Lagrange Formalism for Electromagnetic Field in Terms of Strengths $\vec{E}$ and $\vec{\mathrm{H}}$

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ABSTRACT--- ***In previous works, Weyl´s equation for neutrinos has been written in tensor form, in the form of non-linear Maxwell's like equations, through complex isotropic vector $\vec{F} = \vec{E} + i\vec{H}$. It has been proved that, the complex vector $\vec{F} = \vec{E} + i\vec{H}$ satisfies non-linear condition $\vec{F}^2 = 0$, equivalent to two conditions for real quantities $\vec{E}^2 - \vec{H}^2 = 0$ and $\vec{E}.\vec{H} = 0$, obtained by equating to zero sepatately real and imaginary parts in equality $\vec{F}^2 = 0$. Further, the Lagrange formalism for neutrino field in terms of complex isotropic vectors $\vec{F} = \vec{E} + i\vec{H}$ has been elaborated.***

In this work, by analogy with neutrinos field, we elaborated the Lagrange formalism for electromagnetic field in terms of strengths $\vec{E}$ and $\vec{H}$.

Keywords--- Lagrange formalism, electromagnetic field, strengths.

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1. INTRODUCTION

In previous works, via Cartan map, Weyl´s equation for neutrinos has been written in tensor form, in the form of non-linear Maxwell's like equations, through complex isotropic vector $\vec{\mathrm{F}} = \vec{\mathrm{E}} + i\vec{\mathrm{H}}$, satisfying non-linear condition $\vec{\mathrm{F}}^2 = 0$, equivalent to two conditions for real quantities $\vec{\mathrm{E}}^2 - \vec{\mathrm{H}}^2 = 0$ and $\vec{\mathrm{E}}.\vec{\mathrm{H}} = 0$, obtained by equating to zero sepatately real and imaginary parts in equality $\vec{\mathrm{F}}^2 = 0$.It has been proved, that the vectors $\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$ have the same properties as those of vectors $\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$, components of electromagnetic field. For example, under Lorentz relativistic transformations, they are transformed as components of a second rank tensor $\mathrm{F}_{\mu\nu}$. In addition, it has been proved, that the solution of these non-linear equations for free particle as well fulfils Maxwell's equations for vacuum (with zero at right side). Further, in the works that followed, Maxwell's equations for vacuum also have been written through complex isotropic vector $\vec{\mathrm{F}} = \vec{\mathrm{E}} + i\vec{\mathrm{H}}$ and the Lagrange formalism for neutrinos field and the Lagrange formalism for electromagnetic field in terms of complex isotropic vectors $\vec{\mathrm{F}} = \vec{\mathrm{E}} + i\vec{\mathrm{H}}$ has been elaborated.

In this work, in development of the above mentioned works, we shall elaborate the Lagrange formalism for electromagnetic field in terms of strengths $\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$.

2. RESEARCH METHOD

In previous works, using Cartan map, Weyl´s equation for neutrinos has been written in tensor form, in the form of non-linear Maxwell's like equations. On the basis of these equations, the properties of neutrinos field have been studied. Especially, it has been proved, that neutrinos field has the same properties as electromagnetic field. Using the same method, based on Cartan map, we shall write the Lagrange function for neutrinos field in terms of strengths $\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$, from which we shall derive the Lagrange function for electromagnetic field in terms of strengths $\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$.

3. CARTAN MAP

Definition and Algebraic Properties

We shall denote by C^n, the complex vector space of dimension "n". We shall consider only C^2, $\mathrm{C}^3$and C^4.

Elements of C^2 will be denoted by Geek syllables

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{1}$$

and will be called spinors.

Elements of C^3 will be denoted by Latin syllables

$$\vec{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}, \tag{2}$$

and will be called vectors.

Finally, elements of C^4 will be denoted by Latin syllables

$$j_\mu = \begin{bmatrix} j_0 \\ j_x \\ j_y \\ j_z \end{bmatrix}, \tag{3}$$

and will be called four vectors.

Definition1: Cartan map is a bilinear transformation b from space $C^2 \times C^2$ into space C^4, defined as follows

$$b^0(\xi, \tau) = -(\xi_1\tau_2 - \xi_2\tau_1) \tag{4}$$

$$\vec{b}(\xi, \tau) = \begin{bmatrix} \xi_1\tau_1 - \xi_2\tau_2 \\ i(\xi_1\tau_1 + \xi_2\tau_2) \\ -(\xi_1\tau_2 + \xi_2\tau_1) \end{bmatrix}. \tag{5}$$

From the definitions (4) and (5) follows that b^0 is antisymmetric and $\vec{b}$ is symmetric relative to the permutation ξ and τ, i.e.,

$$b^0(\xi, \tau) = -b^0(\tau, \xi) \tag{6}$$

$$\vec{b}(\xi, \tau) = \vec{b}(\tau, \xi). \tag{7}$$

In particular, for any spinor ξ

$$b^0(\xi, \xi) = 0 \tag{8}$$

Using the definitions (4)-(5), one can prove the following properties of Cartan map

Lemma1: For any spinors ρ, ξ, τ of space C^2, the following identities are verified

$$\vec{b}(\rho, \xi)\, \vec{b}(\tau, \tau) = -2\, b^0(\rho, \tau)\, b^0(\xi, \xi) \tag{9}$$

$$\vec{b}(\rho, \xi)\, \vec{b}(\xi, \tau) = -2\, b^0(\rho, \xi)\, b^0(\xi, \tau) \tag{10}$$

$$\vec{b}(\rho, \tau)\, \vec{b}(\xi, \tau) = \; b^0(\rho, \tau)\, b^0(\xi, \tau) \tag{11}$$

$$\vec{b}(\xi, \xi)\, \vec{b}(\tau, \tau) = -2 b^0(\xi, \tau)^2 \tag{12}$$

$$\vec{b}(\xi, \tau)\, \vec{b}(\tau, \xi) = \; b^0(\xi, \tau)^2 \tag{13}$$

$$\vec{b}(\xi, \xi)\, \vec{b}(\tau, \xi) = 0 \tag{14}$$

Lemma2 : For any two spinors ξ and τ of space C^2, the following identity is verified

$$\vec{b}(\xi, \xi) \times \vec{b}(\tau, \tau) = 2i\, b^0(\xi, \tau)\, \vec{b}(\xi, \tau) \tag{15}$$

Definition2: If

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in C^2 \qquad (16)$$

is a spinor, then the conjugate spinor ξ^* of the spinor ξ is defined as follows

$$\xi^* = \begin{bmatrix} -\overline{\xi_2} \\ \overline{\xi_1} \end{bmatrix} \in C^2 , \qquad (17)$$

Where $\overline{\xi_1}$, $\overline{\xi_2}$ are complex conjugates of spinor components ξ_1 and ξ_2.

Lemma3 : For any two spinors ξ and τ of space C^2, the following identities are verified

$$b^0(\xi, \tau^*) = \overline{b^0(\tau, \xi^*)} \ , \qquad (18)$$

$$\vec{b}(\xi, \tau^*) = \overline{\vec{b}(\tau, \xi^*)} \ , \qquad (19)$$

$$b^0(\xi^*, \tau^*) = \overline{b^0(\xi, \tau)}, \qquad (20)$$

$$\vec{b}(\xi^*, \tau^*) = -\overline{\vec{b}(\xi, \tau)}. \qquad (21)$$

Let us introduce vectors $\vec{F} \in C^3$ and $j_\mu \in C^4$ as follows:

$$\vec{F} = \vec{E} + i\vec{H} = i\vec{b}(\xi, \xi), \qquad (22)$$

$$j_\mu = b_\mu(\xi, \xi^*) \ . \qquad (23)$$

Here $\vec{E}$ and $\vec{H}$ are real vectors.

From formula (21) follows, that

$$\vec{F}^* = \vec{E} - i\vec{H} = \overline{i\vec{b}(\xi, \xi)} = i\,\vec{b}(\xi^*, \xi^*). \qquad (24)$$

Lemma4: From formulas (5) and (22) follows identity

$$\vec{F}^2 = \vec{F}.\vec{F} = 0. \qquad (25)$$

i.e., $\vec{F}$ is isotropic vector.

Formula (25) is equivalent to two conditions, obtained by equating to zero separately real and imaginary parts of equality $\vec{F}^2 = 0$

$$\vec{E}^2 = \vec{H}^2, \qquad (26)$$

$$\vec{E}.\vec{H} = 0. \qquad (27)$$

One can also prove, that

$$j_0 = \left[\frac{\vec{F}.\vec{F}^*}{2}\right]^{1/2} = |\vec{E}|,$$

$$\vec{j} = i\,\frac{\vec{F} \times \vec{F}^*}{2j_0} = \frac{\vec{E} \times \vec{H}}{|\vec{E}|} \ . \qquad (28)$$

Lemma5: For any spinor $\xi \in C^2$, the following identities are verified

$$j_0 = |\vec{E}| = |\xi|^2, \qquad (29)$$

$$\vec{j} = \frac{\vec{E}\times\vec{H}}{|\vec{E}|} = \bar{\xi}^{T}\vec{\sigma}\xi. \qquad (30)$$

Where $\bar{\xi}^{T}$ is the transposed conjugate of the spinor ξ and $\vec{\sigma}=(\sigma_x, \sigma_y, \sigma_z)$ are Pauli spin matrices.

From formulas (29)-(30) follows, that under Lorentz relativistic transformations j_μ transforms as a four vector. Vectors $\vec{E}$ and $\vec{H}$ transform as components of electromagnetic field, i.e., form a second rank tensor $F_{\mu\nu}$.

Lemma6: For any pair of spinors ξ and τ of space C^2 and any vector $\vec{v}$ the following identities are verified

$$b^0(\vec{v}.\vec{\sigma}\xi, \tau) = \vec{v}.\vec{b}(\xi, \tau), \qquad (31)$$

$$\vec{b}(\vec{v}.\vec{\sigma}\xi, \tau) = \vec{v}b^0(\xi, \tau) + i\,\vec{v}\times\vec{b}(\xi, \tau), \qquad (32)$$

$$\vec{b}(\vec{v}.\vec{\sigma}\xi, \xi) = (\vec{v}.\vec{s})\,\vec{b}(\xi, \xi). \qquad (33)$$

Here $\vec{s} = (s_x, s_y, s_z)$ are Proka spin matrices, with $s_i=i(\varepsilon_i)_{jk}$, where ε_{ijk} is the tridimensional antisymmetric tensor Levi-Cevita.

From formula (33) follows, that if ξ is eigenvector of operator $(\vec{v}.\vec{\sigma})$ with eigenvalue λ, then $\vec{b}(\xi, \xi)$ is eigenvector of operator $(\vec{v}.\vec{s})$ with the same eigenvalue λ.

Definition3: Let ξ be a spinor field and $\tilde{A}$, an operator acting on ξ. Let $\vec{b}$ maps spinor ξ on isotropic vector $\vec{F}=i\vec{b}(\xi, \xi)$. We shall say, that the operator $\tilde{A}$ commutes with Cartan map and becomes $\hat{A}$, acting on $\vec{F}$, if:

$$\hat{A}\,\vec{F} = i\hat{A}\,\vec{b}(\xi, \xi) = i\vec{b}(\tilde{A}\xi, \xi). \qquad (34)$$

From formula (34) follows, that if ξ is eigenvector of operator $\tilde{A}$ with eigenvalue λ, then $\vec{F}$ is eigenvector of operator $\hat{A}$ with the same eigenvalue λ; i.e., Cartan map conserves eigenvectors and eigenvalues.

Lemma7 : For any spinor ξ of space C^2, the following identities are verified

$$b^0(\vec{p}\xi, \xi) = -i\{\vec{D}\vec{b}(\xi, \xi)\}.\vec{v}, \qquad (35)$$

$$\vec{b}(\vec{p}\xi, \xi) = \vec{D}\vec{b}(\xi, \xi), \qquad (36)$$

Where $\quad \vec{v} = \frac{\vec{j}}{j_0} = \frac{\vec{E}\times\vec{H}}{\vec{E}^2}$

$$\vec{D} = -i\frac{\hbar}{2}\vec{\nabla}.$$

4. LAGRANGE FORMALISM FOR ELECTROMAGNETIC FIELD IN TERMS OF STRENGTHS

Weyl´s equation for neutrinos is

$$p_0\xi = (\vec{p}\vec{\sigma})\xi. \qquad (37)$$

Via Cartan map, introducing complex isotropic vector

$$\vec{F} = \vec{E} + i\vec{H} = i\vec{b}(\xi, \xi), \qquad (38)$$

equation (37) can be written in vector form as follows

$$D_0\vec{F} = i\vec{D}\times\vec{F} - v_i(\vec{D}F_i). \qquad (39)$$

Here

$$D_0 = \frac{i}{2}\frac{\partial}{\partial t},$$

$$\vec{D} = -\frac{i}{2}\vec{\nabla}, \quad (40)$$

$$\vec{v} = \frac{\vec{E}\times\vec{H}}{\vec{E}^2}.$$

Here we use the natural system of units in which c=ħ=1.

Separating real and imaginary parts in equation (39), we find

$$\begin{cases} \mathrm{rot}\vec{E} + \frac{\partial \vec{H}}{\partial t} = v_i(\vec{\nabla}H_i) \\ \mathrm{rot}\vec{H} - \frac{\partial \vec{E}}{\partial t} = -v_i(\vec{\nabla}E_i) \end{cases}. \quad (41)$$

In terms of isotropic vectors, Maxwell´s equations for vacuum take the form

$$\begin{cases} D_0\vec{F} = i\vec{D}\times\vec{F} \\ \vec{D}\vec{F} = 0 \end{cases}. \quad (42)$$

However, in general case the solution of Maxwell's equations does not satisfy non-linear isotropic condition $\vec{F}^2 = 0$, whereas the solution of equation (39) always satisfies this condition.

Spinor Weyl´s equation (37) can be obtained by variation principle from the Lagrange function

$$L = \frac{i}{2}\left(\xi\sigma^{\mu}\,\partial_{\mu}\xi^* - \partial^{\mu}\xi\sigma_{\mu}\xi^*\right). \quad (43)$$

Written through isotropic vectors $\vec{F} = \vec{E} + i\vec{H}$, formula (43) takes the form

$$L = \frac{i}{2}\left\{\left[D_0\vec{F} - i\vec{D}\times\vec{F} + v_i(\vec{D}F_i)\right]\vec{F}^* - \left[D_0\vec{F}^* + i\vec{D}\times\vec{F}^* + v_i(\vec{D}F_i^*)\right]\vec{F}\right\}/\left(\frac{\vec{F}\vec{F}^*}{2}\right)^{1/2}. \quad (44)$$

From formula (44), we easily obtain the Lagrange function for electromagnetic field by cancelling the non-linear part in formula (44)

$$L = \frac{i}{2}\left[D_0\vec{F} - i\vec{D}\times\vec{F}\right]\frac{\vec{F}^*}{\left(\frac{\vec{F}\vec{F}^*}{2}\right)^{1/2}}. \quad (45)$$

In terms of strengths $\vec{E}$ and $\vec{H}$, formula (45) can be written in the form

$$L = \frac{i}{4|\vec{E}|}\left\{\left[\vec{E}\frac{\partial \vec{H}}{\partial t} - \vec{H}\frac{\partial \vec{E}}{\partial t} + \vec{E}.\vec{\nabla}\times\vec{E} + \vec{H}.\vec{\nabla}\times\vec{H}\right] + i\left[\vec{E}\frac{\partial \vec{E}}{\partial t} + \vec{H}\frac{\partial \vec{H}}{\partial t} - \vec{E}.\vec{\nabla}\times\vec{H} + \vec{H}.\vec{\nabla}\times\vec{E}\right]\right\}. \quad (46)$$

5. FUNDAMENTAL DYNAMICAL VARIABLES

On the basis of Noether's theorem, from Lagrangian (46) we can derive expressions for dynamical variables.

Energy is determined by the formula

$$E = \int T^{00} d^3x, \quad (47)$$

where

$$T^{00} = \frac{\partial L}{\partial \vec{E},0}\frac{\partial \vec{E}}{\partial t} + \frac{\partial L}{\partial \vec{H},0}\frac{\partial \vec{H}}{\partial t}. \quad (48)$$

Replacing expression (46) in formula (48), we find

$$T^{00} = \frac{1}{4|\vec{E}|}\left[-\vec{H}\frac{\partial \vec{E}}{\partial t} + i\vec{E}\frac{\partial \vec{E}}{\partial t} + \vec{E}\frac{\partial \vec{H}}{\partial t} + i\vec{H}\frac{\partial \vec{H}}{\partial t}\right]. \quad (49)$$

Considering the solution in the form of plane waves

$$\vec{E} = \vec{E}^0 e^{-2ikt+2i\vec{k}\vec{r}}, \quad (50)$$

$$\vec{H} = \vec{H}^0 e^{-2ikt+2i\vec{k}\vec{r}}, \quad (51)$$

we obtain

$$T^{00} = k\left|\vec{E}\right|. \quad (52)$$

Similarly for momentum, we have

$$P^j = \int T^{0j} d^3x, \quad (53)$$

where

$$T^{0j} = \frac{\partial L}{\partial \vec{E}_{,0}} \frac{\partial \vec{E}}{\partial x_j} + \frac{\partial L}{\partial \vec{H}_{,0}} \frac{\partial \vec{H}}{\partial x_j}. \quad (54)$$

Using expression (46), we find

$$T^{0j} = \frac{1}{4|\vec{E}|}\left[-\vec{H}\frac{\partial \vec{E}}{\partial x_j} + i\vec{E}\frac{\partial \vec{E}}{\partial x_j} + \vec{E}\frac{\partial \vec{H}}{\partial x_j} + i\vec{H}\frac{\partial \vec{H}}{\partial x_j}\right]. \quad (55)$$

With the relations (50)-(51), we obtain

$$T^{0j} = k_j\left|\vec{E}\right|. \quad (56)$$

For charge, we find

$$Q = \int j^0 d^3x, \quad (57)$$

where

$$j^0 = i\left(\vec{E}\frac{\partial L}{\partial \vec{E}_{,0}} + \vec{H}\frac{\partial L}{\partial \vec{H}_{,0}}\right). \quad (58)$$

Using expression (46) and formulae (50)-(51), we obtain

$$j^0 = -\frac{i}{|\vec{E}|}\left(-\vec{E}\vec{H} + i\vec{E}\vec{E} + \vec{E}\vec{H} + i\vec{H}\vec{H}\right), \quad (59)$$

or

$$j^0 = \left|\vec{E}\right|. \quad (60)$$

In the same way, for the spin pseudo vector we find

$$\vec{S} = \frac{\vec{E}\times\vec{H}}{|\vec{E}|}. \quad (61)$$

6. DISCUSSION AND CONCLUSION

In previous works, spinor Weyl´s equation for neutrinos has been written in tensor form, in the form of non-linear Maxwell's like equations, through complex isotropic vector $\vec{F} = \vec{E} + i\vec{H}$. In the works that followed, these non-linear equations for neutrino have been investigated. Especially, the Lagrange formalism for neutrino field in tensor formalism has been elaborated. In this work, by analogy with neutrino field, we successfully wrote the Lagrange function for electromagnetic field in terms of strengths $\vec{E}$ and $\vec{H}$. From the obtained Lagrangian, we derived expressions for fundamental dynamical variables (energy, momentum, charge and spin) conserved in time.

7. REFERENCES

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