

Some Results on 2-fuzzy n-n Hilbert Space and 2-fuzzy n-n Quasi Inner Product Space

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ABSTRACT---- *The purpose of this paper is to introduce the notion of 2-fuzzy n-n Hilbert space ,2-fuzzy quasi n-n-inner product space and α - quasi n-n-norms.Also some standard results are proved.*

Keywords---- 2-fuzzy quasi n-n- inner product space, α quasi n-n-norms

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1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [22] in 1965 which began a new revolutionary field in mathematics. The theory of 2-norm on a linear space was given by Gahler [9]. In 1984 Katsaras [10] gave the notion of fuzzy norm on a linear space. Several different definitions of fuzzy normed spaces were given by Cheng and Mordeson [2], Bag and Samanta [1]. R.M.Somasundaram and Thangaraj Beaula [16] defined the notion of 2-fuzzy 2-normed linear space $(F(X),N)$, further some standard results were established by Thangaraj Beaula and Angeline Gifita by defining the 2-fuzzy normed linear space in [18] Choonkil Park and Cihangir Alaca [4] defined the concept of 2-fuzzy n-normed linear space. The concept of 2-inner product space was introduced by C.R.Diminnie, S.Gahler and A.White[5]. Further various authors gave definitions of fuzzy inner product space in [6,13,14] and fuzzy normed linear space[7,8,9,12,15,16]. Vijayabalaji and Thillaigovindan introduced fuzzy n-inner product space in [18] as a generalization of the concept of n-inner product space given by Y.J.Cho, M.Matic and J.Pecaric in [3]. Further, Vijayabalaji and Thillaigovindan introduced the notion of Quasi α -n-normed linear space and the concept of ascending family of quasi α -n norms in [18].Thangaraj Beaula and Angeline Gifita introduced the notion 2 fuzzy inner product space in [20] and introduced the notion of orthogonality in 2-fuzzy inner product space and proved some standard results in [21]. In this paper the concept of 2-fuzzy quasi n-n-normed inner product space is introduced and using this α -quasi n-n norms is established and some certain standard results using orthogonality are proved.

2. PRELIMINARIES

Before proceeding further, in this section let us recall some familiar concepts which will be needed in the sequel.

Definition 2.1 ([20])

A fuzzy set in X is a map from X to $[0,1]$, it is an element of $[0,1]^X$

Definition 2.2 ([17])

Let X be a nonempty and $F(X)$ be the set of all fuzzy sets in X. If $f \in F(X)$, $f = \{(x,\mu) | x \in X \text{ and } \mu \in (0,1)\}$. then f is a bounded function for $f(x) \in [0,1]$. ((i.e) $|f(x)| \leq 1$). Let K be the space of real numbers, $F(X)$ is a linear space over the field K where the addition and scalar multiplication are defined by

$$f + g = \{(x,\mu) + (y,\eta)\} = \{(x+y, \mu \wedge \eta) | (x,\mu) \in f \text{ and } (y,\eta) \in g\}$$

$$\text{and } kf = \{(kf, \mu) | (x,\mu) \in f, \} \text{ where } k \in K.$$

The linear space $F(X)$ is said to be normed space if for every $f \in F(X)$, there is associated a non-negative real numbers $\|f\|$ called the norm of f in such a way that:

- (1) $\|f\| = 0$ if and only if $f = 0$. For $\|f\| = 0 \Leftrightarrow \{ \|(x,\mu)\| | (x,\mu) \in f \} = 0, \Leftrightarrow x = 0, \mu \in (0,1] \Leftrightarrow f = 0$.
- (2) $\|kf\| = |k|\|f\|$, $k \in K$. For $\|kf\| = \{ \|k(x,\mu)\| | (x,\mu) \in f \text{ and } k \in K \} = \{ |k| \|(x,\mu)\| | (x,\mu) \in f \} = |k|\|f\|$.

- (3) $\|f + g\| \leq \|f\| + \|g\|$ for every $f, g \in F(X)$.
 For $\|f + g\| = \{ \|(x, \mu) + (y, \eta)\| / x, y \in X, \mu, \eta \in (0, 1) \}$
 $= \{ \|(x+y), (\mu \wedge \eta)\| / x, y \in X, \mu, \eta \in (0, 1) \}$
 $\leq \{ \|(x, \mu \wedge \eta)\| + \|(y, \mu \wedge \eta)\| / (x, \mu) \in f \text{ and } (y, \eta) \in g \} = \|f\| + \|g\|.$

Definition 2.3 ([18])

Let $n \in \mathbb{N}$ and X be a real linear space of dimension greater than or equal to n . A real valued function $\| \cdot, \dots, \cdot \|$ on $X \times \dots \times X$ (n -times) $= X^n$ satisfying the following four properties

- i) $\|x_1, \dots, x_n\| = 0$ if and only if x_1, \dots, x_n linearly dependent.
 - ii) $\|x_1, \dots, x_n\|$ is invariant under any permutation
 - iii) $\|x_1, \dots, \alpha x_n\| = |\alpha| \|x_1, \dots, x_n\|$, for any α is real
 - iv) $\|x_1, \dots, x_{n-1}, y + z\| \leq \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, z\|$
- is called an n -norm on X and the pair $(X, \| \cdot, \dots, \cdot \|)$ is called n -normed linear space.

Definition 2.4 ([5])

Let $F(X^n)$ be a linear space over a real field. A fuzzy subset N of $F(X^n)^n \times \mathbb{R}$ is called fuzzy n -norm if and only if

- (N₁) for all $t \in \mathbb{R}, t \leq 0, N(f_1, \dots, f_n, t) = 0$
 - (N₂) for all $t \in \mathbb{R}, t > 0, N(f_1, \dots, f_n, t) = 1$ if and only if f_1, \dots, f_n are linearly dependent
 - (N₃) $N(f_1, \dots, f_n, t)$ is invariant under any permutation of f_1, \dots, f_n
 - (N₄) for all $t \in \mathbb{R}, t > 0, N(f_1, \dots, cf_n, t) = N(f_1, \dots, f_n, \frac{t}{|c|})$
 - (N₅) for all $s, t \in \mathbb{R}, N(f_1, \dots, f_n + f_n', s+t) \geq \min \{N(f_1, \dots, f_n, s), N(f_1, \dots, f_n', t)\}$
 - (N₆) $N(f_1, \dots, f_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and $\lim_{t \rightarrow \infty} N(f_1, \dots, f_n, t) = 1$
- The $(F(X^n)^n, N)$ is called a 2- fuzzy n -normed linear space.

Definition 2.5 ([20])

Let $F(X)$ be a linear space over the complex field \mathbb{C} . The fuzzy subset η defined as a mapping from $F(X) \times F(X) \times \mathbb{C}$ to $[0, 1]$ such that for all $f, g, h \in F(X), \alpha \in \mathbb{C}$,

- (I₁) for $s, t \in \mathbb{C}, \eta(f+g, h, |t| + |s|) \geq \min \{ \eta(f, h, |t|), \eta(g, h, |s|) \}$
- (I₂) for $s, t \in \mathbb{C}, \eta(f, g, |st|) \geq \min \{ \eta(f, f, |s|^2), \eta(g, g, |t|^2) \}$
- (I₃) for $t \in \mathbb{C}, \eta(f, g, t) = \eta(g, f, t)$
- (I₄) $\eta(\alpha f, g, t) = \eta(f, g, \frac{t}{|\alpha|}), \alpha \neq 0$
- (I₅) $\eta(f, f, t) = 0$ for all $t \in \mathbb{C} \setminus \mathbb{R}^+$
- (I₆) $\eta(f, f, t) = 1$ for all $t > 0$ if and only if $f = 0$
- (I₇) $\eta(f, f, \cdot) : \mathbb{R} \rightarrow I (= [0, 1])$ is a monotonic non-decreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} \eta(f, f, t) = 1$

Then η is said to be a 2-fuzzy inner product space on $F(X)$ and the pair $(F(X), \eta)$ is called a 2-fuzzy inner product space.

3. 2-FUZZY N-N HILBERT SPACE

In this section we introduce the satisfactory notion of 2-fuzzy n -normed inner product space as a generalization of Definition 2.5 as follows:

3.1 2-fuzzy n -norm Inner Product Space

Let $F(X^n)$, be a linear space over \mathbb{C} . Define a fuzzy subset η defined as a mapping from $[F(X^n)]^{n+1} \times \mathbb{C}$ to $[0, 1]$ such that $(f_1, \dots, f_n, f_{n+1}) \in [F(X^n)]^{n+1}, \alpha \in \mathbb{C}$ satisfying the following conditions

- (I₁) for $g, h \in F(X), s, t \in \mathbb{C}$
 $\eta(f_1 + g, h, f_2, \dots, f_n, |t| + |s|) \geq \min \{ \eta(f_1, h, f_2, \dots, f_n, |t|), \eta(g, h, f_2, \dots, f_n, |s|) \}$
- (I₂) for $s, t \in \mathbb{C}$

$$\{\eta(f_1, g, f_2, \dots, f_n, |st|)\} \geq \min \{ \eta(f_1, f_1, f_2, \dots, f_n, |s|^2), \eta(g, g, f_2, \dots, f_n, |t|^2) \}$$

(I₃) for $t \in \mathbb{C}$

$$\eta(f_1, g, f_2, \dots, f_n, |t|) = \eta(g, f_1, f_2, \dots, f_n, |t|)$$

(I₄) $\alpha_1, \alpha_2, \in \mathbb{C}, \alpha_1 \neq 0, \alpha_2 \neq 0$

$$\eta(\alpha_1 f_1, \alpha_2 f_1, f_2, \dots, f_n, t) = \eta(f_1, f_1, f_2, \dots, f_n, \frac{t}{|\alpha_1, \alpha_2|})$$

(I₅) $\eta(f_1, f_1, f_2, \dots, f_n, t) = 0 \quad \forall t \in \mathbb{C} / \mathbb{R}^+$
 $\eta(f_1, f_1, f_2, \dots, f_n, t) = 1 \quad \forall t > 0$ if and only if f_1, \dots, f_n are linearly dependent.

(I₆) $\eta(f_1, g, f_2, \dots, f_n, t)$ is invariant under any permutation of (f_2, \dots, f_n)

(I₇) $\forall t > 0 \quad \eta(f_1, f_1, f_2, \dots, f_n, t) = \eta(f_2, f_2, f_1, f_3, \dots, f_n, t)$

(I₈) $\eta(f_1, g, f_2, \dots, f_n, t)$ is a monotonic non-decreasing function of \mathbb{C} and $\lim_{t \rightarrow \infty} \eta(f_1, g, f_2, \dots, f_n, t) = 1$
 Then η is said to be the 2- fuzzy n-n inner product $F(X)^n$ and the pair $(F(X)^n, \eta)$ is called 2 - fuzzy n-n

IPS.

Example 3.2

Consider the mapping $f : S^n \rightarrow [0,1]$ where S^n in a n-dimensional unit sphere defined as

$$f(x_1, \dots, x_n) = |1 - (x_1^2 + \dots + x_n^2)|$$

Let us define n- dimensional unit sphere, defined as

$$\langle f_1, g, f_2, \dots, f_n \rangle = \begin{vmatrix} f_1 \cdot g & f_1 \cdot f_2 & \dots & f_1 \cdot f_n \\ f_2 \cdot g & f_2 \cdot f_2 & \dots & f_2 \cdot f_n \\ \dots & \dots & \dots & \dots \\ f_n \cdot g & f_n \cdot f_2 & \dots & f_n \cdot f_n \end{vmatrix}$$

where $f_1 \cdot g$ represents the usual innerproduct between two functions defined as

$$f_1 \cdot g = \int f_1(x) g(x) dx, \text{ where } x = (x_1, \dots, x_n).$$

With this innerproduct $(F(X)^n, \langle \dots, \dots, \dots \rangle)$ is an n-n IPS. By considering,

$$\eta(f_1, g, f_2, \dots, f_n, t) = \begin{cases} \frac{t}{t + \langle f_1, g, f_2, \dots, f_n \rangle} & \text{when } t > 0 \\ 0 & \text{when } t \in \mathbb{C} \setminus \mathbb{R}^+ \end{cases}$$

the space $(F(X)^n, \eta)$ is a 2-fuzzy n-n IPS

Definition 3.3

Every 2-fuzzy n-n complex Banach space is a 2-fuzzy n-n Hilbert space

Theorem 3.4

A closed convex subset C of a 2- fuzzy n-n Hilbert space $F(X)^n$ contains a unique vector of smallest norm.

Proof

Let $f, g \in F(X)^n$ the 2-fuzzy n-n Hilbert space where $f = (f_1, f_2, \dots, f_n)$ and $g = (g_1, g_2, \dots, g_n)$

Since C is convex,

$$A\{\lambda f + (1 - \lambda)g\} \geq \min \{ C(f), C(g) \}$$

Let $d = \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t) \geq \alpha\}$

Then there exists a sequence $\{f_n\}$ in C such that $\{f_n\}$ converges to d.

(i.e) $\eta(f_n, f_n, f_2, \dots, f_n, d) = 1$

$$\begin{aligned} \eta(f_n + f_m, f_n + f_m, f_2, \dots, f_n, s+t) &\geq \min \{ \eta(f_n, f_n + f_m, f_2, \dots, f_n, s), \eta(f_m, f_n + f_m, f_2, \dots, f_n, t) \} \\ &\geq \min \{ \min \{ \eta(f_n, f_n, f_2, \dots, f_n, s_1), \eta(f_n, f_m, f_2, \dots, f_n, s_2) \}, \\ &\quad \min \{ \eta(f_m, f_n, f_2, \dots, f_n, t_1), \eta(f_m, f_m, f_2, \dots, f_n, t_2) \} \} \\ &\geq \min \{ \min \{ d, d \}, \min \{ \eta(f_n, f_m, f_2, \dots, f_n, s_2), \eta(f_m, f_n, f_2, \dots, f_n, t_1) \} \} \end{aligned}$$

$$\eta(f_n + f_m, f_n + f_m, f_2, \dots, f_n, s+t) \geq \min \{ d, \eta(f_m, f_n, f_2, \dots, f_n, s_1 + t_1) \} = d$$

hence $\{f_n\}$ is a Cauchy sequence in C . As C is a closed subspace of a Hilbert space $F(X^n)$, C is complete, $\{f_n\}$ converges to some f . (i.e), $\eta(f_n - f, f_n - f, f_2, \dots, f_n, t) = 1$. It follows that, $\lim \eta(f_n - f, f_n - f, f_2, \dots, f_n, t) = 1$

Consider,

$$\eta(f_n - d + d - f, f_n - d + d - f, f_2, \dots, f_n, t) \geq \min\{\eta(f_n - d, f_n - d, f_2, \dots, f_n, t), \eta(d - f, d - f, f_2, \dots, f_n, t)\}$$

$$\geq \min\{1, \eta(d - f, d - f, f_2, \dots, f_n, t)\}$$

Therefore

$$\eta(f_n - f', f_n - f', f_2, \dots, f_n, t) = 1 > \eta(d - f, d - f, f_2, \dots, f_n, t), \text{ which is impossible}$$

and so $\eta(d - f, d - f, f_2, \dots, f_n, t) = 1$, which establishes that f has the smallest norm.

To prove the uniqueness of f , suppose $f' \in C$ such that $\eta(f', f', f_2, \dots, f_n, d) = 1$

Consider,

$$\eta(f - f', f - f', f_2, \dots, f_n, t) \geq \min\{\eta(f, f, f_2, \dots, f_n, t), \eta(f', f', f_2, \dots, f_n, t)\}$$

$$\geq \min\{1, 1\}$$

$$\geq 1$$

$\eta(f - f', f - f', f_2, \dots, f_n, t) > 1$, which is impossible, hence $\eta(f - f', f - f', f_2, \dots, f_n, t) = 1$. Thus there exists a unique f with the smallest norm 'd'.

Theorem 3.5

$$\text{If } 4\eta[(f_1, f_2, \dots, f_n), (f'_1, f'_2, \dots, f'_n), st] = \frac{1}{4} \{ \|f_1 + f'_1, \dots, f_n\|_\alpha^2 + \|f_1 - f'_1, \dots, f_n\|_\alpha^2$$

$$+ i \|f_1 + if'_1, \dots, f_n\|_\alpha^2 - i \|f_1 - if'_1, \dots, f_n\|_\alpha^2 \},$$

then η is a 2-fuzzy inner product on $F(X^n)$.

Proof

$$1) \eta(f+g, h, |t| + |s|) = \eta((f_1 + f'_1, \dots, f_n), h, |t| + |s|)$$

$$\geq \min\{\eta((f_1, \dots, f_n), h, |t|), \eta((f'_1, \dots, f'_n), h, |s|)\}$$

hence,

$$\eta(f+g, h, |t| + |s|) \geq \min\{\eta(f, h, |t|), \eta(g, h, |s|)\}$$

$$2) \text{ To prove, } \eta(f, g, |st|) \geq \min\{\eta(f, f, |s|^2), \eta(g, g, |t|^2)\}$$

$$\eta(f, g, |st|) = \eta((f_1, \dots, f_n), (f'_1, \dots, f'_n), |st|)$$

$$= \frac{1}{4} \{ \|f_1 + f'_1, \dots, f_n\|_\alpha^2 + \|f_1 - f'_1, \dots, f_n\|_\alpha^2 + i \|f_1 + if'_1, \dots, f_n\|_\alpha^2 - i \|f_1 - if'_1, \dots, f_n\|_\alpha^2 \} \quad \text{--- (i)}$$

Consider

$$\min\{\eta(f, f, |s|^2), \eta(g, g, |t|^2)\}$$

$$= \min\{\frac{1}{4} \{ \|f_1 + f_1, \dots, f_n\|_\alpha^2 + \|f_1 - f_1, \dots, f_n\|_\alpha^2 + i \|f_1 + if_1, \dots, f_n\|_\alpha^2 - i \|f_1 - if_1, \dots, f_n\|_\alpha^2 \},$$

$$\frac{1}{4} \{ \|f'_1 + f'_1, \dots, f_n\|_\alpha^2 + \|f'_1 - f'_1, \dots, f_n\|_\alpha^2 + i \|f'_1 + if'_1, \dots, f_n\|_\alpha^2 - i \|f'_1 - if'_1, \dots, f_n\|_\alpha^2 \} \}$$

$$= \min[\frac{1}{4} \{ \inf\{t \in \mathbb{R} : \eta(f_1 + f_1, f_1 + f_1, \dots, f_n, t) \geq \alpha\} + \inf\{t \in \mathbb{R} : \eta(f_1 + if_1, f_1 + if_1, \dots, f_n, t) \geq \alpha\}$$

$$+ \inf\{t \in \mathbb{R} : \eta(f_1 - if_1, f_1 - if_1, \dots, f_n, t) \geq \alpha\} \}, \frac{1}{4} \{ \inf\{t \in \mathbb{R} : \eta(f'_1 + f'_1, f'_1 + f'_1, \dots, f_n, t) \geq \alpha\}$$

$$+ \inf\{t \in \mathbb{R} : \eta(f'_1 + if'_1, f'_1 + if'_1, \dots, f_n, t) \geq \alpha\} + \inf\{t \in \mathbb{R} : \eta(f'_1 - if'_1, f'_1 - if'_1, \dots, f_n, t) \geq \alpha\} \} \}$$

$$= \min[\frac{1}{4} \{ \inf\{t \in \mathbb{R} : \eta(f_1 + f_1, f_1 + f_1, \dots, f_n, t) \geq \alpha\} + \inf\{t \in \mathbb{R} : \eta(f_1 + f_1, f_1 + f_1, \dots, f_n, t) \geq \alpha\}$$

$$+ \frac{1}{4} \{ \inf\{t \in \mathbb{R} : \eta(f'_1 + f'_1, f'_1 + f'_1, \dots, f_n, t) \geq \alpha\} + \inf\{t \in \mathbb{R} : \eta(f'_1 + f'_1, f'_1 + f'_1, \dots, f_n, t) \geq \alpha\} \} \}$$

$$= 2 \|f_1 + f_1, \dots, f_n\|_\alpha^2 + 2 \|f'_1 + f'_1, \dots, f_n\|_\alpha^2 \quad \text{--- (ii)}$$

From (i) and (ii),

$$\eta(f, g, |st|) \geq \min\{\eta(f, f, |s|^2), \eta(g, g, |t|^2)\}$$

$$3) \eta(f, g, |t|) = \frac{1}{4} \{ \|f_1 + f'_1, \dots, f_n\|_\alpha^2 + \|f_1 - f'_1, \dots, f_n\|_\alpha^2 + i \|f_1 + if'_1, \dots, f_n\|_\alpha^2 - i \|f_1 - if'_1, \dots, f_n\|_\alpha^2 \}$$

$$= \frac{1}{4} \{ \|f'_1 + f_1, \dots, f_n\|_\alpha^2 + \|f'_1 - f_1, \dots, f_n\|_\alpha^2 + i \|f'_1 + if_1, \dots, f_n\|_\alpha^2 - i \|f'_1 - if_1, \dots, f_n\|_\alpha^2 \}$$

$$= \eta(g, f, |t|)$$

4) To prove $\eta(\alpha f, \alpha g, |t|) = \eta(f, g, \frac{t}{|\alpha|^2})$

$$\begin{aligned} &\eta(\alpha f, \alpha g, |t|) \\ &= \frac{1}{4} \{ \|\alpha(f_1 + f'_1), \dots, f_n\|_\alpha^2 + \|\alpha(f_1 - f'_1), \dots, f_n\|_\alpha^2 + i \|\alpha(f_1 + if'_1), \dots, f_n\|_\alpha^2 - i \|\alpha(f_1 - if'_1), \dots, f_n\|_\alpha^2 \} \\ &= \frac{1}{4} \{ \inf\{t \in \mathbb{R} : \eta(\alpha(f_1 + f'_1), \alpha(f_1 + f'_1), \dots, f_n, t) \geq \alpha\} + \inf\{t \in \mathbb{R} : \eta(\alpha(f_1 - f'_1), \alpha(f_1 - f'_1), \dots, f_n, t) \geq \alpha\} \\ &\quad + \inf\{t \in \mathbb{R} : \eta(\alpha(f_1 + if'_1), \alpha(f_1 + if'_1), \dots, f_n, t) \geq \alpha\} + \inf\{t \in \mathbb{R} : \eta(\alpha(f_1 - if'_1), \alpha(f_1 - if'_1), \dots, f_n, t) \geq \alpha\} \} \\ &= \frac{1}{4} \{ \inf\{t \in \mathbb{R} : \eta(f_1 + f'_1, f_1 + f'_1, \dots, f_n, \frac{t}{|\alpha|^2}) \geq \alpha\} + \inf\{t \in \mathbb{R} : \eta(f_1 - f'_1, f_1 - f'_1, \dots, f_n, \frac{t}{|\alpha|^2}) \geq \alpha\} \\ &\quad + \inf\{t \in \mathbb{R} : \eta(f_1 + if'_1, f_1 + if'_1, \dots, f_n, \frac{t}{|\alpha|^2}) \geq \alpha\} + \inf\{t \in \mathbb{R} : \eta(f_1 - if'_1, f_1 - if'_1, \dots, f_n, \frac{t}{|\alpha|^2}) \geq \alpha\} \} \end{aligned}$$

So that, $\eta(\alpha f, \alpha g, |t|) = \eta(f, g, \frac{t}{|\alpha|^2})$

5) $\eta(f, f, t) = 0$ for all $t \in \mathbb{C} \setminus \mathbb{R}^+$

$$\begin{aligned} 6) \eta(f, f, t) = 1 &\Rightarrow \|f_1 + f'_1, \dots, f_n\|_\alpha^2 + \|f_1 - f'_1, \dots, f_n\|_\alpha^2 + i \|f_1 + if'_1, \dots, f_n\|_\alpha^2 - i \|f_1 - if'_1, \dots, f_n\|_\alpha^2 = 1 \\ &\Rightarrow \|f_1, \dots, f_n\|_\alpha^2 = 0 \end{aligned}$$

(i.e) f_1, f'_1, \dots, f_n are linearly dependent

7) $\eta(f, f, \dots) : \mathbb{R} \rightarrow I (= [0, 1])$ is a monotonic non decreasing function of \mathbb{R} and $\lim \eta(f, f, t) = 1$ as $t \rightarrow \infty$

Hence η is a 2-fuzzy inner product.

Theorem 3.6

Let M be a closed linear subspace of $[F(X^n), \eta]$. Let f does not belong to M . Let 'd' be the distance from f to M . Then there exists a unique g_0 in M such that $\eta(f - g_0, f - g_0, \dots, f_n, d) = 1$.

Proof

Let $C = f + M$, the space C is closed, convex and let d be the distance of origin 0 to C . We know that there exists a unique f_0 in C such that, $\eta(f_0, f_0, \dots, f_n, d) = 1$.

Suppose $f - f_0 = g_0 \in M$

We $\eta(f - g_0, f - g_0, \dots, f_n, d) = \eta(f_0, f_0, \dots, f_n, d) = 1$. Now to prove that this g_0 is unique.

Suppose there exists a $g_1 \in M$, such that $g_1 \neq g_0$ and $\eta(f - g_1, f - g_1, \dots, f_n, d) = 1$. Then $f - g_1 = f_1$ is an element in C such that $f_1 \neq f_0$ and $\eta(f_1, f_1, \dots, f_n, d) = 1$ which is a contradiction to the uniqueness of f_0 . Hence g_0 is unique.

Theorem 3.7

If M is a proper closed linear subspace of $[F(X^n), \eta]$ then there exists a f_0 such that

$$\eta(f_0, f_0, \dots, f_n, t) = 1 \text{ in } F(X^n) \text{ such that } f_0 \perp M$$

Proof

Let f does not belong to M . Let 'd' be the distance of f to M , there exists a unique g_0 in M such that $\eta(f - g_0, f - g_0, \dots, f_n, d) = 1$. Define $f_0 = f - g_0$ in such a way that $\eta(f_0, f_0, \dots, f_n, d) = 1$. Let $g \in M$ to assert that $f_0 \perp g$. For some scalar α ,

$$\begin{aligned} \eta(f_0 - \alpha g, f_0 - \alpha g, \dots, f_n, t) &= \eta(f - g_0 - \alpha g, f - g_0 - \alpha g, \dots, f_n, t) \\ &= \eta(f - (g_0 + \alpha g), f - (g_0 + \alpha g), \dots, f_n, t) \\ &\geq \eta(f_0, f_0, \dots, f_n, d) \text{ since } d \leq t \end{aligned}$$

It follows that $\eta(f_0 - \alpha g, f_0 - \alpha g, \dots, f_n, t) - \eta(f_0, f_0, \dots, f_n, d) \geq 0$

$$\Rightarrow \sup_{t_1+t_2=t} \{ \min[\eta(f_0, f_0 - \alpha g, \dots, f_n, t_1), \eta(\alpha g, f_0 - \alpha g, \dots, f_n, t_2)] \} - \eta(f_0, f_0, \dots, f_n, d) \geq 0$$

$$\Rightarrow \sup_{t_1+t_2=t} \{ \min\{ \sup_{t_3+t_4=t_1} \{ \min[\eta(f_0, f_0, \dots, f_n, t_3), \eta(f_0, \alpha g, \dots, f_n, t_4)] \}, \sup_{t_5+t_6=t_2} \{ \min[\eta(\alpha g, f_0, \dots, f_n, t_5), \eta(\alpha g, \alpha g, \dots, f_n, t_6)] \} \} \} - \eta(f_0, f_0, \dots, f_n, d) \geq 0$$

$$\Rightarrow \sup_{t_1+t_2=t} \{ \min\{ \eta(f_0, \alpha g, \dots, f_n, t_4), \eta(\alpha g, f_0, \dots, f_n, t_5) \} \} \geq 0$$

$$\Rightarrow \sup_{t_1+t_2=t} \{ \min\{ \eta(f_0, g, \dots, f_n, \frac{t_4}{|\alpha|}), \eta(g, f_0, \dots, f_n, \frac{t_5}{|\alpha|}) \} \} \text{ which is either equal to } 0 \text{ or } 1$$

The last inequality holds for

$$\eta(f_0, g, \dots, f_n, \frac{t_4}{|\alpha|}) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

$$\eta(g, f_0, \dots, f_n, \frac{t_5}{|\alpha|}) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

By definition it follows immediately that $f_0 \perp g$.

Theorem 3.8

If M and N are closed linear subspaces of $[F(X^n), \eta]$ such that $M \perp N$, then the linear subspace $M+N$ is also closed.

Proof

Let a be the limit point of $M+N$. The aim is to show that $a \in M+N$. Automatically there exists a sequence

$\{a_k\} \in M+N$ such that $a_k \rightarrow a$. Since $M \perp N$, each a_k can be written as $a_k = f_k + g_k$ where $f_k \in M$ and $g_k \in N$. By using Pythagorean theorem,

$$\begin{aligned} \eta(a_k - a_1, a_k - a_1, \dots, a_n, t) &= \eta(f_k + g_k - (f_1 + g_1), f_k + g_k - (f_1 + g_1), \dots, f_n + g_n, t) \\ &= \eta(f_k - f_1 + g_k - g_1, f_k - f_1 + g_k - g_1, \dots, f_k + g_n, t) \\ &= \eta(f_k - f_1, f_k - f_1, \dots, f_n, t) * \eta(g_k - g_1, g_k - g_1, \dots, g_n, t) \end{aligned}$$

So $\{f_k\}$ and $\{g_k\}$ are Cauchy sequences in M and N. M and N are closed and therefore complete.

so there exists f in M and g in N such that $f_k \rightarrow f$ and $g_k \rightarrow g$.

Therefore,

$$\begin{aligned} a &= \lim_{k \rightarrow \infty} a_k \\ &= \lim_{k \rightarrow \infty} (f_k + g_k) \\ &= \lim_{k \rightarrow \infty} f_k + \lim_{k \rightarrow \infty} g_k \\ &= f + g \end{aligned}$$

then $f+g \in M+N$ and so $M+N$ is closed.

Theorem 3.9

If M and M^\perp are closed linear subspaces of $[F(X^n), \eta]$, then $[F(X^n), \eta] = M \oplus M^\perp$

Proof

Let M and M^\perp be the closed linear subspaces of $[F(X^n), \eta]$. We know that $M+M^\perp$ is also a closed linear subspace of $[F(X^n), \eta]$. Now to prove that $[F(X^n), \eta] = M \oplus M^\perp$

Assume $[F(X^n), \eta] \neq M+M^\perp$, there exists a g_0 in $M+M^\perp$ such that $g_0 \perp (M+M^\perp)$

$\Rightarrow g_0 \in M^\perp \cap (M+M^\perp)^\perp$ which is not possible.

Therefore $[F(X^n), \eta] = M+M^\perp$

Since M and M^\perp are disjoint, $[F(X^n), \eta] = M+M^\perp$ can be strengthened to $M \oplus M^\perp$.

4. 2-FUZZY N-N QUASI INNER PRODUCT SPACE

As a consequence of Definition 3.1 we introduce the notion of ascending family of α -quasi n-n on $F(X^n)$ corresponding to fuzzy quasi n-n inner products.

Definition 4.1

Let $F(X^n)$, be a linear space over \mathbb{C} , a fuzzy subset η defined as a mapping from $[F(X^n)]^{n+1} \times \mathbb{C}$ to $[0,1]$ such that $(f_1, \dots, f_n, f_{n+1}) \in [F(X^n)]^{n+1}$ with $\alpha \in \mathbb{C}$ satisfying the following conditions.

(I₁) for $g, h \in F(x)$, and $s, t \in \mathbb{C}$

$$\eta(f_1 + g, h, f_2, \dots, f_n, |t| + |s|) \geq \min \{ \eta(f_1, h, f_2, \dots, f_n, |t|), \eta(g, h, f_2, \dots, f_n, |s|) \}$$

(I₂) for $s, t \in \mathbb{C}$

$$\{ \eta(f_1, g, f_2, \dots, f_n, |st|) \geq \min \{ \eta(f_1, f_1, f_2, \dots, f_n, |s|^2), \eta(g, g, f_2, \dots, f_n, |t|^2) \}$$

(I₃) for $t \in \mathbb{C}$

$$\eta(f_1, g, f_2, \dots, f_n, |t|) = \eta(g, f_1, f_2, \dots, f_n, |t|)$$

(I₄) $\alpha_1, \alpha_2, \in \mathbb{C}$, $\alpha_1 \neq 0, \alpha_2 \neq 0$

$$\eta(\alpha_1 f_1, \alpha_2 f_1, f_2, \dots, f_n, t) = \eta(f_1, f_1, f_2, \dots, f_n, \frac{t}{|\alpha_1 \alpha_2|^p}) \text{ where } 0 < p < 1$$

(I₅) $\eta(f_1, f_1, f_2, \dots, f_n, t) = 0 \quad \forall t \in \mathbb{C} / \mathbb{R}^+$

$\eta(f_1, f_1, f_2, \dots, f_n, t) = 1 \quad \forall t > 0$ if and only if f_1, \dots, f_n are linearly dependent.

(I₆) $\eta(f_1, g, f_2, \dots, f_n, t)$ is invariant under any permutation of (f_2, \dots, f_n)

(I₇) $\forall t > 0 \quad \eta(f_1, f_1, f_2, \dots, f_n, t) = \eta(f_2, f_2, f_1, f_3, \dots, f_n, t)$

(I₈) $\eta(f_1, g, f_2, \dots, f_n, t)$ is a monotonic non decreasing function of \mathbb{C} and $\lim_{t \rightarrow \infty} \eta(f_1, g, f_2, \dots, f_n, t) = 1$

is said to be the 2- fuzzy n-n quasi inner product and $(F(X^n), \eta)$ is called a 2 - fuzzy n-n quasi IPS.

Theorem 4.2

Let $(F(X^n), \eta)$ be a 2-fuzzy n-n quasi IPS satisfying the condition that, $\eta(f_1, f_1, f_2, \dots, f_n, t^2) > 0$ when $t > 0$ implies f_1, f_2, \dots, f_n are linearly dependent. Then for all $\alpha \in (0,1)$, $\|f_1, \dots, f_n\|_\alpha = \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha\}$ is an ascending family of real numbers, the quasi n-n norms on $F(X^n)$. These quasi n-n norms are called the α -quasi n-n norms on $F(X^n)$ corresponding to fuzzy quasi n-n inner products.

Proof

$$\begin{aligned} 1) \|f_1, \dots, f_n\|_\alpha &= 0 \\ &\Rightarrow \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha\} = 0 \\ &\Rightarrow \text{for all } t \in \mathbb{R}, t > 0, \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha > 0 \\ &\Rightarrow f_1, \dots, f_n \text{ are linearly dependent.} \end{aligned}$$

Conversely assume that f_1, \dots, f_n are linearly dependent then $\eta(f_1, f_1, f_2, \dots, f_n, t^2) = 1$ for all $t > 0$ implies that for all $\alpha \in (0,1)$, $\inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha\} = 0$. Therefore, $\|f_1, \dots, f_n\|_\alpha = 0$

2) As $\eta(f_1, f_1, f_2, \dots, f_n, t^2)$ is invariant under any permutation it follows that $\|f_1, \dots, f_n\|_\alpha$ is invariant under any permutation.

$$\begin{aligned} 3) \text{ For all } \alpha \in F, 0 < p < 1, \|cf_1, \dots, f_n\|_\alpha &= \inf\{s: \eta(cf_1, cf_1, f_2, \dots, f_n, s^2) \geq \alpha\} \\ &= \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, \frac{s^2}{|c|^{2p}}) \geq \alpha\} \end{aligned}$$

$$\begin{aligned} \text{Let } t &= \frac{s}{|c|^{2p}} \text{ then } \|cf_1, \dots, f_n\|_\alpha = \inf\{t|c|^{2p}: \eta(cf_1, cf_1, f_2, \dots, f_n, t^2) \geq \alpha\} \\ &= |c|^{2p} \inf\{t: \eta(cf_1, cf_1, f_2, \dots, f_n, t^2) \geq \alpha\} \\ &= |c|^{2p} \|f_1, \dots, f_n\|_\alpha \end{aligned}$$

$$\begin{aligned} 4) \|f_1, \dots, f_n\|_\alpha + \|f'_1, \dots, f_n\|_\alpha &= \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha\} + \inf\{s: \eta(f'_1, f'_1, f_2, \dots, f_n, s^2) \geq \alpha\} \\ &= \inf\{t+s: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha, \eta(f'_1, f'_1, f_2, \dots, f_n, s^2) \geq \alpha\} \\ &\geq \inf\{t+s: \eta(f_1 + f'_1, f_1 + f'_1, f_2, \dots, f_n, s^2) \geq \alpha\} \\ &= \|f_1 + f'_1, \dots, f_n\|_\alpha \end{aligned}$$

hence $\|f_1 + f'_1, \dots, f_n\|_\alpha \leq \|f_1, \dots, f_n\|_\alpha + \|f'_1, \dots, f_n\|_\alpha$

Thus $\{\|\cdot, \dots, \cdot\|_\alpha, \alpha \in (0,1)\}$ is a quasi α -n-n norm on $F(X^n)$.

$$\begin{aligned} \text{Let } 0 < \alpha_1 < \alpha_2 < 1, \text{ then } \|f_1, \dots, f_n\|_{\alpha_1} &= \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha_1\} \\ \|f_1, \dots, f_n\|_{\alpha_2} &= \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha_2\} \end{aligned}$$

$$\begin{aligned} \text{As } \alpha_1 < \alpha_2, \{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha_1\} &\supset \{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha_2\} \\ \Rightarrow \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha_2\} &\geq \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha_1\} \\ \Rightarrow \|f_1, \dots, f_n\|_{\alpha_2} &\geq \|f_1, \dots, f_n\|_{\alpha_1} \end{aligned}$$

Therefore $\{\|\cdot, \dots, \cdot\|_\alpha | \alpha \in (0,1)\}$ is an ascending family of quasi α -n-n norms on $F(X^n)$.

Theorem 4.3

Let $\{\|\cdot, \dots, \cdot\|_\alpha | \alpha \in (0,1)\}$ be an ascending family of quasi norms corresponding to $(F(X^n), \eta)$. Now define a function, $\eta' : [F(X^n)]^{n+1} \times \mathbb{R} \rightarrow [0,1]$ by,

$$\eta'(f_1, f_1, f_2, \dots, f_n, t^2) = \begin{cases} \sup\{\alpha \in (0,1) : \|f_1, \dots, f_n\| \leq t, \text{ when } f_1, \dots, f_n \text{ are linearly independent} \\ 0 & \text{otherwise} \end{cases}$$

Then $(F(X^n), \eta')$ is a 2-fuzzy quasi n-n IPS

Proof

Consider the following two lemmas

Lemma 1

If $(F(X^n), \eta)$ be a 2-fuzzy n-n IPS satisfying all the conditions and $\{\|\cdot, \dots, \cdot\|_\alpha, \alpha \in (0,1)\}$ be an ascending family of α - n-n norms on $F(X^n)$ defined as,

$$\|f_1, \dots, f_n\|_\alpha = \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha\} \text{ for all } \alpha \in (0,1) \text{ ---- (1)}$$

Then for $g_1, \dots, g_n \in F(X^n)$, $\eta(g_1, g_1, \dots, g_n, \|g_1, \dots, g_n\|_\alpha^2) \geq \alpha$ for all $\alpha \in (0,1)$

Proof

According to (1), suppose $\|g_1, \dots, g_n\|_\alpha^2 = A$, then $A > 0$ it asserts that there exists a sequence $\{a_n\}$, $a_n > 0$, such that $\eta(g_1, g_1, \dots, g_n, a_n^2) \geq \alpha$ and a_n converges to A .

Hence $\lim_{n \rightarrow \infty} \eta(g_1, g_1, \dots, g_n, a_n^2) = \eta(g_1, g_1, \dots, g_n, \lim_{n \rightarrow \infty} a_n^2) \geq \alpha \Rightarrow \eta(g_1, g_1, \dots, g_n, A) \geq \alpha$ which implies that $\eta(g_1, g_1, \dots, g_n, \|g_1, \dots, g_n\|_\alpha^2) \geq \alpha$ for all $\alpha \in (0, 1)$.

Lemma 2

If $(F(X^n), \eta)$ be a 2-fuzzy n-n IPS satisfying all the conditions and $\{\|\cdot, \dots, \cdot\|_\alpha / \alpha \in (0, 1)\}$ be an ascending family of α - n-n norms on $F(X^n)$ defined as,

$$\|f_1, \dots, f_n\|_\alpha = \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha\} \text{ for all } \alpha \in (0, 1)$$

then for $g_1, \dots, g_n \in F(X^n)$, $\alpha \in (0, 1)$ and $a' > 0 \in \mathbb{R}$, $\|g_1, \dots, g_n\|_\alpha = a'$ if and only if

$$\eta(g_1, g_1, \dots, g_n, a'^2) = \alpha$$

Proof

Let $\alpha \in (0, 1)$, and $a' = \|g_1, \dots, g_n\|_\alpha = \inf\{s: \eta(g_1, g_1, \dots, g_n, s^2) \geq \alpha\}$, there exists a sequence $\{s_n\}$ such that, $s_n \rightarrow a'$.

$$\begin{aligned} \lim \eta(g_1, g_1, \dots, g_n, s_n^2) &\geq \alpha = \eta(g_1, g_1, \dots, g_n, \lim s_n^2) \geq \alpha \\ &= \eta(g_1, g_1, \dots, g_n, a'^2) \geq \alpha \end{aligned}$$

so, $\eta(g_1, g_1, \dots, g_n, a'^2) \geq \alpha$ --- (i)

Also $\eta(g_1, g_1, \dots, g_n, a'^2) \leq \eta(g_1, g_1, \dots, g_n, s^2)$ if $\eta(g_1, g_1, \dots, g_n, s^2) \geq \alpha$ for all $\alpha \in (0, 1)$,

Assume $\eta(g_1, g_1, \dots, g_n, a'^2) > \alpha$, by continuity and since $\eta(g_1, g_1, \dots)$ is strictly increasing at a' , there exists $a'' < a'$ such that $\eta(g_1, g_1, \dots, g_n, a''^2) > \alpha$ which is impossible for $a' = \inf\{s: \eta(g_1, g_1, \dots, g_n, s^2) \geq \alpha\}$.

Thus $\eta(g_1, g_1, \dots, g_n, a'^2) \leq \alpha$ ---- (ii)

from (i) and (ii), $\eta(g_1, g_1, \dots, g_n, a'^2) = \alpha$, (i.e) $a' = \|g_1, \dots, g_n\|_\alpha$ implies $\eta(g_1, g_1, \dots, g_n, a'^2) = \alpha$ --- (iii)

If $\eta(g_1, g_1, \dots, g_n, a'^2) = \alpha$ for all $\alpha \in (0, 1)$, by definition,

$$\|g_1, \dots, g_n\|_\alpha = \inf\{s: \eta(g_1, g_1, \dots, g_n, s^2) \geq \alpha\} = a' \text{ --- (iv), from (iii) and (iv) for } g_1, \dots, g_n \in F(X^n),$$

$\alpha \in (0, 1)$ and $a' > 0 \in \mathbb{R}$, $\|g_1, \dots, g_n\|_\alpha = a'$ if and only if $\eta(g_1, g_1, \dots, g_n, a'^2) = \alpha$

Now to the proof of the theorem,

Let $\eta': [F(X^n)]^{n+1} \times \mathbb{R} \rightarrow [0, 1]$ and $\eta(g_1, g_1, \dots, g_n, s^2) = \alpha_0$, consider,

$$\|g_1, \dots, g_n\|_\alpha = \inf\{s: \eta(g_1, g_1, \dots, g_n, s^2) \geq \alpha\} \text{ for all } \alpha \in (0, 1)$$

And $\eta'(g_1, g_1, \dots, g_n, s^2) = \sup\{\alpha \in (0, 1) | \|g_1, \dots, g_n\|_\alpha \leq s\}$ where $s > 0$.

Consider the following cases,

Case (i)

$s \leq 0$ and g_1, \dots, g_n are linearly dependent ,
 $\eta(g_1, g_1, \dots, g_n, s^2) = 1$ and $\eta'(g_1, g_1, \dots, g_n, s^2) = 1$,
 so, $\eta(g_1, g_1, \dots, g_n, s^2) = \eta'(g_1, g_1, \dots, g_n, s^2)$

Case (ii)

$s > 0$ and g_1, \dots, g_n are linearly dependent
 $\eta(g_1, g_1, \dots, g_n, s^2) = 1$ and $\eta'(g_1, g_1, \dots, g_n, s^2) = 1$,
 so, $\eta(g_1, g_1, \dots, g_n, s^2) = \eta'(g_1, g_1, \dots, g_n, s^2)$

Case (iii)

$s \leq 0$ and g_1, \dots, g_n are linearly dependent
 $\eta(g_1, g_1, \dots, g_n, s^2) = 1$ and $\eta'(g_1, g_1, \dots, g_n, s^2) = 1$,
 so, $\eta(g_1, g_1, \dots, g_n, s^2) = \eta'(g_1, g_1, \dots, g_n, s^2)$

Case (iv)

g_1, \dots, g_n are not linearly dependent, $s > 0$ and $\eta(g_1, g_1, \dots, g_n, s^2) = 0$
 By lemma 1, $\eta(g_1, g_1, \dots, g_n, \|g_1, \dots, g_n\|_\alpha^2) \geq \alpha$ for all $\alpha \in (0, 1)$,
 since $\eta(g_1, g_1, \dots, g_n, s^2) = 0 < \alpha$ it follows that $s < \|g_1, \dots, g_n\|_\alpha$ for all $\alpha > 0$
 so, $\eta'(g_1, g_1, \dots, g_n, s^2) = \sup\{\alpha \in (0, 1) | \|g_1, \dots, g_n\|_\alpha \leq s\}$
 $= \sup \emptyset$
 $= 0$

so $\eta(g_1, g_1, \dots, g_n, s^2) = \eta'(g_1, g_1, \dots, g_n, s^2)$

Case (v)

g_1, \dots, g_n are not linearly dependent, $0 < \eta(g_1, g_1, \dots, g_n, s^2) < 1$
 Then let, $\eta(g_1, g_1, \dots, g_n, s^2) = \alpha_0$, $0 < \alpha_0 < 1$
 Now $\eta'(f_1, f_1, \dots, f_n, t^2) = \sup\{\alpha \in (0, 1) | \|f_1, \dots, f_n\|_\alpha \leq t\}$ when $t > 0$ ---- (i)
 And $\|f_1, \dots, f_n\|_\alpha = \inf\{t: \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha\}$ for all $\alpha \in (0, 1)$ ---- (ii)
 Since, $\eta(g_1, g_1, \dots, g_n, s^2) = \alpha_0$, then from (ii), $\|g_1, \dots, g_n\|_\alpha \leq s$ ---- (iii)
 Using (iii) and from (1), $\eta'(g_1, g_1, \dots, g_n, s^2) \geq \alpha_0$
 which implies, $\eta'(g_1, g_1, \dots, g_n, s^2) \geq \eta(g_1, g_1, \dots, g_n, s^2)$ --- (iv)

From lemma 2, if and only if $\eta(g_1, g_1, \dots, g_n, s^2) = \alpha_0$ if and only if $\|g_1, \dots, g_n\|_\alpha = s$

For $1 > \alpha > \alpha_0$ let $\|g_1, \dots, g_n\|_\alpha = a'$ then $a' \geq s$

By lemma 2, $\eta(g_1, g_1, \dots, g_n, a'^2) = \alpha$

so, $\alpha = \eta(g_1, g_1, \dots, g_n, a'^2) = \alpha \geq \alpha_0 = \eta(g_1, g_1, \dots, g_n, s^2)$,

Since $\eta(g_1, g_1, \dots)$ is strictly increasing in $H = \{s: \eta(g_1, g_1, \dots, g_n, s^2) \geq \alpha\}$, $\alpha \in (0, 1)$

$a', s \in H$ and $\eta(g_1, g_1, \dots, g_n, a'^2) > \eta(g_1, g_1, \dots, g_n, s^2)$ implies $a' > s^2$

For $1 > \alpha > \alpha_0$ $\|g_1, \dots, g_n\|_\alpha = a' > \alpha_0$, and we get $\eta'(g_1, g_1, \dots, g_n, s^2) < \alpha_0 = \eta(g_1, g_1, \dots, g_n, s^2)$ ---- (v)

from (vi) and (v), $\eta'(g_1, g_1, \dots, g_n, s^2) = \eta(g_1, g_1, \dots, g_n, s^2)$

Case(vi)

g_1, \dots, g_n are not linearly dependent and $\eta(g_1, g_1, \dots, g_n, s^2) = 1$

$\eta'(f_1, f_1, \dots, f_n, t^2) = \sup\{\alpha \in (0, 1) \mid \|f_1, \dots, f_n\|_\alpha \leq t\}$ when $t > 0$

$\|f_1, \dots, f_n\|_\alpha = \inf\{t: \eta(f_1, f_1, \dots, f_n, t^2) \geq \alpha\}$ for all $\alpha \in (0, 1)$

It follows that $\|g_1, \dots, g_n\|_\alpha \leq S$ for all $\alpha \in (0, 1)$

which implies $\eta'(g_1, g_1, \dots, g_n, s^2) = 1$, so $\eta'(g_1, g_1, \dots, g_n, s^2) = \eta(g_1, g_1, \dots, g_n, s^2)$

Thus $[F(X^n), \eta']$ is a 2-fuzzy quasi n-n IPS

5. CONCLUSION

In this paper we consider the notion of 2-fuzzy inner product space introduced by Thangaraj Beaula and R.A.S.Gifta [20] and develop the concept of 2-fuzzy n-n inner product space as a generalization.. As a consequence we introduce the notion of 2-fuzzy quasi n-n inner product space and establish certain results regarding these concepts.

6. REFERENCES

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