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Buoyancy Driven Convection in a Liquid Layer with Insulating Permeable Boundaries

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ABSTRACT--- ***In this paper, we investigate the onset of buoyancy driven thermal convection in a horizontal layer of fluid with thermally insulating permeable boundaries, using the classical linear stability analysis. It is proved that the principle of exchange of stabilities is valid. The eigenvalue problem is solved by using the Galerkin method. Results are obtained and discussed for a wide range of values of the boundary parameters characterizing the permeable nature of boundaries. Attention is focused on a situation where the value of the critical Rayleigh number is less than that for the case when one of the boundaries is rigid while the other one is free and the convection is not maintained in general. In the case, when permeability parameter of either one of the two boundaries varies inversely to that of the other, we discover that the critical Rayleigh number decreases and goes through a lowest minimum at a certain value of the permeability parameter and this situation pertains when the critical wave number is zero. In addition, existing results for various combinations of the boundary conditions namely, when both the bounding surfaces are either dynamically free or rigid and when either one of them is dynamically free while the other one is rigid, are obtained as limiting cases of the boundary parameters.***

Keywords--- Buoyancy; Convection; Insulating; Linear stability; Permeable; Slip velocity

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1. INTRODUCTION

Convective phenomenon is very common in nature and holds a key role in various scientific and engineering fields and play an important part as many physical processes such as the weather, the Earth's mantle, solar flares and oceanic currents to name a few. Stimulated by Bénard ([1], [2]) experiments, the mathematical foundation of thermal instability has been laid down by Rayleigh [3] who explained the phenomenon in terms of buoyancy. The buoyancy driven thermal convection in a horizontal layer of fluid heated underside is also known as Rayleigh Bénard convection. Rayleigh's theory was further generalized and extended mainly in the nature of refinements of the boundary conditions and methods of solutions by many authors and are discussed in the monograph by Chandrasekhar [4]. The onset of convection in a horizontal layer of fluid heated from below with thermally conducting permeable boundaries was first discussed by Nield [5]. However, Nield's analysis has the limitation as it gives no insight into the effect of the permeable boundaries which may not be identically same and that has been addressed by Gupta et al. [6].

In this paper, we investigate the onset of convection in a horizontal layer of fluid with thermally insulating case of permeable boundaries which is more relevant and physically significant. We establish mathematically that the principle of exchange of stabilities is valid for the present problem. The single term Galerkin method is used to solve the eigenvalue problem. We focus our attention on a situation where the critical Rayleigh number is less than that for the case when either one of the boundaries is rigid while the other one is free, and the convection is not maintained in general. We discover that when the permeability parameter of either one of the two boundaries varies inversely to that of the other one, value of the critical `Rayleigh number decreases and goes through a lowest minimum at a certain value of the permeability parameter, and this situation pertains to the zero critical wave number (single cell formation) on the onset of convection. In addition, we found that existing results are the limiting cases of the permeability parameters namely, when both the bounding surfaces are either dynamically free or rigid, and either one of them is dynamically free while the other one is rigid.

2. THE EIGENVALUE PROBLEM IN NON-DIMENSIONAL FORM

We consider an infinite horizontal layer of viscous Boussinesq fluid of uniform thickness d heated from below, whose both boundary surfaces are thermally insulating and permeable on which the boundary conditions of the type proposed by Beavers and Joseph [7] are applicable. We choose a Cartesian coordinate system with x and y axes in the plane of the lower boundary and positive direction of the z axis along the vertically upward direction so that the fluid layer is confined between the planes at $z=0$ and $z=d$ (Fig. 1). A uniform temperature gradient is maintained across the layer by maintaining the lower boundary surface at a uniform temperature T_0 and the upper one at T_1. We wish to examine the hydrodynamic stability of the system using linear stability theory under the force field of gravity.

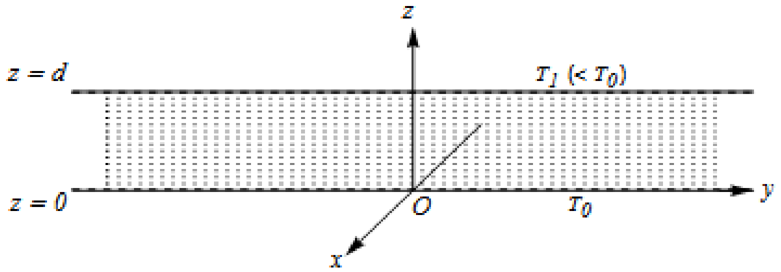

Fig. 1. Schematic representation of a fluid layer heated from below.

Following the usual procedure for obtaining the linearized perturbation equations (Chandrasekhar [4]), the non-dimensional form of the governing equations are given as

$$(D^2-a^2)(D^2-a^2-p)w = Ra^2\theta\,, \tag{1}$$

$$(D^2-a^2-pP_r)\theta = -w\,. \tag{2}$$

where w is the z-component of the perturbation velocity, θ is the temperature perturbation, a is the horizontal wave number, $P_r = \nu/\kappa$ is the Prandtl number, $R = g\alpha\beta d^4/\kappa\nu$ is the Rayleigh number, α is the volume coefficient of thermal expansion, $\beta = (T_0 - T_1)/d$ is the maintained temperature gradient, g is the gravitational acceleration, ν is the kinematic viscosity, κ is the thermal diffusivity, $p = p_r + ip_i$ represents the growth rate of perturbations (a complex constant in general), as p_r and p_i are real constants, and $D = d/dz$. We have chosen d, d^2/ν, ν/d and $\beta d\nu/\kappa$ as the units of length, time, velocity and temperature respectively.

Since both the lower and upper boundary planes are fixed and thermally insulating, the associated boundary conditions are:

$$w=0 \text{ and } D\theta = 0 \text{ at } z=0 \text{ and } z=1\,. \tag{3}$$

Further, Beavers and Joseph [7] proposed that at a permeable boundary the normal derivative of the tangential velocity is directly proportional to that velocity and if the normal is taken into the fluid then the constant of proportionality is positive. As described by Gupta et al. [6], the appropriate boundary conditions at the lower permeable boundary and the upper permeable boundary are respectively given by

$$D^2w(0) - K_0Dw(0) = 0\,, \text{ at } z=0\,, \tag{4}$$

and

$$D^2w(1) + K_1Dw(1) = 0\,, \text{ at } z=1\,. \tag{5}$$

where K_0 and K_1 are non-negative dimensionless parameters, characterizing the permeable nature of the lower and upper boundary respectively.

Equations (1) and (2) together with boundary conditions (3)-(5) pose a double eigenvalue problem for p, for prescribed values of a, P_r, R, K_0 and K_1. The given normal mode is stable, neutral or unstable according as the real part p_r of p is negative, zero or positive respectively. Further, the marginal state of the system is defined by $p_r = 0$, and if $p_r = 0$ implies that $p_i = 0$ for every wave number a then the ensuing thermal convection is neutral and the 'principle of exchange of stability' is valid. Otherwise, we will have over-stability at least when instability sets in as a certain mode.

3. CHARACTERIZATION OF THE MARIGINAL STATE

The technique of Pellew and Southwell [8] for characterization of the marginal state is applicable to equations (1)-(2) and boundary conditions **(**3)-(5) with the result proved in the following theorem:

Theorem 1. If $R>0$, a necessary condition for the existence of nontrivial solutions for w and θ satisfying equations (1)-(2) and boundary conditions (3)-(5) is that $p_i=0$.

Proof. Multiplying both sides of equation (1) by w^* (the complex conjugate of w) and integrating from $z=0$ to 1, and substituting in this equation for $\int_0^1 w^*\theta\, dz$ using equation (2), we have

$$\langle\ w^*(D^2-a^2)(D^2-a^2-p)w\ \rangle=-Ra^2\langle\ \theta(D^2-a^2-p^*P_r)\theta^*\ \rangle, \tag{6}$$

where angular bracket denotes $\langle....\rangle$ the integration with respect to z from 0 to 1. Integrating each term of equation (6) by parts, for a suitable number of times and making use of boundary conditions (3)-(5), we have

$$K_1\left|Dw(1)\right|^2+K_0\left|Dw(0)\right|^2+\langle\ \left|D^2w\right|^2+(2a^2+p)\left|Dw\right|^2+a^2(a^2+p)\left|w\right|^2\ \rangle=Ra^2\langle\ \left|D\theta\right|^2+(a^2+p^*P_r)\left|\theta\right|^2\ \rangle, \tag{7}$$

Comparing the imaginary parts of equation (7), we get

$$p_i\{\langle\left|Dw\right|^2+a^2\left|w\right|^2+Ra^2P_r\left|\theta\right|^2\rangle\}=0, \tag{8}$$

For $R>0$, from equation (8), it follows that $p_i=0$.

This proves that the principle of exchange of stabilities is valid for the problem under consideration. Hence the marginal state is stationary and it is characterized by $p=0$. When the marginal state is stationary, the governing equations (1)-(2) become

$$(D^2-a^2)^2w=Ra^2\theta\,, \tag{9}$$

$$(D^2-a^2)\theta=-w\,. \tag{10}$$

Equations (9)-(10) together with boundary conditions (3)-(5) can now be treated as eigenvalue problem of order six in R for prescribed values of a, K_0 and K_1.

4. SOLUTION OF THE PROBLEM

The single term Galerkin method as described by Finlayson [9] is convenient for solving the present problem. Accordingly, the unknown variables w and θ are written as

$$w=Aw_1 \quad\text{and}\quad \theta=B\theta_1 \tag{11}$$

where A, B are constants, w_1 and θ_1 are the trial functions which are chosen suitably satisfying the boundary conditions (3)-(5).
Multiplying equation (9) by w and equation (10) by θ, integrating each term of the equations with respect to z from 0 to 1 using the boundary conditions (3)-(5). Substituting for w and θ from equation (11) and we obtain the following system of linear homogeneous algebraic equations:

$$[K_1(Dw_1(1))^2+K_0(Dw_1(0))^2+\langle(D^2w_1)^2+2a^2(Dw_1)^2+a^4(w_1)^2\rangle]A-Ra^2\langle w_1\,\theta_1\rangle B=0\,, \tag{12}$$

$$\langle w_1\,\theta_1\rangle A-\langle(D\theta_1)^2+a^2(\theta_1)^2\rangle B=0\,. \tag{13}$$

The system of equations given by (12)-(13) will have a non-trivial solution if and only if

$$R = \frac{1}{a^2 \langle w_1 \theta_1 \rangle^2} [K_1 (Dw_1(1))^2 + K_0 (Dw_1(0))^2 + \langle (D^2 w_1)^2 + 2a^2 (Dw_1)^2 + a^4 (w_1)^2 \rangle] \times [\langle (D\theta_1)^2 + a^2 (\theta_1)^2 \rangle]. \quad (14)$$

where the angular bracket $\langle ... \rangle$ denotes integration with respect to z from 0 to 1. We select the trial functions satisfying the boundary conditions (3)-(5) as

$$w_1 = z^4 - 2\frac{K_0 K_1 + 5K_0 + 3K_1 + 12}{K_0 K_1 + 4(K_0 + K_1) + 12} z^3 + \frac{K_0 (K_1 + 6)}{K_0 K_1 + 4(K_0 + K_1) + 12} z^2 + 2\frac{K_1 + 6}{K_0 K_1 + 4(K_0 + K_1) + 12} z, \quad (15)$$

$$\theta_1 = 1. \quad (16)$$

Substitution of trial functions given by (15)-(16) into the equation (14) yields R in terms of a, K_0 and K_1 given by

$$\begin{aligned} R = &\frac{10}{7\{K_0(K_1+9)+9(K_1+8)\}^2} \times \\ &[504\{K_0(K_1+4)+4(K_1+3)\}\{K_0(K_1+9)+9(K_1+8)\} \\ &+24a^2[72\{K_1(K_1+13)+51\}+3K_0\{5K_1(K_1+14)+312\}+K_0{}^2\{K_1(K_1+15)+72\}] \\ &+a^4[76K_1(K_1+15)+K_0\{17K_1(K_1+16)+1140\}+K_0{}^2\{K_1(K_1+17)+76\}+4464]]. \end{aligned} \quad (17)$$

For given values of K_0 and K_1, equation (17) gives the Rayleigh number R as a function of wave number a. The minimum of R is the critical Rayleigh number R_c and the value of a at which R attains minimum is the critical wave number a_c.

5. RESULTS AND DISCUSSION

A close observation of the expression for R given by equation (17), shows that R attains its minimum when $a = 0$ for any fixed values of the pair (K_0, K_1). We put $a = 0$ on the right hand side of the expression for R in equation (17) and obtain its minimum R_c as given by

$$R_c = 720 \left[\frac{K_0 K_1 + 4(K_0 + K_1) + 12}{K_0 K_1 + 9(K_0 + K_1) + 72} \right]. \quad (18)$$

Following results are obtained on using the equation (18) for various limiting cases of boundary parameters K_0 and K_1.

Case 1**.** It is easily seen from equations (9)-(10) and boundary conditions (3)-(5) that for the limiting case when $K_0 \to 0$ and $K_1 \to 0$, we have governing equations for the Rayleigh Bénard problem with both boundaries as dynamically free and thermally insulating. In this case, from equation (18), we find that

$$R_c = 720 \times \frac{12}{72} = 120. \quad (19)$$

This is in fact the known exact value for R_c obtained by Nield [5].

Case 2. For the case when $K_0 \to \infty$ and $K_1 \to \infty$, we find that equations (9)-(10) and boundary conditions (3)-(5) coincide with the governing equations for the Rayleigh Bénard problem with both boundaries as rigid and thermally insulating. In this case, from equation (18), we find that

$$R_c = 720 \times \frac{1}{1} = 720. \quad (20)$$

This is in fact the known exact value for R_c obtained by Sparrow et al. [10].

Case 3. For the case when either $K_0 \to 0$ and $K_1 \to \infty$, or $K_0 \to \infty$ and $K_1 \to 0$, we find that equations (9)-(10) and boundary conditions (3)-(5) coincide with the governing equations for the Rayleigh Bénard problem with either one of the boundaries is dynamically free while the other is rigid and each boundary being thermally insulating. In this case, from equation (18), we find that

$$R_c = 720 \times \frac{4}{9} = 320. \quad (21)$$

This is in fact the known exact value for R_c obtained by Sparrow et al. [10].

Consequently, limiting cases of the parameters K_0 and K_1, give rise to the various combinations of boundary conditions namely, when both the boundaries are dynamically free or when both the boundaries are rigid, and when the lower boundary is free and upper one is rigid or when the lower boundary is rigid and upper one is free.

In Table 1 we have listed the numerical values of R_c for various fixed values of the pair (K_0, K_1) computed on using the equation (18). Table 1 shows that as K_1 increases (for fixed K_0) the value of R_c increases, and when K_0 increases (for fixed K_1) again the value of R_c increases.

Table 1. Values of R_c for various fixed values of K_0 and K_1.

K_0	K_1	R_c
10^{-6}	10^{-6}	120.00
10^{-6}	10^{-1}	122.47
10^{-6}	1	142.22
10^{-6}	10	231.11
10^{-6}	10^{6}	320.00
10^{-1}	10^{-6}	122.47
10^{-1}	10^{-1}	124.96
10^{-1}	1	144.88
10^{-1}	10	234.58
10^{-1}	10^{6}	324.39
1	10^{-6}	142.22
1	10^{-1}	144.88
1	1	166.15
1	10	262.54
1	10^{6}	360.00
10	10^{-6}	231.11
10	10^{-1}	234.58
10	1	262.54
10	10	392.73
10	10^{6}	530.52
10^{6}	10^{-6}	320.00
10^{6}	10^{-1}	324.39
10^{6}	1	360.00
10^{6}	10	530.52
10^{6}	10^{6}	720.00

Fig. 2 illustrates the variation of R_c with both K_0 and K_1. Both table 1 and Fig.2 show that increase in value of either of the two boundary parameters for fixed value of the other one has stabilizing effect on the onset of convection.

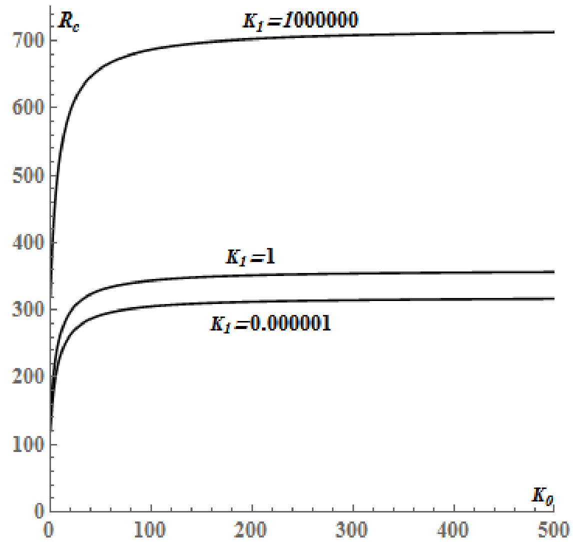

Fig. 2. Variation of R_c with T_0 when $K_0 \to 0$, $K_1 = 1$ and $K_1 \to \infty$.

In addition, we consider a new theoretical case of possible practical interest which has not been discussed in literature so far despite its importance in problems related to science, engineering and technological fields. In this case, we consider that value of either of the two parameters characterizing the permeability varies inversely to that of the other, and we let $K_1 = K_0^{-1}$. Equation (18) then yields

$$R_c = 720\left[\frac{4K_0^{\,2} + 13K_0 + 4}{9K_0^{\,2} + 73K_0 + 9}\right]. \tag{22}$$

In Table 2, we have listed numerical values of R_c for various values of K_0. Table 2 shows that as K_0 increases from 0 to ∞, R_c decrease from 320 to 166.15, attains the lowest minimum at $K_0 = 1$ and then increases to 320. A similar variation of R_c with K_0 was found by Gupta et al. [6] for the conducting case of boundary conditions, using the Chandrasekhar [4] method. They found that the lowest minimum in the conducting case was $R_c = 761.22$ attained at $K_0 = 1$. Compared with the insulating boundary conditions, the constant temperature (conducting) condition is more restricting, so that in the latter case there is a greater potential for reduction in the eigenvalue when permeable boundary parameters are varied inversely.

Table 2**.** Values of R_c for various values of K_0 when $K_1 = K_0^{-1}$.

K_0	R_c
10^{-6}	320.00
10^{-1}	234.58
0.9	166.32
1.0	166.15
1.1	166.29
10	234.58
10^{6}	320.00

Fig. 3 is plotted using the relation (22) and illustrates the variation of R_c with K_0 (when $K_1 = K_0^{-1}$). It clearly shows that increasing values of K_0 from 0 to 1 has the destabilizing effect on the onset of convection, and that the system becomes most unstable when $K_0 = 1$.

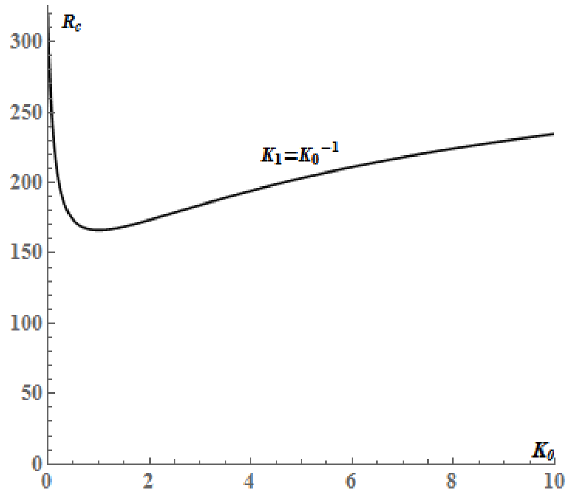

Fig. 3. Variation of R_c with K_0 when $K_1 = K_0^{-1}$.

6. CONCLUSION

The linear stability analysis of the Rayleigh Bénard convection problem with insulating permeable boundaries has been studied theoretically and the following results are obtained.

1. The principle of exchange of stabilities is valid for the present problem.

2. The limiting cases of the boundary parameters K_0 and K_1, give rise to the various particular cases namely, when both the boundaries are dynamically free ($K_0 \to 0$, $K_1 \to 0$) or when the lower boundary is free and upper boundary is rigid ($K_0 \to 0$, $K_1 \to \infty$) or the lower boundary is rigid and the upper boundary is free ($K_0 \to \infty$, $K_1 \to 0$) or when both boundary rigid ($K_0 \to \infty$, $K_1 \to \infty$). Table 1 shows that as K_1 increases (for fixed K_0) the value of R_c increases, and when K_0 increases (for fixed K_1) again the value of R_c increases. Thus, for fixed value of any one of the two permeability parameters, increasing values of the other parameter has stabilizing effect on the onset of convection.
3. For the case when $K_1 = K_0^{-1}$, increasing values of K_0 from 0 to 1 has the destabilizing effect on the onset of convection and the system becomes most unstable when $K_0 = 1$.

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