Grid ramification of set-based multiset ordering

Chinedu Peter[1] and Dasharath Singh[2]

[1]Mathematical Science & IT Department, Federal University Dutsin-ma
(Dutsin-ma, Nigeria).
macpee3@yahoo.com

[2](Former professor, Indian Institute of Technology, Bombay)
Department of Mathematics, Ahmadu Bello University (Zaria, Nigeria).
mathdss@yahoo.com

ABSTRACT—***The paper presents a grid form of the Jouannaud-Lescanne set-based multiset ordering, otherwise known as the grid-based set-based multiset ordering. A relatively more applicable definition of multiset ordering is presented. The grid approach has been used in this paper to prove some assertions, the pair-wise equality theorem for multisets, in particular.***

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1. Introduction

Since the appearance of a fundamental work of Dershowitz and Manna [1], various ramifications of multiset orderings have appeared ([2],[3], just to mention a few). The main objective of this paper is to present an extension of set-based multiset ordering of Jouannaud and Lescanne [3].

Intuitively, similar to the manner in which *reference lines* on a *map* are used in finding specific points, the elements of a multiset can be *referenced* by a *grid*. A grid can be viewed as representing a multiset partition. Each element (in this case, set) of the partition represents a *reference* on the grid. We introduce the concept of *difference grid* of two multisets in which some references are allowed to be empty. We thereafter define multiset ordering based on difference grid. As some references may be empty, the approach can be seen as an extension of the method used by Jouannaud and Lescanne [3] in which each set in the partition contains at least one element of the partitioned multiset. The flexibility potential of the approach is demonstrated in the proofs of a number of results. In particular, the pair-wise equality theorem for multisets is proved.

2. Preliminaries

Definition 1. *Multiset*

A *multiset* (*mset,* for short) is an unordered collection of objects, in which, unlike a set, multiple occurrences of objects are admitted. Formally, a multiset M, built upon a set S (called a domain set of M), is a mapping $M : S \to \mathbb{N}$, the set of all non-negative integers. $M(x)$ denotes the number of occurrences of x in M. The empty multiset over S, defined $M(x) = 0\ \forall x \in S$, is denoted by $\varnothing$. An object x is a member of M if $M(x) > 0$. Each individual occurrence of x in M is called an element of M. The set of all finite multisets built upon a domain set S is denoted by $\mathfrak{M}(S)$. It follows that a set is a multiset M if $M(x) = 1$ for each $x \in S$. The cardinality of a multiset, denoted $|M|$, is the sum of the multiplicities of all the objects in M. We shall assume all the multisets dealt with to be finite. ([4], for some further details).

Definition 2. *Equality of multisets*

Let E be a domain set of multsets M and N. Then $M = N$ if and only if $M(x) = N(x)$ for all $x \in E$.

Definition 3. *Order relation*

Let S be a set and R be a relation on S. R is a *quasi-order* (or *pre-order*) if it is reflexive and transitive; a *proper* (or *strict*) order if it is irreflexive and transitive; a *partial order* (or simply an *order*) if it is reflexive, antisymmetric and transitive; a *total order* if it is a partial order and *connected* (or *determinate*). A strict order is also called a *strict partial order* to distinguish it from a total order.

Definition 4. *Set ordering*

We define set ordering as follows: A set S is greater than a set T (written $S \ggg T$) if and only if $\forall x \in T\backslash S,\ \exists y \in S\backslash T$ such that $y > x$.

Assume, in the following examples, that alphabets represent incomparable elements while numbers represent comparable elements. Assume also that the alphabets are incomparable to the numbers. By this definition, the following relations hold:

(a) $\{4\} \ggg \{3, 2, 1, 0\}$
(b) $\{a, b, c\} \ggg \{a, b\}$
(c) $\{3, 2, a, b, c\} \ggg \{3, a, b, c\}$
(d) $\{4, a, b, c\} \ggg \{3, 2, a, b, c\}$

It can be verified that the set ordering so defined is a partial ordering.

Definition 5. *Incomparability relation [3]*

Let $\preccurlyeq$ be a partial ordering on a set S. We write $x \# y$ to mean the following:

$$\neg(x \prec y \text{ or } y \prec x \text{ or } x = y) \tag{1}$$

read as "x and y are *incomparable* under '$\preccurlyeq$'"for some elements x and y of S. Moreover, if there exists incomparable sets U and V, that is $\neg(U \lll V$ or $V \lll U$ or $U = V)$, then we write $U \# V$. Furthermore, we write $U \neg\lll V$ to mean U is not less than V. The same operators are valid for multisets. Consider now the following result.

Lemma 1. *Let the relation $S \# T$ hold for sets S and T. Then there exist incomparable elements x and y such that $x \in S \backslash T$, $y \in T$ and there exist incomparable elements v and w such that $v \in T \backslash S$ and $w \in S$.*

Proof. Given that S and T are sets such that $S \# T$ holds. By Definition 5, $S \neg\lll T$, $T \neg\lll S$ and $S \neq T$. Since $S \neg\lll T$, by Definition 4 $S \nsubseteq T$ and $\nexists y \in T$ such that $y \succ x$ $\forall x \in S \backslash T$. It follows that $\exists x \in S \backslash T$ such that $x \succ y$, $x = y$ or $x \# y$ for some $y \in T$. If $x \succ y$ $\forall x \in S \backslash T$, then $T \lll S \backslash T$. This implies $T \cup (S \cap T) \lll S \backslash T \cup (S \cap T)$, which implies $T \lll S$; and this is a contradiction of $S \# T$. Therefore, the statement $\exists x \in S \backslash T$ such that $x = y$ for some $y \in T$ or the statement $\exists x \in S \backslash T$ such that $x \# y$ for some $y \in T$, holds. If $x = y$ $\forall x \in S \backslash T$, then $S \backslash T \subseteq T$, which implies $S \subseteq T$. Again, this contradicts $S \# T$. Therefore $\exists x \in S \backslash T$ such that $x \# y$. Similarly since $T \neg\lll S$, there exists incomparable elements v and w such that $v \in T \backslash S$ and $w \in S$.

Definition 6. *The Dershowitz-Manna definition of multiset ordering [1]*

Let S be a domain set of multisets M and N. $M \lll N$ if there exist two multisets X and Y in $\mathfrak{M}(S)$ satisfying

(i) $\varnothing \neq X \subseteq N$,
(ii) $M = (N \backslash X) + Y$, and
(iii) $(\forall y \in Y)(\exists x \in X)[y \prec x]$.

In other words, $M \lll N$ if M is obtained from N by removing none or at least one element (those in X) from N, and replacing each such element x by zero or any finite number of elements (those in Y), each of which is strictly less than (in the ordering $\prec$) one of the elements x that have been removed. Informally, we say that M is smaller than N in this case. Similarly, $\ggg$ on $\mathfrak{M}(S)$ can be defined. For example, let $S = (\{0, 1, 2, ...\} = \aleph)$. Then, under the corresponding multiset ordering $\lll$ over $\aleph$, each of the following multisets $[3, 4]$, $[3, 2, 2, 1, 1, 1, 4, 0]$ and $[3, 3, 3, 3, 2, 2]$ is less than the multiset $[3, 3, 4, 0]$. The empty multiset $\varnothing$ is smaller than any multiset. It is also easy to see that for all y in N, if $[\exists x \in M \wedge x \succ y]$, then $M \ggg N$. We shall denote this ordering by $\lll_{\mathcal{DM}}$. See [3] and [5] for various ramifications of the Dershowitz-Manna Ordering.

Definition 7. *The Huet-Oppen definition of multiset ordering [2]*

Let M and N be multisets over a domain set S, then
$M \lll N$ if and only if
$M \neq N$ & $[M(x) > N(x) \implies (\exists y \in S,\ x \prec y$ & $M(y) < N(y))]$.

We shall denote this ordering by $\lll_{\mathcal{HO}}$.

Definition 8. *The Jouannaud-Lescanne set-based multiset ordering [3]*

Let M be a multiset over a domain set S. $\overline{M} = \{M_i,\ i = 1, 2, ..., p\}$ is the set-based partition of M in lexicographical order if and only if the following properties are satisfied:

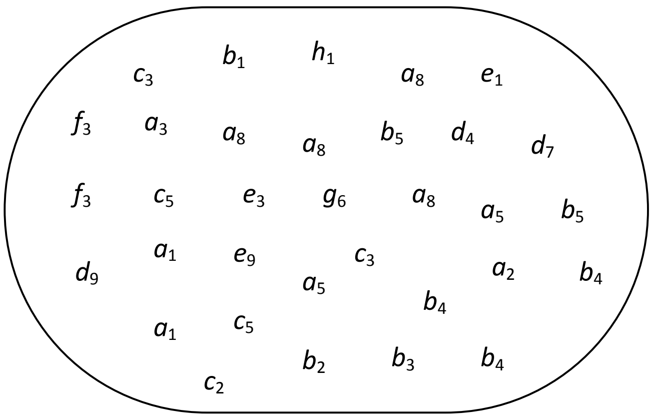
(i) $M_i(x) \leqslant 1$,
(ii) $x \in M_i$ and $y \in M_i \implies x$ and y are incomparable and
(iii) $\forall i \in [2, ..., p],\ x \in M_i \implies \exists y \in M_{i-1}$ such that $x \preccurlyeq y$.

Let M and N be multisets over a domain set S, and let $\overline{M}$ and $\overline{N}$ be the respective set-based partitions of M and N in lexicographical order, then $M \lll N$ if and only if $\overline{M}$ is lexicographically less than $\overline{N}$. We shall denote this ordering by $\lll_{\mathcal{JL}}$.

3. The grid of a partially ordered multiset

Let us consider the following diagrammatic representation of a multiset:

$M =$



Suppose that the elements of M are ordered as follows:

(i) $a_{i+1} \prec a_i,\ b_{i+1} \prec b_i,\ c_{i+1} \prec c_i,\ d_{i+1} \prec d_i,\ e_{i+1} \prec e_i,\ f_{i+1} \prec f_i,$ $g_{i+1} \prec g_i$ and $h_{i+1} \prec h_i$.
(ii) Elements with different letterings are incomparable.
(iii) Elements with the same lettering and index are equal.

Consider the following monotonically non-increasing sequences of elements of M:

$a_1 = a_1 \succ a_2 \succ a_3 \succ a_5 = a_5 \succ a_8 = a_8 = a_8 = a_8,$
$b_1 \succ b_2 \succ b_3 \succ b_4 = b_4 = b_4 \succ b_5 = b_5,$
$c_2 \succ c_3 = c_3 \succ c_5 = c_5,$
$d_4 \succ d_7 \succ d_9,$
$e_1 \succ e_3 \succ e_9,$
$f_3 = f_3,$
$g_6,$
$h_1.$
Let G_i be the set of all the i^{th} elements from the sequences. Thus,
$G_1 = \{a_1,\ b_1,\ c_2,\ d_4,\ e_1,\ f_3,\ g_6,\ h_1\},$
$G_2 = \{a_1,\ b_2,\ c_3,\ d_7,\ e_3,\ f_3\},$
$G_3 = \{a_2,\ b_3,\ c_3,\ d_9,\ e_9\},$
$G_4 = \{a_3,\ b_4,\ c_5\},$
$G_5 = \{a_5,\ b_4,\ c_5\},$
$G_6 = \{a_5,\ b_4\},$
$G_7 = \{a_8,\ b_5\},$
$G_8 = \{a_8,\ b_5\},$
$G_9 = \{a_8\},$
$G_{10} = \{a_8\}.$
It is immediate to see that G_i's are submultisets of M, which are, in fact, sets containing incomparable elements of M. Consequent upon the incomparability of the elements of each G_i, the number of submultisets of M in $[G_i]$ containing an element x equals the multiplicity of x in M, henceforth denoted by $\alpha_{M(x)}$. Moreover, from the above, we have

$$[G_i] = [G_1 \ggcurly G_2 \ggcurly G_3 \ggcurly G_4 \ggcurly G_5 \ggcurly G_6 \ggcurly G_7 = G_8 \ggcurly G_9 = G_{10}].$$

That is, $[G_i]$ is a non-increasing sequence. A permutation (an ordered sequence of elements with repetition allowed) is usually enclosed in square brackets. We call $[G_i]$ the *set-based grid of* M and each G_i, a *set-based grid reference* of M (or, simply a *reference* of M or M*-reference*). We give below, a formal definition of the concept. Viz: *Set-based grid of a partially ordered multiset.*

Definition 9.

Let $\preccurlyeq$ be a partial ordering defined on a set S and let M be a multiset of cardinality n over S. The permutation $[M_i]$ of subsets $M_1, M_2, ...M_m$ of M, $m \leqslant n$, is called *the set-based grid of* M if the following properties are satisfied:

(i) $\forall(x, y, i)[x \in M_i, y \in M_i] \implies [x \# y]$ (Incomparability property). From (1) it follows that M_i's are sets.

(ii) $\forall(i < j)[x \in M_j] \implies \exists y(y \in M_i)[x \preccurlyeq y]$ (Order property).

If for all x in M_j, $x \prec y$ (in the strict sense of the ordering) for some y in M_i, we say that M_j is strictly dominated by M_i, denoted $M_j \llcurly M_i$. Obviously,

$M_i = M_j$ if both contain the same elements. If $\prec$ is a total ordering on S, then any two elements of M are comparable (either equal or one greater than the other), and each M_i would contain a single element. In such a case, $m = n$. If $\prec$ is a strict partial ordering on S then $\exists i$ for which M_i contains at least two incomparable elements of M, and $m < n$. If no two elements of M are comparable, then all the elements of M belong to the only reference available, which is M itself. The references of M are ordered by $\ggcurly$. Thus, the grid of M is the monotonically non-increasing sequence of subsets of incomparable elements of M, in which the i^{th} submultiset contains the i^{th} element from each of all the longest possible monotonically non-increasing sequences of comparable elements of M. This construct is a variant of [3].

We next show that the second property in the above definition is equivalent to Definition 4 in the theorem that follows.

Lemma 2. *Let S and T be sets. The following statements are equivalent.*
(1) $S \ggcurly T$ *iff* $S \neq T$ *and* $\forall x \in T\ \exists y \in S$ *such that* $y \succcurlyeq x$.
(2) $S \ggcurly T$ *iff* $(x \in T \implies x \notin S) \implies (\exists y \in S \backslash T,\ y \succ x)$.

Proof. (1) $\implies$ (2). Let $x \in T$. $\exists y \in S$ such that $y \succcurlyeq x$. Thus, we have the following union of sets: $\{x \in T : y \succ x,\ y \in S\} \cup \{x \in T : y = x,\ y \in S\}$. By the additional constraint $x \notin S$ from (2), we get $\{x \in T : y \succ x,\ y \in S\} \cup \{x \in T : y = x,\ y \in S,\ x \notin S\}$. This reduces the set to $\{x \in T : y \succ x,\ y \in S\} \cup \varnothing$. Hence, $y \succ x$. We claim $y \notin T$. Suppose the contrary and assume $y \in T$. By the hypothesis, $\exists y_1 \in S$ such that $y_1 \succcurlyeq y$ and by the result obtained, $y \notin S$. This contradicts the hypothesis. Hence, $y \notin T$ holds.

(2) $\implies$ (1). It is easy to see that $T \neq S$. Let $x \in T$. If $x \notin S$, then from (2) $\exists y \in S \backslash T$ such that $y \succ x$. Also, if $x \in S$, then $\exists y \in S$ such that $y = x$ holds. The two statements imply $\exists y \in S$ such that $y \succcurlyeq x$. $\square$

4. Difference Grid

Definition 10.

Let $[M_i]$ and $[N_j]$ be the grids of the multisets M and N for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$, respectively. Let $q = \max\{m, n\}$ and, let $k = 1, 2, ..., q$. We construct the *difference grid* $[M_k, N_k]$ of M and N as follows:

(i) $M_q \neq \varnothing$ or $N_q \neq \varnothing$.

(ii) If $M_k \neq \varnothing$ and $N_k \neq \varnothing$ for any given k then $M_{k-1} \neq \varnothing$ and $N_{k-1} \neq \varnothing$.

M_k are the references of M in $[M_k, N_k]$ while N_k are the references of N in $[M_k, N_k]$. It is easy to see from (i) that $M_q \neq \varnothing$ if $n < m$, $N_q \neq \varnothing$ if $m < n$, and both are non-empty if $m = n$. In other words, if $m < n$ then $q = n$ and M_k is empty for $k = m + 1, m + 2, ..., n$, and if $n < m$ then $q = m$ and N_k is empty for $k = n + 1, n + 2, ..., m$. Unlike the references in the grid of a multiset, the references in a difference grid of two multisets are empty up to the number of references with which the grid with more references exceeds the grid with fewer references. Also, the difference grid of two multisets is a collection of all the references from the individual grids of the multisets.

Consider the multisets $M = \{5, 4, 2, 2, 1, a, a, a, a, b, b\}$ and $N = \{6, 3, 3, a, a, a, b, b, c\}$. The difference grid of M and N is

$\{5, a, b\}$	$\{4, a, b\}$	$\{2, a\}$	$\{2, a\}$	$\{1\}$
$\{6, a, b, c\}$	$\{3, a, b\}$	$\{3, a\}$	$\varnothing$	$\varnothing$

where the following two tables are tabular forms of the grids of M and N, respectively.

$\{5, a, b\}$	$\{4, a, b\}$	$\{2, a\}$	$\{2, a\}$	$\{1\}$

$\{6, a, b, c\}$	$\{3, a, b\}$	$\{3, a\}$

.

Definition 11.

Let $[M_k, N_k]$ be a difference grid of multisets M and N. A property p of references is said to be pair-wise iff whenever p is attributed to a reference M_i of $[M_k]$, then p is also attributed to the corresponding reference N_i of $[N_k]$. For instance, if p stands for 'non-empty', and M_i and N_j are non-empty for $i = j$ then M_i and N_j are pair-wise non-empty. Similarly, if p stands for 'disjoint', and M_i and N_j are disjoint for $i = j$ then M_i and N_j are pair-wise disjoint. We prove the following theorem:

Theorem 3. (Pair-wise equality theorem for multiset)
Let M and N be multisets. $M = N$ if and only if $M_i = N_j$ for all $i = j$ where M_i and N_j are the respective references of M and N in the difference grid $[M_k, N_k]$ of M and N.

Proof. Let M and N be multisets over a domain set S and suppose $M_i = N_j$ for all $i = j$. Since M_i are the references of M in the difference grid $[M_k, N_k]$, an element x of M belongs to M_i for some i. Since only incomparable elements belong to any set-based reference of a multiset, only one occurrence of x belongs to a reference containing x in the grid of M. Thus, $M(x) = \alpha_{M(x)}$. By hypothesis, N_j contains x for some j and only one occurrence of x belongs to N_j. Thus, $N(x) = \alpha_{N(x)}$. Hence, $\alpha_{M(x)} = \alpha_{N(x)}$ and it follows that $M(x) = N(x)\ \forall x \in S$. Therefore, $M = N$.

Conversely, let $M = N$. We claim $M_i = N_j$ for all $i = j$. Suppose the contrary holds. Let l_0 be the smallest positive integer for which the claim is not true. Then, either $M_{l_0} \ggg N_{l_0}$ or $M_{l_0} \lll N_{l_0}$ or $M_{l_0} \# N_{l_0}$. Suppose $M_{l_0} \ggg N_{l_0}$. By Definition 4, $(x \in N_{l_0} \implies x \notin M_{l_0}) \implies (\exists y \in M_{l_0} \backslash N_{l_0},\ y > x)$. By Property (ii) of Definition 9, $y \notin N_l\ \forall l > l_0$. Hence, $M(y) > N(y)$, thereby contradicting the equality of M and N. Again by Definition 4, if on the other hand $(x \in N_{l_0} \implies x \in M_{l_0}\ \forall x)$, then $N_{l_0} \subset M_{l_0}$. Let $u \in M_{l_0} \backslash N_{l_0}$. If $u \notin N_l$ for all $l > l_0$ then $M(u) > N(u)$. This is a contradiction to the equality of M and N. If, on the other hand, $\exists l > l_0$ such that $u \in N_l$, then by Property (ii) of Definition 9, there exists a positive integer k_1 less than l and an element z_1 of N_{k_1} such that $z_1 \geqslant u$. Again, there exists a positive integer k_2 less than k_1 and an element z_2 of N_{k_2} such that $z_2 \geqslant z_1 \geqslant u$. Continuing this way, it can be seen that there exists an element z of N_{l_0} such that $z \geqslant ... \geqslant z_2 \geqslant z_1 \geqslant u$. This implies $z \geqslant u$. Since $N_{l_0} \subset M_{l_0}$ then $z \in M_{l_0}$. This contradicts the incomparability of u and z in M_{l_0}. Similar argument holds if $M_{l_0} \lll N_{l_0}$. If $M_{l_0} \# N_{l_0}$ then by Lemma 1, there exists x and y where $x \# y$, such that $x \in M_{l_0} \backslash N_{l_0}$ and $y \in N_{l_0}$. If $x \notin N_l \; \forall l > i_0$, then $M(x) > N(x)$, and this is a contradiction of the equality of M and N. If on the other hand, there exists l where $l > l_0$ such that $x \in N_l$, then by Property (ii) of Definition 9, $\exists z \in N_{l_0}$ such that $z > x$. Also, $z \notin M_{l_0} \; \forall l \geqslant l_0$. Furthermore, $M_i = N_i \; \forall i < l_0$. It follows that $N(z) > M(z)$. This is a contradiction of the equality of M and N. Therefore, $M_i = N_j$ for all $i = j$. □

Theorem 4. *The grid of a partially ordered multiset is unique.*

Proof. Assume $[M_i]$ and $[N_j]$ are two grids of a partially ordered multiset M. By the pairwise equality theorem $M_i = N_j$ for all $i = j$. Thus, $[M_i] = [N_j]$. Therefore, M has a unique grid. □

5. Grid approach to set-based multiset ordering

Definition 12

Let M and N be multisets. $M \lll N$ if and only if the following property is satisfied:

If $M_i = N_i$ for all i such that $M_i \neq \varnothing$ and $N_i \neq \varnothing$, then $M_i = \varnothing$ for the remaining i; otherwise if $\exists i$ such that $M_i \neg\lll N_i$ then $\exists j$ with $j < i$ such that $M_j \lll N_j$, where M_i are the references of M, while N_i and N_j are the references of N in the difference grid $[M_k, N_k]$ of M and N.

The semicolon separates the definition into first and second parts. In the second part, '$\neg\lll$' denotes 'not less than'. Also in this part, the word *otherwise* emphasises that the first part does not hold. That is, $M_i = N_i$ does not hold for all i such that $M_i \neq \varnothing$ and $N_i \neq \varnothing$.

In the definition, the empty multiset $\varnothing$ can be seen to be smaller than every arbitrary non-empty multiset N. This is obvious, since as all the references in the grid of $\varnothing$ are empty, Properties (i) and (ii) of Definition 10 imply all the references in the grid of N are non-empty. It can also be seen that no element of $\varnothing$ is greater than or equal to any element of N. Also, the relations $\varnothing \lll \varnothing$ and $N \lll N$ do not exist by this definition. We shall refer to this variant by $\lll_{\mathcal{PS}}$.

Theorem 5. $\lll_{\mathcal{PS}}$ *is equivalent to* $\lll_{\mathcal{JL}}$.

Proof. Let $\lll_{\mathcal{JL}}$ be the Jouannaud-Lescanne set-based (defined using set-based partition) multiset ordering and let $\lll_{\mathcal{PS}}$ be the multiset ordering in Definition

12. Suppose M and N are multisets such that $M \lll_{\mathcal{PS}} N$. We first show that $M \neq N$. Let $[M_k, N_k]$ be the difference grid of M and N for $k = 1, 2, ..., p$. By the first property of Definition 10, $M_p \neq \emptyset$ or $N_p \neq \emptyset$. If either M_p or N_p is non-empty then $M \neq N$. If both M_p and N_p are non-empty then by the second part of Definition 12 or by the contrapositive of the first part of Definition 12, $M_k \neq N_k$ for some k. Therefore, by Theorem 3, $M \neq N$.

The monotonically non-increasing sequence of all the non-empty references in the grid of M from $[M_k, N_k]$ is a set-based partition $\overline{M}$ of M in lexicographic order, each set containing incomparable elements. Also, Property (iii) of $\lll_{\mathcal{JL}}$ is embedded in Property (ii) of Definition 9. Thus, Properties (i) and (ii) of Definition 9 for $\lll_{\mathcal{PS}}$ hold.

From the first part of Definition 12, $M_i = N_i$ for all i such that $M_i \neq \emptyset$ and $N_i \neq \emptyset$ implies $M_i = \emptyset$ for the remaining i, then the number of sets in the partition of N is greater than the number of sets in the partition of M. Thus, N is greater than M by lexicographic ordering. From the second part of Definition 12, $M_i \neg\lll_{\mathcal{PS}} N_i$ implies $\exists j$ with $j < i$ such that $M_j \lll_{\mathcal{PS}} N_j$, then the partition of N is greater than the partition of M by lexicographic ordering. Therefore, $M \lll_{\mathcal{JL}} N$.

Conversely, suppose $M \lll_{\mathcal{JL}} N$. Let p and q be the number of sets in the partitions of M and N, respectively. Lexicographic extension of the ordering entails the following:

(i) $M_i = N_i$ for all $i = j$ implies $p < q$.

(ii) $\exists i$ such that $M_i \neg\lll N_i$ implies $\exists j < i$ such that $M_j \lll N_j$.

Given (ii), either of the permutations $[M_i, \emptyset, \emptyset, ..., \emptyset, N_j]$ where $\emptyset$ appears $q - p$ times or, $[M_i, N_j, \emptyset, \emptyset, ..., \emptyset]$ where $\emptyset$ appears $p - q$ times is the difference grid of M and N. However, only the former is the difference grid of M and N for (i). □

Theorem 6. *$\lll_{\mathcal{PS}}$ is stronger than $\lll_{\mathcal{HO}}$.*

Proof. Let $\lll_{\mathcal{HO}}$ be the Huet-Oppen multiset ordering and let $\lll_{\mathcal{PS}}$ be the multiset ordering in Definition 12. Suppose $M \lll_{\mathcal{HO}} N$ holds. Let i_0 be such that $[M_{i_0}, N_{i_0}]$ is the first pairwise unequal references in the difference grid of M and N. We claim $N_{i_0} \ggg M_{i_0}$. Suppose the contrary, that is $N_{i_0} \neg\ggg M_{i_0}$. Consider the contrapositive of Definition 4 on M_{i_0} and N_{i_0}. Thus, $x \in M_{i_0} \backslash N_{i_0}$ implies there exist w and x such that either $w \in N_{i_0} \backslash M_{i_0}$ and $w \nsucc x$ or $w \succ x$ and $w \notin N_{i_0} \backslash M_{i_0}$. Either case implies $x \notin N_l \; \forall l > i_0$ by Property (ii) of Definition 9. Furthermore, $M_i = N_i \; \forall i < i_0$. Hence, $M(x) > N(x)$. This contradicts $M \lll_{\mathcal{HO}} N$.

Next, we show that the converse is false. Let $M = \{2, 1\}$ and $N = \{2, a\}$ be such that $a \# 1$ and $a \# 2$. $M \lll_{\mathcal{PS}} N$ since the reference $\{2, a\}$ of N is greater than either of the two references $\{2\}$ and $\{a\}$ of M. However, M and N are incomparable under $\lll_{\mathcal{HO}}$. □

Theorem 7. *$\lll_{\mathcal{PS}}$ is stronger than $\lll_{\mathcal{DM}}$.*

Proof. [3] contains a proof of the Dershowitz-Manna definition being equivalent to the Huet-Oppen definition. Since by Theorem 6, Definition 12 is stronger than the Huet-Oppen definition, it follows that Definition 12 is stronger than the Dershowitz-Manna definition. $\square$

Theorem 8. *Let $\mathfrak{M}(S)$ be the set of all finite multisets on S. Then $\lll_{\mathcal{PS}}$ is well-founded on $\mathfrak{M}(S)$ if and only if $\prec$ is well-founded on S.*

Proof. The 'only if' part is trivial. For the 'if' part, since $\lll_{\mathcal{PS}}$ is equivalent to $\lll_{\mathcal{JL}}$ and $\lll_{\mathcal{JL}}$ is wellfounded on $\mathfrak{M}(S)$, then $\lll_{\mathcal{PS}}$ is wellfounded on $\mathfrak{M}(S)$. $\square$

6. Concluding remarks

It is shown in Theorem 5 that the Jouannaud-Lescanne set-based multiset ordering and the grid-based set-based multiset ordering are equivalent. However, in view of Theorems 3 and 6, the grid-based multiset ordering seems to have a greater potential for applications.

7. References

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