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Degree of Approximation of a Function Belonging to $Lip(\xi(t),r)$ Class by (E,1)(C,2) Summability Means

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ABSTRACT— ***In this paper, we determine the degree of approximation of a function*** $f \in Lip(\xi(t),r)$***, where*** $\xi(t)$ ***is nonnegative and increasing function of t, by (E,1)(C,2) product operators on Fourier series associated with f.***

Keywords— Degree of approximation, $Lip(\xi(t),r)$ class of function, (E,1) means, (C,2) means, (E,1)(C,2) product means, Fourier series, Lebesgue integral.

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1. INTRODUCTION

Let f(x) be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series of f(x) is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

with n[th] partial sum $s_n(f:x)$.

$L_\infty -$ norm of a function $f: R \to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x \in R\}$

$$L_r - \text{norm is defined by } \|f\|_r = \left(\int_0^{2\pi}|f(x)|^r dx\right)^{\frac{1}{r}}, r \geq 1 \tag{1.2}$$

The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial t_n of order n under sup norm $\| \ \|_\infty$ is defined as

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min\|t_n - f\|_r \text{ (Zygmund [13])} \tag{1.3}$$

A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O\left(|t|^\alpha\right) \text{ for } 0 < \alpha \leq 1 \tag{1.4}$$

$f \in Lip(\alpha, r)$ if

$$\left(\int_0^{2\pi}|f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\left(|t|^\alpha\right) \quad 0 < \alpha \leq 1, \text{ and } r \geq 1 \tag{1.5}$$

(Definition 5.38 of Mc Fadden [6], 1942).

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi}|f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)) \tag{1.6}$$

If $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class coincides with the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ reduces to the $Lip\alpha$.

We observe that

$$Lip\alpha \subseteq Lip(\alpha, r) \subseteq Lip(\xi(t), r) \text{ for } 0<\alpha\le 1,\ \ r\ge 1.$$

This method of approximation is called trigonometric Fourier approximation (TFA).

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n[th] partial sums $\{s_n\}$.

The (C, 2) transform is defined as the n[th] partial sum of (C, 2) summability and is given by

$$t_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1)\ s_n \to s \ \ as \ \ n\to\infty \tag{1.8}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by (C,2) method.

If

$$(E,1) = E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k \to s \ \ as \ \ \ n\to\infty \tag{1.9}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable (E,1) to the definite number s (Hardy[3]).

The (E,1) transform of the (C,2) transform defines (E,1)(C,2) transform and we denote it by $E_n^1 C_n^2$.

Thus if

$$E_n^1 C_n^2 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} C_k^2 \to s \quad \text{as } n\to\infty \tag{1.10}$$

where E_n^1 denotes the (E,1) transform of s_n and C_n^2 denotes the (C,2) transform of s_n, then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (E,1)(C,2) means or summable (E,1)(C,2) to a definite number s.

We use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$K_n(t) = \frac{1}{\pi 2^n} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k-\nu+1) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}$$

2. MAIN THEOREM

Alexits [1], Sahney and Goel [11], Chandra [2], Qureshi and Neha [9], Liendler [5] and Rhoades [10] have determined the degree of approximation of a function belonging to $Lip\alpha$ class by Cesáro, Nörlund and generalized Nörlund single summability methods. Working in the same direction Sahney and Rao [12], Khan [4] and Qureshi [7,8] have studied the degree of approximation of function belonging to $Lip(\alpha, r)$ class by Nörlund and generalized Nörlund single summability methods. But nothing seems to have been done so far in the direction of present work. The $Lip(\xi(t), r)$ class is a generalization of $Lip\alpha$ class and $Lip(\alpha, r)$ class. Therefore, in present paper, a theorem on degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by (C,2)(E,1) product summability means of Fourier series has been established in the following form:

2.1 Theorem 1

If f is a 2π-periodic function, Lebesgue integrable on [0,2π], belonging to the $Lip(\xi(t), r$ class then its degree of approximation by (E,1)(C,2) summability means on Fourier series is given by

$$\left\| E_n^1 C_n^2 - f \right\|_r = O\left[(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{(n+1)} \right) \right] \tag{2.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)} \right)^r dt \right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1} \right), \tag{2.2}$$

and

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)} \right)^r dt \right\}^{\frac{1}{r}} = O\left\{ (n+1)^{\delta} \right\} \tag{2.3}$$

where δ is an arbitrary number such that $0 \neq \delta s + 1 < s$, $\frac{1}{r} + \frac{1}{s} = 1$, conditions (2.2) and (2.3) hold uniformly in x and $E_n^1 C_n^2$ is (E,1)(C,2) means of the series (1.1).

3. LEMMAS

For the proof of our theorems, following lemmas are required:

3.1 Lemma 1

$$|K_n(t)| = O(n+1) \quad \text{for} \quad 0 \le t \le \frac{1}{n+1}$$

Proof: For $0 \le t \le \frac{1}{n+1}$, $\sin nt \le n \sin t$

$$|K_n(t)| = \frac{1}{\pi 2^n} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k-\nu+1) \frac{\sin\left(\nu + \frac{1}{2}\right)t}{\sin\frac{t}{2}} \right] \right|$$

$$\le \frac{1}{\pi 2^n} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k-\nu+1) \frac{(2\nu+1)\sin\frac{t}{2}}{\sin\frac{t}{2}} \right] \right|$$

$$\le \frac{1}{\pi 2^n} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \frac{1}{(k+1)(k+2)} (2k+1) \sum_{\nu=0}^{k} (k-\nu+1) \right] \right|$$

$$\le \frac{1}{\pi 2^n} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \frac{1}{(k+1)(k+2)} (2k+1) \frac{(k+1)(k+2)}{2} \right] \right|$$

$$= \frac{1}{\pi 2^{n+1}} \sum_{k=0}^{n} \left[\binom{n}{k} (2k+1) \right]$$

$$= \frac{1}{\pi 2^{n+1}} \{ 2^n (n+1) \}$$

$$= O(n+1)$$

3.2 Lemma 2

$$|K_n(t)| = O\left(\frac{1}{t}\right), \quad \text{for } \frac{1}{n+1} \le t \le \pi$$

Proof: For $\frac{1}{n+1} \le t \le \pi,$ by applying Jordan's lemma

$$\sin\frac{t}{2} \ge \frac{t}{\pi} \quad and \quad \sin nt \le 1$$

$$|K_n(t)| \le \frac{1}{\pi 2^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\frac{1}{(k+1)(k+2)}\sum_{\nu=0}^{k}(k-\nu+1)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin(t/\pi)}\right]\right|$$

$$\le \frac{1}{\pi 2^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\frac{1}{(k+1)(k+2)}\sum_{\nu=0}^{k}(k-\nu+1)\left(\frac{1}{t/\pi}\right)\right]\right|$$

$$\le \frac{1}{t\ 2^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\frac{1}{(k+1)(k+2)}\sum_{\nu=0}^{k}(k-\nu+1)\right]\right|$$

$$= \frac{1}{t\ 2^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\frac{1}{(k+1)(k+2)}\frac{(k+1)(k+2)}{2}\right]\right|$$

$$= \frac{1}{t\ 2^{n+1}}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\right]\right|$$

$$= \frac{1}{t\ 2^{n+1}} 2^n$$

$$= O\left(\frac{1}{t}\right)$$

4. PROOF OF THEOREM 1

Following Titchmarsh [12] and using Riemann-Lebesgue theorem, $s_n(f;x)$ of the series (1.1) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Therefore using (1.1), the (C, 2) transform C_n^2 of $s_n(f;x)$ is given by

$$C_n^2 - f(x) = \frac{1}{\pi(n+1)(n+2)}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}(n-k+1)\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Now denoting (E,1)(C,2) transform of $s_n(f;x)$ by $E_n^1C_n^2$, we write

$$E_n^1C_n^2 - f(x) = \frac{1}{\pi 2^n}\sum_{k=0}^{n}\left[\binom{n}{k}\frac{1}{(k+1)(k+2)}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\left\{\sum_{\nu=0}^{k}(k-\nu+1)\sin\left(\nu+\frac{1}{2}\right)t\right\}dt\right]$$

$$= \int_0^{\pi}\phi(t)K_n(t)dt$$

$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)dt$$

$$= I_1 + I_2 \quad \text{(say)} \tag{4.1}$$

We consider,

$$|I_1| \le \int_0^{\frac{1}{n+1}}|\phi(t)|\,|K_n(t)|\,dt$$

Using Hölder's inequality and the fact that $\phi(t) \in Lip(\xi(t), r)$,

$$|I_1| \le \int_0^{\frac{1}{n+1}}|\phi(t)||K_n(t)|\;dt$$

$$\le \left[\int_0^{\frac{1}{n+1}}\left\{\frac{t|\phi(t)|}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)\,|K_n(t)|}{t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)\,|K_n(t)|}{t}\right\}^s dt\right]^{\frac{1}{s}} \qquad \text{by (2.2)}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{(n+1)\,\xi(t)}{t}\right\}^s dt\right]^{\frac{1}{s}} \qquad \text{by Leema 1}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$I_1 = O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}}\frac{dt}{t^s}\right]^{\frac{1}{s}} \quad \text{for some } 0 \le \in < \frac{1}{n+1}$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-s+1}}{-s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}}$$

$$=O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{1-\frac{1}{s}}\right\}$$

$$=O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \quad \because \frac{1}{r}+\frac{1}{s}=1 \tag{4.2}$$

Using Hölder's inequality,

$$|I_2| \le \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||K_n(t)|\,dt$$

$$\le \left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)\,|K_n(t)|}{t^{-\delta}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$=O\{(n+1)^{\delta}\}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)\,|K_n(t)|}{t^{-\delta}}\right\}^s dt\right]^{\frac{1}{s}} \qquad \text{by (2.3)}$$

$$=O\{(n+1)^{\delta}\}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta}}\right\}^s dt\right] \qquad \text{by Leema 2}$$

Now putting $t=1/y$,

$$I_2=O\{(n+1)^{\delta}\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{(y)^{\delta-1}}\right\}^s \frac{dy}{y^2}\right]^{\frac{1}{s}}.$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$I_2=O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1}\frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}} \quad \text{for some } \frac{1}{\pi}\le\eta\le n+1$$

$$=O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{(n+1)}\right)\right\}\left[\int_{1}^{n+1}\frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}} \quad \text{for } \frac{1}{\pi}<1\le n+1.$$

$$=O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{y^{s(1-\delta)-1}}{s(1-\delta)-1}\right\}_1^{n+1}\right]^{\frac{1}{s}}$$

$$=O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1-\delta)-\frac{1}{s}}\right]$$

$$=O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{1-\frac{1}{s}}\right\}$$

$$= O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\} \qquad \because \frac{1}{r}+\frac{1}{s}=1 \tag{4.3}$$

Combining (4.1), (4.2) and (4.3),

$$\left|E_n^1 C_n^2 - f(x)\right| = O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}$$

Now using L_r - norm we get,

$$\left\|E_n^1 C_n^2 - f(x)\right\|_r = \left\{\int_0^{2\pi} \left|E_n^1 C_n^2 - f(x)\right|^r dx\right\}^{\frac{1}{r}}$$

$$= O\left[\left\{\int_0^{2\pi} \left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}\right]$$

$$= O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right]$$

$$= O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of the theorem.

6. APPLICATIONS

The following corollaries can be derived from our main theorem:

Corollary 1

If $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$, then the class $Lip(\xi(t), r)$, $r \ge 1$, reduces to the class $Lip(\alpha, r)$ and the degree of approximation of a function $f \in Lip(\alpha, r)$, $\frac{1}{r} < \alpha < 1$, is given by

$$\left|E_n^1 C_n^2 - f\right| = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right)$$

Proof:

We have

$$\left\|E_n^1 C_n^2 - f\right\|_r = O\left\{\int_0^{2\pi} \left|E_n^1 C_n^2 - f\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\} = O\left\{\int_0^{2\pi} \left|E_n^1 C_n^2 - f\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$O(1) = O\left\{\int_0^{2\pi} \left|E_n^1 C_n^2 - f\right|^r dx\right\}^{\frac{1}{r}} . \; O\left\{\frac{1}{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)}\right\}$$

Hence

$$\left|E_n^1 C_n^2 - f\right| = O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}$$

for if not the right-hand side will be O(1), therefore

$$\left|E_n^1 C_n^2 - f\right| = O\left\{\left(\frac{1}{n+1}\right)^{\alpha} (n+1)^{\frac{1}{r}}\right\}$$

$$= O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right)$$

Corollary 2

If $r \to \infty$ in corollary 1, then the class $Lip(\alpha, r)$ reduces to the class $f \in Lip\alpha$ and the degree of approximation of a function $f \in Lip\alpha$, $0 < \alpha < 1$ is given by

$$\left\|E_n^1 C_n^2 - f\right\|_{\infty} = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}$$

Remark: An independent proof of above corollaries 1 can be obtained along the same lines of our theorem.

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8. REFERENCES

[1] G. Alexits, "Convergence problems of orthogonal series", Pergamon Press, London, 1961.
[2] P. Chandra, "Trigonometric approximation of functions in L_p - norm", J. Math. Anal. Appl., vol. 275, no. 1, pp.13-26, 2002.
[3] G. H. Hardy, "Divergent series, first edition", Oxford University Press, pp70, 1949.
[4] H. H. Khan, "On degree of approximation of functions belonging to the class Lip(α, p)'', Indian J. Pure App. Math., vol. 5, no.2, pp. 132-136, 1974.
[5] L. Leindler, "Trigonometric approximation in L_p - norm", J. Math. Anal. Appl.. vol. 302, pp. 129-136, 2005.
[6] L. McFadden, "Absolute Nörlund summability", Duke Math. J. vol. 9, pp. 168-207, 1942.
[7] K. Qureshi, "On the degree of approximation of a periodic function f by almost Nörlund means", Tamkang J. Math. 12(1981), no. 1, 35-38.
[8] K. Qureshi, "On the degree of approximation of a function belonging to the class Lip α ", Indian J. pure Appl. Math., Vol. 13, no. 8, pp. 560-563, 1982.
[9] K. Qureshi, and H. K. Neha, "A class of functions and their degree of approximation", Ganita, vol. 41, no. 1, pp.37, 1990.
[10] B. E. Rhodes, "On degree of approximation of functions belonging to Lipschitz class", Tamkang J. Math., vol. 34, no. 3, pp. 2450-247, (2003.
[11] B. N. Sahney, and D. S. Goel, "On the degree of continuous functions", Ranchi, University Math. Jour., vol. 4, pp. 50-53, 1973.
[12] E. C. Titchmarsh, "The Theory of functions", Oxford University Press, pp. 402-403, 1939.
[13] A. Zygmund, "Trigonometric series", 2nd rev. ed., Cambridge Univ. Press, Cambridge, vol. 1, 1959.