Asian Journal of Fuzzy and Applied Mathematics (ISSN: 2321 – 564X)
Volume 01– Issue 02, August 2013

A Rereading and a Mathematical Formulation of Deformations and Strains in Elasticity

Raoul Domingo Ayissi

Department of Mathematics, Faculty of Science, University of Yaoundé I, Cameroon

ABSTRACT— ***In this paper, we study, using tensors and a rigorous mathematical formulation, elastic strains in Cartesian coordinates. We also generalize this study to curvilinear coordinates. We then apply the results obtained in thermodynamics and we study an example.***

Keywords— Deformations, strains, displacements, fundamental metric tensor, isotropic body, invariant of strains, potential energy density.

1. INTRODUCTION

The problem of elasticity is one of the first to be met in the history of physical theories, and the scholars who have devoted themselves to this study for more than a century are numerous. The modern theories have thrown new light and allow the solution of a certain number of serious difficulties or old paradoxes. Despite all these works, there remain still enough serious uncertainties on the exact nature of the laws of interaction of the particles, the collection of which constitute a solid body. Arguments based on the fiction of a continuous and homogeneous solid retain then a real value. We know they can be applied correctly only if the deformations of the solid are very slowly variable from one point to another, but with this approximation, these methods still make their contribution. The continuous theory of solids will be correct so long as the deformations can be considered uniform in domains of some tens of angstroms. Under these conditions, the properties of the solid differ only a very little from what one observes on a large scale, and we can abstract from the details of internal structure.

The objective of the present work is to study, using tensors and rigorous mathematical formulations, strains in Cartesian coordinates, and to generalize the results obtained to curvilinear coordinates. We also show how we can apply those results in thermodynamics using the important notion of potential energy density for a strained solid.

The paper organizes as follows:

In section 2, we study deformations in Cartesian coordinates.

In section 3, we give the general definition of strains.

In section 4, we study the invariants of the strain.

2. DEFORMATIONS IN CARTESIAN COORDINATES

Latin indexes $i, j, k, \ldots$ range from 1 to 3. We adopt the Einstein summation convention:

$$\sum_\alpha A^\alpha B_\alpha = A^\alpha B_\alpha.$$

Let us take orthogonal rectilinear axes x^1, x^2, x^3 and consider in the undeformed solid two neighboring points P and *P'* with coordinates $x^i, x^i + dx^i$. The square of the distance between them is given by:

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \qquad (1)$$

The *fundamental metric tensor* g_{ik} is reduced to the table:

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we can, in these circumstances, cease to distinguish covariant and contravariant components of a tensor.
Suppose now that each of the points of the body is subjected to a displacement **u**, varying from one point to another. The point *P* comes to a point Q with coordinates X^i such that :

$$X^i = x^i + u^i. \qquad (2)$$

The displacement u^i will not be treated as infinitely small. It can have any magnitude at all. These displacements have as their result a certain motion of the whole body and a *deformation.*
A **deformation** is the transformation of a body from a reference configuration to a current configuration. A configuration is a set containing the positions of all particles of the body.
We want to isolate the deformation terms. *Mathematically, the body is deformed if the distance between two neighboring points is changed*. The square of the distance *PP'* was ds^2 before the deformation. The square dS^2 of the distance *QQ'* between two displaced points is:

$$dS^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \qquad (3)$$

We will suppose that the displacements u^i are continuous differentiable functions of the original coordinates x^i. We can then set:

$$dX^l = dx^l + du^l = dx^l + \frac{\partial u^l}{\partial x^k} dx^k \qquad (4)$$

Hence:

$$dS^2 = ds^2 + \frac{\partial u^l}{\partial x^k} dx^l dx^k + \frac{\partial u^l}{\partial x^i}\frac{\partial u^l}{\partial x^k} dx^i dx^k. \qquad (5)$$

In this equation, in conformity with our usual procedure, we suppose the summations are made *independently* over all the indices $l, k,$ and i. A term $dx^1 dx^2$ appears twice, with $k = 1, l = 2$ and $l = 1, k = 2$. This result may be rewritten as follows:

$$\delta(ds)^2 = dS^2 - ds^2 = e_{ik} dx^i dx^k$$

$$e_{ik} = \frac{\partial u^i}{\partial x^k} + \frac{\partial u^k}{\partial x^i} + \frac{\partial u^l}{\partial x^i}\frac{\partial u^l}{\partial x^k}. \qquad (6)$$

The substitution of the e_{ik} in (6) gives back (5) except for a change of notation, dummy index being replaced in the first group of terms.

When the displacements **u** are given, the collection of quantities e_{ik} forms a tensor with two indices, attached to the point Q of the strained body and referred to the Cartesian coordinate X^i of this point. This tensor e_{ik} defines *mathematically the strain*.

A ***strain*** is a normalized measure of deformation representing the displacement between particles in the body relative to a reference length.
We distinguish:

$$\text{extensions:} \quad e_{11} = 2\frac{\partial u^1}{\partial x^1} + \left(\frac{\partial u^1}{\partial x^1}\right)^2 + \left(\frac{\partial u^2}{\partial x^1}\right)^2 + \left(\frac{\partial u^3}{\partial x^1}\right)^2$$

$$\text{shears:} \quad e_{12} = \frac{\partial u^1}{\partial x^2} + \frac{\partial u^2}{\partial x^1} + \frac{\partial u^1}{\partial x^1}\frac{\partial u^1}{\partial x^2} + \frac{\partial u^2}{\partial x^1}\frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^1}\frac{\partial u^3}{\partial x^2}. \qquad (7)$$

This tensor is obviously symmetric so that in reality it has six independent components.
If we want to obtain a general result, we have to keep these expressions in their totality with the second-degree terms. So, the equations are rigorous. If one were interested only in small strains, the partial derivatives $\frac{\partial u^i}{\partial x^k}$ could be supposed infinitesimals of the first order, and one would arrive at the elementary formulas:

$$e_{11} = 2\frac{\partial u^1}{\partial x^1} \; (= 2e_1), \quad e_{12} = \frac{\partial u^1}{\partial x^2} + \frac{\partial u^2}{\partial x^1} \; (= e_6). \qquad (8)$$

Voigt's notation is often used for the six components of the strain. It has the inconvenience of bringing in a factor 2 which makes components differ from the components of the tensor:

$$\text{strain tensor:} \quad e_{11}\; e_{22}\; e_{33}\; e_{23}\; e_{31}\; e_{12}$$

$$\text{W.voigt's notation:} \quad 2e_1\; 2e_2\; 2e_3\; e_3\; e_5\; e_6. \qquad (9)$$

Up to now we have considered the coordinates X^i of a point of the strained body referred to invariable orthogonal axes. We can try to proceed in another way and identify a point of the strained body by means of coordinates x^i which define the initial position of the point before the strain. This comes to the same thing as choosing new coordinate surfaces $\bar{X}^i$ in the strained body which will be curvilinear. These surfaces are derived from the original rectangular coordinate planes by supposing them to be carried along by the body in its motion and deformed with it. In these deformed curvilinear coordinates, the coordinates of a point keep numerically the same values:

$$\bar{X}^i = x^i. \qquad (10)$$

The coordinate transformation for the strained body would be made according to the following equations which allow us to pass from the rigid axes X^i to the deformed axes $\bar{X}^i$:

$$dX^r = a_i^r dx^i = a_i^r d\bar{X}^i \qquad (11)$$

with coefficients :

$$a_i^r = \delta_i^r + \frac{\partial u^r}{\partial x^i}.$$

In these curvilinear axes carried along by the body in the course of its deformation, the square of the distance between two neighboring points is:

$$dS^2 = h_{ik} dx^i dx^k = h_{ik} d\bar{X}^i d\bar{X}^k \qquad (12)$$

with

$$h_{ik} = \delta_{ik} + e_{ik}.$$

The result comes directly from application of equations (6).

We will find the geometrical significance of the deformation coefficients from this expression. If at a point in space we have units of length l_i along the different axes and angles Θ_{ik} between these axes which are not in general orthogonal, we find:

$$g_{ii} = l_i^2, \quad g_{ik} = l_i l_k \cos\Theta_{ik}. \qquad (13)$$

We apply these results to the curvilinear coordinates $\bar{X}^i$ whose metric tensor is h_{ik}:

$$h_{ii} = l_i^2 = 1 + e_{ii}, \quad h_{ik} = l_i l_k \cos\Theta_{ik} = e_{ik}. \qquad (14)$$

Before the deformation we were in an orthogonal Cartesian system with a common unit of length for the three axes: $l_1 = l_2 = l_3 = 1.$

After the deformation we have, then, variations of length of the segments which were originally equal to 1 and directed along the axes:

$$\vartriangle l_1 = \sqrt{1 + e_{11}} - 1, \qquad (15)$$

and the angles Θ_{ik} are no longer right angles. Let us limit ourselves to the case of a very small strain. The equations reduced to (8) will be sufficient, and we will have:

$$\vartriangle l_1 \approx \frac{1}{2} e_{11} = e_1 = \frac{\partial u^1}{\partial x^1}. \qquad (16)$$

The extension along the 1- axis will then be given by e_1, and it is just in this way that this quantity is usually defined for very small strains.

For the cosines of the angles we find:

$$l_1 l_2 \cos\Theta_{12} \approx \cos\Theta_{12} = e_{12} = \frac{\partial u^1}{\partial x^2} + \frac{\partial u^2}{\partial x^1}. \qquad (17)$$

This agrees with the elementary definition of the shears:

$$\Theta_{12} = \frac{\pi}{2} - \alpha_{12}, \qquad \alpha_{12} \approx e_{12}. \qquad (18)$$

According to the problem we are treating, we will sometimes be interested in supposing the strained body is referred to rigid orthogonal axes $\left(X^k\right)$ and sometimes to the curvilinear coordinates $\left(\bar{X}^k\right)$ embedded in the body. The transformation equations (11) allow us to write immediately the transition from the first to the second system of components for an arbitrary tensor. For example, according to whether we have contravariant or covariant tensors:

$$T^{rs} = a_i^r a_j^s \bar{T}^{ij}$$
$$\bar{T}_{ij} = a_i^r a_j^s T_{rs}. \qquad (19)$$

Let us point out an important conclusion: we cannot have in a solid a completely arbitrary combination of strains. If we take the case of the small strains (8), we find the relations:

$$\frac{\partial^2 e_{22}}{\partial x_3^2}+\frac{\partial^2 e_{33}}{\partial x_2^2}-2\frac{\partial^2 e_{23}}{\partial x_2 \partial x_3}=0$$

and two others of the type. These equations are useful to recall, for they set up connections between the possible combinations of strains.

3. THE GENERAL DEFINITION OF STRAINS

The discussion we have just written in Cartesian coordinates may be written without too much trouble for any coordinates.

Let us consider in our space a solid at rest referred to coordinates $k(x^1,x^2,x^3)$. We also consider two neighboring points:

$$P(x^1,x^2,x^3) \quad \text{and} \quad P'\left(x^1+dx^1,x^2+dx^2,x^3+dx^3\right).$$

The square of distance between them is given by the quadratic form:

$$ds^2 = g_{ij}dx^i dx^j, \qquad (20)$$

where the g_{ij} stand for the *fundamental metric tensor*.

We now suppose that the body is subjected to a *deformation* in the sense that the distance between two neighboring points is changed during a motion of the whole body. The points *P*

and *P'* of the body will occupy different positions, *Q* and *Q'*. We take the coordinates of the point *Q* to be X^i and *Q'* to be $X^i + dX^i$ with respect to the coordinate system *k*, about which we make the essential supposition that it remains invariant during the deformation. The square of the distance *QQ'* is now :

$$dS^2 = g_{RS}X^R X^S, \qquad (21)$$

the notation g_{RS} showing that we are dealing with values of the *g* taken at the point $Q(X^i)$ and not $P(x^i)$.

To define the strain at every point we must give ourselves X^1, X^2, X^3 as functions of the x^1,x^2,x^3. We can then develop equation (21) and write :

$$dS^2 = g_{RS}a_i^R a_j^S dx^i dx^j = h_{ij}dx^i dx^j \qquad (22)$$

where we set:

$$a_i^R = \frac{\partial X^R}{\partial x_i}, \quad dX^R = a_i^R dx^i. \qquad (23)$$

To avoid all confusion, we must recall that these equations do not represent a change of coordinates but a displacement of the solid body with respect to a rigid coordinate system.

From the method itself by which we have derived them, we see that the coefficients h_{ij} form a twice covariant tensor when the law of strain is given. We obtain the ***strain tensor*** by taking the difference:

$$e_{ij} = h_{ij} - g_{ij}. \qquad (24)$$

We must now try to go into details of the general expressions (24) which define the strains. To that end, we introduce the displacement u^i of a point of the body by setting:

$$X^i = x^i + u^i. \qquad (25)$$

We note immediately that u^i is a contravariant tensor only if the strain is infinitely small. The a_i^R do not form a tensor. We obtain then for these last coefficients the following values:

$$a_i^R = 1 + \frac{\partial u^i}{\partial x^i}\ (R = i)$$

$$a_i^R = \frac{\partial u^R}{\partial x^i}\ (R \neq i).$$

In the expression for the e_{ij} we must then distinguish the particular terms for which the two indices of a coefficient *a* become equal. This allows us to write:

$$e_{ij} = a_i^R a_j^S g_{RS} - g_{ij}$$

$$= g_{IJ} - g_{ij} + \frac{\partial u^S}{\partial x^j} g_{IS} + \frac{\partial u^R}{\partial x^i} g_{JR} + \frac{\partial u^R}{\partial x^i}\frac{\partial u^S}{\partial x^j} g_{RS}. \qquad (26)$$

The exceptional terms written in the first place come from the cases *R=i* and *S=j*. We will develop this expression, keeping terms of the first and the second order. We will have from the Taylor series development:

$$g_{RS} = g_{rs} + \frac{\partial g_{rs}}{\partial x^p} u^p + \frac{1}{2}\frac{\partial^2 g_{rs}}{\partial x^p x^q} u^p u^q + \cdots$$

Hence we get:

$$e_{ij} = \frac{\partial g_{ij}}{\partial x^p} u^p + \frac{\partial u^s}{\partial x^j} g_{ir}$$

$$+\frac{\partial u^r}{\partial x^i}\frac{\partial u^s}{\partial x^j} g_{rs} + \frac{\partial u^r}{\partial x^i} u^p \frac{\partial g_{rj}}{\partial x^p} + \frac{\partial u^s}{\partial x^j} u^p \frac{\partial g_{si}}{\partial x^p} + \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^p x^q} u^p u^q + \cdots$$

The first-order terms grouped in the first line may be rewritten without difficulty in the following form:

$$e_{ij} = \frac{Du_i}{Dx^j} + \frac{Du_j}{Dx^i}, \quad \text{(very small strains)}. \qquad (27)$$

Likewise, after some simple calculations, we can arrive at the following form:

$$e_{ij} = \frac{Du_i}{Dx^j} + \frac{Du_j}{Dx^i} + g_{rs}\frac{Du^r}{Dx^i}\frac{Du^s}{Dx^j}$$

$$+u^p \Gamma_{ps}^r \left[g_{ir}\frac{\partial u^s}{\partial x^j} + g_{jr}\frac{\partial u^s}{\partial x^i} \right] + u^p u^q \left[\frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^p x^q} - g_{rs}\Gamma_{ip}^r \Gamma_{jq}^s \right] + \cdots \qquad (28)$$

The second line and all terms of higher order (which we have not written) disappear if the space is Euclidian and we take rectilinear axes, for all the derivatives of *g* and the Γ then vanish. Recall that the Γ stand for the *Christoffel symbols of the second kind* associated to our fundamental metric tensor.

The global value of the strain given by equation (28) is rather difficult to discuss, but we can arrange things so as to reduce it all to the simple form (27) of the small deformations. The point we wish to show is the following: if we take a body already under strain and make a new displacements δu in the $\bar{k}$ system of axes, embedded in the solid and deformed by the whole preceding strain, then the variations of the strain tensor are given by a simple equation of type (27):

$$\delta e_{ij} = \frac{D\delta u_i}{D\bar{X}^j} + \frac{D\delta u_j}{D\bar{X}^i}. \qquad (29)$$

We consider a body which has undergone a first deformation. The neighboring points with original coordinates x^i and $x^i + dx^i$ now occupy new positions X^i and $X^i + dX^i$.

After this strain, the square of their separation has the value:

$$dS^2 = g_{RK} X^R X^K, \qquad (30)$$

according to formula (21).

Suppose we make a very small variation of the deformation: the coordinates X^i are increased by $\delta X^i = \delta u^i$. Those of the point $X^i + dX^i$ become $X^i + dX^i + d\delta u^i$. The dS^2 then undergoes a variation :

$$dS^2 = g_{RK} X^r d\delta u^k + g_{RK} X^k d\delta u^r + \delta g_{RK} X^r dX^k, \qquad (31)$$

or by changing the names of dummy indices of the last two terms

$(r \to l, \quad k \to s \text{ in the second term}, k \to s \text{ in the third term})$:

$$\delta dS^2 = \left\{ g_{RK}\frac{\partial \delta u^k}{\partial X^s} + g_{SL}\frac{\partial \delta u^l}{\partial X^r} + \frac{\partial g_{RS}}{\partial X^p}\delta u^p \right\} dX^r dX^s.$$

$$\delta dS^2 = \left\{\frac{D\delta u_r}{DX^s} + \frac{D\delta u_s}{DX^r}\right\} dX^r dX^s. \qquad (32)$$

But we can also express the dS^2 by means of the dx^k by using equation (22). During a variation of the strain the dx^k remain unchanged and we obtain:

$$\delta dS^2 = \delta h_{ij} dx^i dx^j = \delta e_{ij} dx^i dx^j. \qquad (33)$$

If we compare the expressions (32) and (31), we draw from them immediately:

$$\delta e_{ij} = \left\{\frac{D\delta u_r}{DX^s} + \frac{D\delta u_s}{DX^r}\right\} a_i^r a_j^s. \qquad (34)$$

The coefficients a_i^r are, as before, the derivatives $\frac{\partial X^r}{\partial x^i}$. This equation will be very useful in what follows. We will see how it can be interpreted. Up to now we have always dealt with the coordinates X^i of the strained body referred to invariable *k* axes. We can proceed in another way and define a point of the strained body by the three numbers x^1, x^2, x^3 which gave its position in the initial state. Thus we choose in the strained body new coordinate surfaces forming a system of axes $\bar{K}$, and in reference to these, the coordinate of a point is numerically equal to x^1, x^2, x^3. This system of $\bar{K}$ axes is that which is derived from the initial axes *k* by supposing them embedded in the body and deformed with it. For these $\bar{K}$ axes we will have then:

$$\bar{X}^i = x^i, \qquad dS^2 = h_{ij} dx^i dx^j = h_{ij} d\bar{X}^i d\bar{X}^j \quad . \qquad (35)$$

The coordinate transformation is then made according to the formulas:

$$dX^r = a_i^r dx^i = a_i^r d\bar{X}^i, \qquad (36)$$

and a tensor t_{ij} transforms by the rule:

$$\bar{t}_{ij} = a_i^r a_j^s t_{rs}. \qquad (37)$$

If we compare equations (34) and (37), we arrive at the relation (29) which we have stated.
These equations represent the generalization for curvilinear coordinates of the expressions we have given in Cartesian coordinates.

4. ISOTROPIC BODY AND INVARIANTS OF THE STRAIN

When we consider a body that is initially homogeneous and isotropic, it is of interest to see what invariant combinations, with respect to coordinate transformations we can form with the components of the strain tensor. These invariants alone ought to enter the expression for the potential energy, for example. The tensor notations show us immediately that the following combinations are invariant. They are written with the help of mixed components:

$$e_i^k = e_{ij} g^{kj}.$$

We will have:

$$\begin{aligned} &\text{First degree invariant:} \quad I_1 = e_i^i = e_1^1 + e_2^2 + e_3^3 \\ &\text{Second degree invariant:} \quad I_2 = e_i^k e_k^i \\ &\text{Third degre invariant:} \quad I_3 = e_i^k e_k^l e_l^i, \end{aligned} \qquad (38)$$

etc.
As another invariant of the second degree we will have also I_1^2; and of the third degree we find $I_1^3, I_1 I_2$, etc.
Let us now write the complete expressions for I_2 and I_3, for these will prove to be very useful:

$$\begin{aligned} I_2 &= e_i^k e_k^i = e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{23}^2 + 2e_{31}^2 \\ I_3 &= e_i^k e_k^l e_l^i = e_{11}^3 + e_{22}^3 + e_{33}^3 + 3e_{11}(e_{12}^2 + e_{13}^2) + 3e_{22}(e_{21}^2 + e_{31}^2) \\ &\quad + 3e_{33}(e_{31}^2 + e_{32}^2) + 6e_{12}e_{23}e_{31}. \end{aligned} \qquad (39)$$

In these expressions we have assumed orthogonal Cartesian axes so that the covariant, contravariant, and mixed components are all equal among themselves. The expressions are obtained by taking into account that a coefficient e_{12}, for example, comes in several times in the sum (38), since e_{12} and e_{21} are equal.
None of these invariants gives the variation of the volume. This is obtained most easily from the functional determinant:

$$\Delta = \frac{D(X^1X^2X^3)}{D(x^1x^2x^3)} = |a_i^r| = \begin{vmatrix} 1+\frac{\partial u^1}{\partial x^1} & \frac{\partial u^1}{\partial x^2} & \frac{\partial u^1}{\partial x^3} \\ \frac{\partial u^2}{\partial x^1} & 1+\frac{\partial u^2}{\partial x^2} & \frac{\partial u^2}{\partial x^3} \\ \frac{\partial u^3}{\partial x^1} & \frac{\partial u^3}{\partial x^2} & 1+\frac{\partial u^3}{\partial x^3} \end{vmatrix}. \qquad (40)$$

In the first approximation this determinant is equal to:

$$1+\frac{\partial u^1}{\partial x^1}+\frac{\partial u^2}{\partial x^2}+\frac{\partial u^3}{\partial x^3} \approx 1+\frac{1}{2}I_1. \qquad (41)$$

But we must not be content with this result since we are trying throughout to set up rigorous formulas. The original volume was:

$$dv = \sqrt{g}dx^1dx^2dx^3,$$

where g stands for the determinant of the g_{ik}.

The final volume, in the embedded and strained axes $\bar{X}^i = x^i$, is written:

$$dv' = \sqrt{h}d\bar{X}^1d\bar{X}^2d\bar{X}^3 = \sqrt{h}dx^1dx^2dx^3,$$

and we have the equation:

$$\Delta = \frac{\sqrt{h}}{\sqrt{g}}$$

which represents a particular case of the general case concerning the transformation of the determinant g under changes of axes. We have, on the other hand, according to (24):

$$h_{ij} = g_{ij} + e_{ij}, \qquad (42)$$

and this allows us to form the expression for the determinant h.

We make the calculation only for orthogonal Cartesian axes where equations simplify as follows:

$$|g| = 1$$

$$\Delta^2 = |h_{ij}| = |\delta_{ij}+e_{ij}| = \begin{vmatrix} 1+e_{11} & e_{12} & e_{13} \\ e_{21} & 1+e_{22} & e_{23} \\ e_{31} & e_{32} & 1+e_{33} \end{vmatrix} \qquad (43)$$

$$= 1+I_1+\frac{1}{2}\left(I_1^2-I_2\right)-\frac{I_1I_2}{2}+\frac{I_1^3}{6}+\frac{I_3}{3}.$$

We then obtain Δ^2 as a function of the invariants I of the strain, and we derive from it, by taking the square root, an approximate development:

$$\Delta = 1+\frac{I_1}{2}-\frac{I_2}{4}+\frac{I_1^2}{8}+\cdots \qquad (44)$$

The expression (43) is exact, but (44) represents only a first approximation. We see we must not confuse the invariant I_1 with twice the cubical dilation $(\Delta - 1)$.

The relative variation of the volume is:

$$\frac{\delta v}{v} = \Delta - 1 = \frac{I_1}{2}-\frac{I_2}{4}+\frac{I_1^2}{8}+\cdots \qquad (45)$$

5. POTENTIAL ENERGY DENSITY FOR A STRAINED SOLID

The principles of thermodynamics allow the definition of the energy of a given body as a function of the mechanical variables (strains) and the temperature. For, if we call dT_e the work of the external forces, dQ, the heat supplied, dE_{kin}, the variation of the total kinetic energy of the body during an infinitely small transformation, we have:

$$dQ + dT_e = dE_{kin} + dU,$$

where U is the internal energy of the body under consideration.

The kinetic energy E_{kin} is that which corresponds to the motion of the solid body as a whole; the energy of thermal agitation is calculated in U.

On the other hand, the second law gives us for a reversible transformation:

$$dQ - TdS = 0,$$

S being the entropy of the body being studied. We derive hence the following relations:

$$dU - TdS = dT_e - dE_{kin} \tag{46}$$

$$d(U - TS) + SdT = dT_e - dE_{kin}. \tag{47}$$

We will see that we can eliminate the temperature variable in the two following cases :

1. Adiabatic transformations.

The body is supposed to be thermally insulated so that it cannot exchange energy with the exterior. This case is realized practically in the study of rapid vibratory motions such that heat flow does not have time to equalize temperatures. We use the relation (46) with dS.

We see then that the function U plays the role of a potential energy for the solid body. By dividing the body into small volume elements we can define an *energy density* $\mathcal{E}$ which will satisfy the condition:

$$U = \int_V \mathcal{E} d\tau. \tag{48}$$

The coordinates used here are, of course, the *coordinates embedded* in the body in its motion so that the limits of equation (47) are fixed and do not depend on the strains. The symbol $\mathcal{E}$ represents the energy of the strained system *referred to an initial unit volume* v_0.

2. Isothermal transformation

The body is supposed to be contained in a constant temperature bath. The motions must be slow enough that the heat exchanges between the body and the thermostat always have time to take place to keep the temperature uniform. We use then the relation (47)

with $dT = 0$ and see that the function $(U - TS)$ (thermodynamic potential) plays the role of a potential energy. We define an *energy density* n by the condition:

$$U - TS = \int_V n d\tau \tag{49}$$

in the embedded coordinates.

We see that in either of these two cases, we can define a density $\mathcal{E}$ or n. Since these two quantities play exactly the same role for the following calculations, we will not distinguish them any more, and we will speak of an *energy density* $\mathcal{E}$. To obtain the laws of elasticity, there remains one more step to take by specifying the way the energy density is expressed in terms of the strains. We can proceed at this point only by approximations and form a series development holding for small deformations starting from a given state. The choice of the initial state is the natural state of the solid body when it is subjected to no external force.

The restriction is not a happy one, for it limits many applications of the formulas that are thus established. We prefer to suppose that the initial state is quite arbitrary, either natural or strained. We must distinguish two categories of external forces :

I. Steady external forces, which give the initial strain of the body and which are supposed to act in a permanent way on the body.

II. Accidental external forces, which can introduce increments in the strain starting from the initial state of the body (strained by the steady forces of the first group).

We will make a development with respect to the strains for the internal elastic energy density of the body, measured from the initial state we have just defined. This development will be made up of terms $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots,$ homogeneous of degrees $0, 1, 2, 3, \ldots$ in the strains. For the moment we limit ourselves to second-degree terms:

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \cdots = \mathcal{E}_0 + \Omega^{ij} e_{ij} + \Lambda^{ij,hk} e_{ij} e_{hk} + \cdots \tag{50}$$

The linear terms $\mathcal{E}_1$ grouped in the sum $\Omega^{ij} e_{ij}$ will exist only if in the initial state the body is already subjected to external forces, that is, if the steady external forces of group *I* do not vanish. For, when these external forces *I* vanish, the energy density $\mathcal{E}$ ought to start with second-degree terms. Every departure, positive or negative ($e_{ij} > 0$ or < 0), from the initial state ought to be accompanied by an increase of the internal potential energy in order that the initial state be stable.

The coefficients Ω^{ij} form by their very definition a *tensor density*, since multiplied by the elements e_{ij} (tensor), they must give a scalar density $\mathcal{E}$.

Since the e_{ij} are symmetric in ij, we can take the Ω^{ij} also to be symmetric. This does not limit their generality at all since the antisymmetric part would play no role.

Likewise, the coefficients $\Lambda^{ij,hk}$ form a tensor density, symmetric in ij and hk and for the interchange of the two groups of indices:

$$\Lambda^{ij,hk} = \Lambda^{hk,ij}. \qquad (51)$$

The external forces of group I can be supposed to be derived from a particular potential energy, the energy of the mechanical system inducing the permanent strain. We refer this energy to the original volume of the solid and obtain an energy density *H* which, as the preceding, has a series development in terms of the strains of the body:

$$H = H_0 + H_1 + H_2 + \cdots = H_0 + \Omega_1^{ij} e_{ij} + \Lambda_1^{ij,hk} e_{ij} e_{hk} + \cdots \qquad (52)$$

Let us look an example: suppose the body is immersed in a liquid medium which exerts on it a uniform hydrostatic pressure *p*. Let us assume that this pressure is constant, whatever may be the additional strains imposed on the body by the accidental forces of group *II*. This case will be realized if the body is immersed in a large volume of liquid at a pressure *p*.The total energy coming from this pressure will be *pV*, where *V* is the total volume of the solid. We can write an energy density *H* of the form:

$$H = pV = p[1 + \delta v], \qquad V = 1 + \delta v. \qquad (53)$$

v is the volume occupied after the additional strain by a portion of the body whose initial volume was 1(under the influence of the pressure p, the only representative of the type *I* forces).

Making use of the development (45), we express δv as a function of the invariants of the additional strains:

$$H = p\left[1 + \frac{I_1}{2} - \frac{I_2}{4} + \frac{I_1^2}{8} + \cdots\ \right]. \qquad (54)$$

We must now say something about the *stability of the initial state*. If we impose a small additional strain e_{ij} of arbitrary sign, the sum $\mathcal{E} + H$ of the energies of the body and the external forces of group *I* must increase. This sum must contain only second-degree terms. The first-degree terms of $\mathcal{E}$ and H must cancel each other exactly:

$$\mathcal{E} + H = \mathcal{E}_0 + H_0 + \mathcal{E}_2 + H_2 + \cdots$$
$$\mathcal{E}_1 + H_1 = 0 \qquad (55)$$
$$\Omega^{ij} = -\Omega_1^{ij}.$$

Let us take up again the example of the solid under a constant hydrostatic pressure *p*. The above condition gives us, in rectangular Cartesian coordinates:

$$\Omega^{ij} = -\Omega_1^{ij} = \begin{vmatrix} -\frac{p}{2} & 0 & 0 \\ 0 & -\frac{p}{2} & 0 \\ 0 & 0 & -\frac{p}{2} \end{vmatrix} \qquad (56)$$

since using (54), the terms of the first degree H_1 of the external forces are:

$$H_1 = \frac{p}{2} I_1 = \frac{p}{2}(e_{11} + e_{22} + e_{33}). \qquad (57)$$

The pressure *p* is balanced, as far as the first-degree terms are concerned, by the reaction of the solid. But it still has a role in the second-degree terms, since the sum of these terms is:

$$\mathcal{E}_2 + H_2 = \Lambda^{ij,hk} e_{ij} e_{hk} + p\left[-\frac{I_2}{4} + \frac{I_1^2}{8}\right]. \qquad (58)$$

When we measure the forces necessary to induce further strains in the body, we obtain not only the coefficients Λ increased or diminished by *kp* (*k*, numerical coefficient), in short, the sum $\mathcal{E}_2 + H_2$ and not $\mathcal{E}_2$ alone.

If the body is subjected to a pressure which depends on its volume, the second-degree terms H_2 will not have the simple form (58), and the measurements will be very difficult to interpret.

6. CONCLUSION

In the most general case, we will have six distinct coefficients Ω^{ij} to define the initial *stresses*. Their number is that of the components of a second rank symmetric tensor. We must here make an important remark which justifies the manner of definition we choose for the strains. We insisted on the importance of keeping all the necessary terms of higher order than the first in the development of the strain tensor (equation (7) and (28)). It is easy to see that if we develop the energy density up to terms of the nth degree in the strains ($n = 2$ in equation (55)), it will be necessary, in the terms linear in the strains, to write the development of the strains up to the terms of the nth degree in the displacements $\mathbf{u}$ or their derivatives. In particular, if we do not wish to neglect terms of the same order as those kept somewhere else, it is indispensible in equation (55) to keep the expressions of the strains up to terms of the second degree in $\mathbf{u}$ and $\frac{\partial u}{\partial x}$.

7. REFERENCES

[1] Arnold, V; Les méthodes Mathématiques de la Mécanique classique, Edtions MIR. Moscou, 1976
[2] Bergmann,P.G; Introduction to the Theory of Relativity. Prentice-Hall, Englewood Cliffs, New jersey, 1942
[3] Cartan, E; Leçons sur la Géométrie des espaces de Riemann. 2nd ed. Gauthier-Villars, Paris, 1945.
[4] Eddington, A.S; Space, Time and Gravitation. cambridge univ. Press, london and New York, 1960.
[5] Ehlers, J; A survey of general relativity theory, Astrophysics and cosmology. Ed.W.Israel 14-26.
[6] Eisenhart, L.P; Riemannian Geometry. Princeton Univ. Press, Princeton, New Jersey, 1949.
[7] Glassey, R.T, Strauss, W.A; Absence of shocks in a initial dilute collisionless plasma, Comm.Math.Phys; 113, 191-208, 1987.
[8] Glassey, R.T, Strauss, W.A; Singularity Formation in Collisionless plasma could occur only at high velocities, Archive for rational mechanics and analysis, vol 92, \mathrm{n}^{0} 1, 1986.
[9] Malliavin, P; Géométrie differentielle, herman, Enseignement des sciences, Paris, 1972.
[10] Mizohata,S;The theory of partial differential equations. Cambridge University press. 1973.
[11] Schutz, B.F ; Geometrical method of mathematical physics, Cambridge university press. 1980
[12] Spain, B; Tensor Calculus. Wiley (Interscience), New York, 1953.
[14] Synge, J.L; and Schild, A; Tensor Calculus. Univ. of Toronto Press, Toronto, 1949.