed by $E\left(X^k\right)=M_k$, where $E\left(X^k\right)$ is the kth (theoretical) moment of the distribution and M_k is the kth sample moment, for k = 1, 2, ...

$$M_k=\frac{1}{n}\sum_{i=1}^{n}X_i^k=\int_{-\infty}^{x}t^k f\left(t;\alpha,\delta,\lambda,k,1\right)\eta dt=E\left(X^k\right),k=1,2,3,4,5.$$

The solutions to the above equations denoted by $\hat{\lambda},\hat{\alpha},\hat{\beta},\hat{k},\hat{\eta}$ yields the moment estimators of $\alpha,\delta,\lambda,k,\eta$.[8].

4.2.3 Failure Time Models and The Extended Weibull Families of Distributions

The extended weibull families of distributions has the cumulative distribution function $F(x;\alpha,\delta,\lambda,k,\eta)=1-\exp\left\{-\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\}$, and so the survivor function $S(x;\alpha,\delta,\lambda,k,\eta)$ is $S(x;\alpha,\delta,\lambda,k,\eta)=\exp\left\{-\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\}$. Taking the derivative of the cumulative distribution function yields the density function,;

$$f(x;\alpha,\delta,\lambda,k,\eta)=\frac{\alpha}{\delta}\left(\frac{x-\eta}{\delta}\right)^{\alpha-1}\exp\left\{-\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\};$$

and so the hazard rate is:

$$h(x;\alpha,\delta,\lambda,k,\eta)=\frac{f(x;\alpha,\delta,\lambda,k,\eta)}{S(x;\alpha,\delta,\lambda,k,\eta)}=\frac{f(x;\alpha,\delta,\lambda,k,1)\eta}{F(x;\alpha,\delta,\lambda,k,\eta)}=\frac{\alpha}{\delta}\left(\frac{x-\eta}{\delta}\right)^{\alpha-1};$$

Thus the hazard rate is a power function of the duration with parameters with derivative given by:

$$h'(x;\alpha,\delta,\lambda,k,\eta)=-\frac{\alpha(\alpha-1)}{\delta^{2}}\left(\frac{x-\eta}{\delta}\right)^{\alpha-2}$$

The form of the duration dependence is based on the parameters $x=\eta$ If . η,α,δ then the hazard rate equal zero, increases with survival time if $\alpha(\alpha-1)\succ 0$ then there is positive duration dependence (the hazard takes a non-monotonic shape increasing from zero to a maximum, and decreasing thereafter). Is a decreasing function with survival time if $\alpha(\alpha-1)\prec 0$, there is negative duration dependence (the hazard is monotonic decreasing from infinity). If $\alpha=0$ or $\alpha=1$, there is no duration dependence (which is the exponential case). So, depending on the value of this parameter, three different patterns of behavior for the hazard rate can be explained.

5.Generating Random Times from a Five-parameters Weibull Distribution

The Goodness-of-Fit of generated from a uniform random number generator of five-parameters weibull random variable was proposed with survival function is given by the following equation;

$$F(x;\alpha,\delta,\lambda,k,\eta)=1-\exp\left\{-\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\},$$

$$F_X(x;\alpha,\delta,\lambda,k,\eta)=U\xrightarrow{?}F(x;\alpha,\delta,\lambda,k,\eta)=1-\exp\left\{-\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\}$$

Let $u=1-\exp\left\{-\left[\left(\frac{x-\eta}{\delta}\right)^{\alpha}+\lambda^{k}\right]\right\}$ then $\left(\frac{x-\eta}{\delta}\right)^{\alpha}=-\lambda^{k}-\ln(1-u)$; if $U\sim\text{uniform}[0,1]$ then $1-U\sim\text{uniform}[0,1]$ we can firstly generate $U\sim\text{uniform}(0.1)$. Let then $X\sim Weibull(\alpha,\delta,\lambda,k,\eta)$, the inverse function of cumulative distribution function is:

$$F^{-1}_{(\alpha,\delta,\lambda,k,\eta)}(u)=\eta+\delta\sqrt[\alpha]{-\lambda^{k}+\ln\left(\frac{1}{(1-u)}\right)}$$

The five-parameters weibull conditional reliability function is given by:

$$R_{(\alpha,\delta,\lambda,k,\eta)}(t/T)=\frac{R_{(\alpha,\delta,\lambda,k,\eta)}(T+t)}{R_{(\alpha,\delta,\lambda,k,\eta)}(T)}=\exp\left\{-\left[\left(\frac{(T+t)-\eta}{\delta}\right)^{\alpha}\left(\frac{T-\eta}{\delta}\right)^{\alpha}\right]\right\},$$

The random time would be the solution for t wherea $,R_{(\alpha,\delta,\lambda,k,\eta)}(t/T)=U_R[0.1]$ uniform random number from 0 to 1, $U_R[0.1]$, is first obtained. The random time from five-parameters weibull distribution then you can generate variates t_i from the translated Weibull with the following:

$$x_i\equiv F^{-1}_{(\alpha,\delta,\lambda,k,\eta)}(u_i)=\eta+\delta\sqrt[\alpha]{-\lambda^{k}+\ln\left(\frac{1}{(1-u_i)}\right)},$$

where the u_i are sequence uniform random numbers.

6.Applications

The recent applications of weibull families distributions is useful for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rainfall, sea, electronic , manufacturing products, aerospace equipments, navigation and transportation control. The probability of earthquake occurrence is discussed on the assumption that strain of the earth's crust increases linearly with time and the earthquake occurrence time is governed by a Weibull distribution.

The theories and tools of reliability engineering is applied into widespread fields such as electronic and manufacturing products, aerospace equipments, earthquake and volcano forecasting, communication systems, navigation and transportation control, medical treatment to the survival analysis of human being or biological species and so on (Weibull) [23].

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