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Linear System of the Volterra Integral Equations with a Polar Kernel

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ABSTRACT— ***This study are related Volterra integral equation with a polar kernel. Initial value problems for hyperbolic equations with function coefficients provides integral equation with 3-D Volterra type. Existence and uniqueness theorems of the Volterra integral equation a polar kernel are proved. Method of successive approximation used in the solutions of singular integral equations, existence and uniqueness theorems are emphasized.***

Key words— System of Volterra integral equations, polar kernel, successive approximation.

1. INTRODUCTION

The theory of partial differential equations with constant coefficients is very well developed. Many problems for partialdifferential equations with constant coefficients may be solved explicitly. For example, the initial value problem for the d'Alambert equation may be solved byd'Alambert formula and the Cauchy problem for the wave equation is given by Kirchoff's formula [7]. The situation is changed if some coefficients of partial differential equations are functions. There is no explicit formulae in this case. But some problems for partial differential equations with function coefficients are reducible to the solution of an integral equations [5].

The theory of linear hyperbolic equations with function coefficients is very well developed. There are general existence and uniqueness theorems for weak and classical solutions of initial value, see, for example [1,3,6]. Some particular cases of hyperbolic equations have interesting properties which are useful for numerical methods, inverse problems theory and others. We showed before that the initial value problems for Klein-Gordon-Fock equation with the function coefficient are reduced to the integral equations of Volterra type and these presentations efficiently were used for the study ofinverse problems [8]. The case in which the function coefficients appear in the principal part is more complicated. However if the speed coefficient depending on three space variables is a smooth function then the solution of the Cauchy problem for the wave equation satisfies a 3-D Volterra integral equation. This result was obtained by S.Sobolev [11] and is a generalization of Kirchhoff's formula. This Sobolev result was generalized for some hyperbolic equations [9,5].

This study is related to a linear system of the integral equations with a polar kernel. The main of this paper is to solution of polar kernel and system by the method successive approximations [1], [2], [3], [8]. Consider the linear system of the Volterra integral equations of the form

$$\bar{v}(t) = \bar{f}(t) + \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} K(t,\tau)\bar{v}(\tau)d\tau, \qquad (1)$$

where$\bar{v}(t) = \left(v_1(t), v_2(t), \ldots, v_n(t)\right)$ and $\bar{f}(t) = \left(f_1(t), f_2(t), \ldots, f_n(t)\right)$ are vector functions, $K(t,\tau) = \left(K_{ij}(t,\tau)\right)_{n\times n}$ is the matrix of the order $n \times n$, and

$$\int_0^t \bar{v}(t) = \left(\int_0^t v_1(t)d, \int_0^t v_2(t)dt, \cdots, \int_0^t v_n(t)dt\right).$$

The system (1) can be written in the form

$$v_m(t) = f_m(t) + \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} \sum_{j=1}^n K_{mj}(t,\tau)v_j(\tau)d\tau, \qquad m = 1, 2, 3, \dots, n.$$

We assume that $f_m(t) \in C[0,T]$, $K_{mj}(t,\tau) \in C(0 \le \tau \le t \le T)$, m, $j = 1, 2, 3, \dots, n$, and such that $K(t, tz)$ is continuous with respect to t, z for $t \in [0,T]$, $z \in [0,1]$. We will show that (1) determines a continuous solution for $t \in [0,T]$. We seek solution in the form of the Nuemann series [4]

$$\bar{v}(t) = \sum_{k=0}^{\infty} \bar{v}^{(k)}(t), \tag{2}$$

where

$$\sum_{k=0}^{\infty} \bar{v}^{(k)}(t) = \left(\sum_{k=0}^{\infty} v_1{}^{(k)}(t), \sum_{k=0}^{\infty} v_2{}^{(k)}(t), \dots, \sum_{k=0}^{\infty} v_n{}^{(k)}(t)\right),$$

$$\bar{v}^{(0)}(t) = \bar{f}(t), \ \bar{v}^{(k)}(t) = \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} K(t,\tau)\bar{v}^{(k-1)}(\tau)d\tau, \quad k = 1, 2, \dots, n. \tag{3}$$

The relation (3) for $k = 1, 2, \dots, n$ may be written in the form [9,10].

$$\bar{v}^{(k)}(t) = \int_0^1 \frac{1}{\sqrt{1-z^2}} K(t, tz)\bar{v}^{(k-1)}(tz)dz, \quad k = 1, 2, \dots, n. \tag{4}$$

Example 1. It is easy to show that

$$\int_\tau^s \frac{t}{\sqrt{(s^2-t^2)(t^2-\tau^2)}} dt = \frac{\pi}{2}, \ \tau \le s. \tag{5}$$

Solution. Changing the variable *t* by *x*, as follows

$$x = \frac{2t^2 - s^2 - \tau^2}{s^2 - \tau^2},$$

we find

$$2tdt = \frac{s^2-\tau^2}{2} dx,$$

$$\sqrt{(s^2 - t^2)(t^2 - \tau^2)} = \frac{s^2-\tau^2}{2} \sqrt{1 - x^2}.$$

The left hand size of (5) may be written as

$$\frac{1}{2}\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}. \tag{6}$$

Using substituting $x = sin\theta$ in the last integral we find

$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{cos\theta d\theta}{\sqrt{1-sin^2\theta}} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}.$$

This completed to solution of Example 1.

Let us numerical approximation for singular integral (6). The exact value is $\frac{\pi}{2} = 1.570796326794897$.You may find other sources, even MATLAB. Whatever sources you use, the points and weights should be given with 15 digit accuracy.

```
%Improper Integral

fun = f(x) (2*sqrt(1-x.*x)).^(-1);

q = integral (fun,-1,1)
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The numerical result is 1.570796326794671. Finally, the absolute error can be found by

| approximation value—exact value | =2.253752739989068e-13.

2. EXISTENCE, UNIQUENESS AND THEOREM OF SYSTEM OF VOLTERRA

Theorem 1. Let T be a given positive number and $f_m(t)$and $K_{mj}(t,\tau)$, $m = 1,2,\dots,n$; $j = 1,2,\dots,n$; satisfy above mentioned conditions. Then there exists a unique solution $\bar{v}(t) \in C[0,T]$,of the integral system (1) and this solution may be constructed by the successive approximation method.The proof of the existence theorem is based on the following cases [11], [12].

Proof. Here we will give five cases in the following:

Case 1. Let conditions of the Theorem 1 take the place and

$$K_0 = \max_{1\le i\le n} \max_{1\le j\le n} \max_{0\le\tau\le t\le T} |K_{ij}(t,\tau)|,$$

$$F_0 = \max_{t\in[0,T]} \sum_{j=1}^{n} |f_j(t)|,$$

$$M_1 = T(nF_0K_0)^2\frac{\pi}{2},$$

$$F_1 = max\{F_0, K_0F_0T\pi\},$$

Letting $l = 0$in (3) we find

$$\left|v_m{}^{(0)}(t)\right| = |f_m(t)| \leq \max_{t\in[0,T]} \sum_{m=1}^{n} |f_m(t)| = F_0 \leq F_1 , \quad m = 1,2,\ldots,n.$$

Using (3) for $l = 1$ we find

$$v_m{}^{(1)}(t) = \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} \sum_{j=1}^{n} K_{mj}(t,\tau) f_j(\tau) d\tau, \qquad t \in [0,T].$$

We find from the last relation

$$\left|v_m{}^{(1)}(t)\right| = K_0 \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} \sum_{j=1}^{n} \left|f_j(\tau)\right| d\tau = K_0 F_0 T \frac{\pi}{2}, \qquad m = 1,2,\ldots,n; \quad t \in [0,T].$$

The relations (3) can be written in the form

$$v_m{}^{(k)}(t) = \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} \sum_{j=1}^{n} K_{mj}(t,\tau) v_j{}^{(k-1)}(\tau) d\tau, \ \ m = 1,2,\ldots,n; \ t \in [0,T]. \quad (7)$$

Denoting $y^{(k)}(t) = \sum_{j=1}^{n} \left|v^{(k)}(t)\right|$, $\quad k = 1,2,\ldots,n,$ we find from (7) the inequality

$$\left|v_m{}^{(k)}(t)\right| \leq F_0 K_0 \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} \sum_{j=1}^{n} \left|v_j{}^{(k-1)}(\tau)\right| d\tau,$$

$$= F_0 K_0 \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} y^{(k-1)}(\tau) d\tau, \qquad m = 1,2,\ldots,n; \ \ t \in [0,T].$$

Summing the last relations we have

$$y^{(k)}(t) = \sum_{m=1}^{n} \left|v_m{}^{(k)}(t)\right|$$

$$\leq nF_0 K_0 \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} y^{(k-1)}(\tau) d\tau, \quad k = 2,3,\ldots,n; \qquad t \in [0,T]. \qquad (8)$$

Multiplying equation (8) by $\ (s^2 - t^2)^{-\frac{1}{2}}$ and integrating from 0 to s using the similar reasoning which we did before, we find

$$\int_0^s \frac{\left|y^{(k)}(t)\right|}{\sqrt{s^2-t^2}} dt \leq nF_0 K_0 \int_0^s \int_0^t \frac{t}{\sqrt{s^2-t^2}} \frac{\left|y^{(k-1)}(\tau)\right|}{\sqrt{t^2-\tau^2}} d\tau dt,$$

$$= nF_0 K_0 \frac{\pi}{2} \int_0^s \left|y^{(k-1)}(\tau)\right| d\tau , \qquad t \in [0,T],$$

or

$$\int_0^t \frac{|y^{(k)}(\tau)|}{\sqrt{t^2-\tau^2}} d\tau \leq nF_0 K_0 \frac{\pi}{2} \int_0^t |y^{(k-1)}(\tau)| d\tau, \; t \in [0,T], \; k = 1,2,\dots,n. \tag{9}$$

Using (8) ve (9) , we find

$$y^{(k)}(t) \leq nF_0 K_0 \int_0^t \frac{t}{\sqrt{t^2-\tau^2}} y^{(k-1)}(\tau) d\tau$$

$$\leq TnF_0 K_0 \left(nF_0 K_0 \frac{\pi}{2}\right) \int_0^t y^{(k-2)}(\tau) d\tau, \qquad t \in [0,T].$$

As a result we have

$$y^{(k)}(t) \leq M_1 \int_0^t y^{(k-2)}(\tau) d\tau, \qquad k = 2,3,4,\dots,n; \; t \in [0,T], \tag{10}$$

whereM_1 is defined in the Case 1.

The relations (10) may be written as the following two relations

$$\left|v_m{}^{(2l)}(t)\right| \leq y^{(2l)}(t) \leq M_1 \int_0^t y^{(2l-2)}(\tau) d\tau \leq F_1 \frac{(M_1 t)^l}{l!}, \qquad l = 1,2,3,\dots,n,$$

$$\left|v_m{}^{(2l+1)}(t)\right| \leq y^{(2l+1)}(t) \leq M_1 \int_0^t y^{(2l-1)}(\tau) d\tau \leq F_1 \frac{(M_1 t)^l}{l!}, \qquad l = 1,2,3,\dots,n.$$

Case 2. The summing of two series of $\sum_{k=0}^{\infty} v_m{}^{(k)}(t)$

$$\sum_{k=0}^{\infty} v_m{}^{(k)}(t) = \sum_{l=0}^{\infty} (v_m{}^{(2l)}(t) + v_m{}^{(2l+1)}(t)) = \sum_{l=0}^{\infty} v_m{}^{(2l)}(t) + \sum_{l=0}^{\infty} v_m{}^{(2l+1)}(t).$$

From Case1, we find that series $\sum_{k=0}^{\infty} v_m{}^{(k)}(t)$ for $t \in [0,T]$ is majorized by a convergent numerical series. According to the first Weierstrass Theorem $\sum_{k=0}^{\infty} v_m{}^{(k)}(t)$is uniformly convergent on $[0,T]$.

Case 3. Under conditions of the Theorem 1 the series $v_m{}^{(k)}(t)$ is continuous function on $[0,T]$. $K(t,tz)$is a continuous with respect to t and z for $t \in [0,T]$, $z \in [0,1]$ and

$$\frac{K(t,tz)}{\sqrt{1-z^2}},$$

is integrable with respect to z over (-1,1) for any $t \in [0,T]$. Using the theorem (Lavrent'ev, 1997, sf. 99), [4] and formula (3), (4), each term of the series is continuous on $[0,T]$.

Case 4. Using the Cases 2 and 3 and second Weierstrass Theorem, we can reach $\sum_{k=0}^{\infty} \bar{v}^{(k)}(t)$is uniformly convergent to a continuous function$\bar{v}(t)$, $m = 1,2,..,n$

Case 5. Under conditions of the Theorem 1 $\bar{v}(t) = \sum_{k=0}^{\infty} \bar{v}^{(k)}(t)$ is a solution of the integral equation (1).Consider (3)and summing from $k = 1$to $n = N$, then

$$\sum_{k=1}^{N}\bar{v}^{(k)}(t)=\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}K(t,\tau)\sum_{k=1}^{N}\bar{v}^{(k-1)}(\tau)d\tau,\quad k\geq 1,\ \ t\in[0,T].$$

Adding the term $\bar{v}^{(0)}(t)=\bar{f}(t)$ for each side of the last equation

$$\bar{v}^{(0)}(t)+\sum_{k=1}^{N}\bar{v}^{(k)}(t)=\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}K(t,\tau)\sum_{k=1}^{N}\bar{v}^{(k-1)}(\tau)d\tau+\bar{f}(t),$$

or

$$\sum_{k=0}^{N}\bar{v}^{(k)}(t)=\bar{f}(t)+\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}K(t,\tau)\sum_{k=0}^{N-1}\bar{v}^{(k)}(\tau)d\tau,\ \ k\geq 1.$$

Turning $N\to\infty$ and using second Weierstrass Theorem

$$\lim_{N\to\infty}\sum_{k=0}^{N}\bar{v}^{(k)}(t)=\bar{f}(t)+\lim_{N\to\infty}\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}K(t,\tau)\sum_{k=0}^{N-1}\bar{v}^{(k)}(\tau)d\tau,$$

$$\bar{v}(t)=f(t)+\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}K(t,\tau)\left[\lim_{N\to\infty}\sum_{k=0}^{N-1}\bar{v}^{(k)}(\tau)\right]d\tau,$$

$$\bar{v}(t)=\bar{f}(t)+\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}K(t,\tau)\bar{v}(\tau)d\tau.$$

We find that $\bar{v}(t)=\sum_{k=0}^{\infty}\bar{v}_m{}^{(k)}(t)$ satisfies the equation (1).

Theorem 2. Under conditions of the Theorem 1, the solution of (1) is a unique in the class $C[0,T]$, [10].

Proof: Let$\bar{v}_1(t)$ and $\bar{v}_2(t)$ are two solutions (1). Denote $\tilde{v}(t)=\bar{v}_1(t)-\bar{v}_2(t)$, then

$$\tilde{v}(t)=\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}K(t,\tau)\tilde{v}(\tau)d\tau,$$

or

$$\tilde{v}_m(t)=\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}\sum_{j=1}^{n}K_{mj}(t,\tau)\tilde{v}_j(\tau)d\tau,\quad m=1,2,\dots,n;\ \ t\in[0,T].$$

Last we have this inequality

$$|\tilde{v}_m(t)|\leq\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}\sum_{j=1}^{n}\left|K_{mj}(t,\tau)\right|\left|\tilde{v}_j(\tau)\right|d\tau,\ \ t\in[0,T],\qquad(11)$$

$$\leq K_0\int_0^t\frac{t}{\sqrt{t^2-\tau^2}}\sum_{j=1}^{n}\left|\tilde{v}_j(\tau)\right|d\tau,\qquad m=1,2,\dots,n;\ \ t\in[0,T].$$

Multiplying equation (11) by $(s^2-t^2)^{-\frac{1}{2}}$ and integrating 0 to s then we find

$$\int_0^s\frac{|\tilde{v}_m(t)|}{\sqrt{s^2-t^2}}dt\leq K_0\int_0^s\frac{t}{\sqrt{s^2-t^2}}\int_0^t\frac{1}{\sqrt{t^2-\tau^2}}\sum_{j=1}^{n}\left|\tilde{v}_j(\tau)\right|d\tau dt,$$

$$=K_0\int_0^s\sum_{j=1}^{n}\left|\tilde{v}_j(\tau)\right|d\tau\int_\tau^s\frac{t}{\sqrt{(s^2-t^2)(t^2-\tau^2)}}dt,$$

$$= K_0 \frac{\pi}{2} \int_0^s \sum_{j=1}^{n} \left| \tilde{v}_j(\tau) \right| d\tau, \quad m = 1, 2, \ldots, n; \quad s \in [0, T].$$

As we result of it, we have

$$\int_0^s \frac{|\tilde{v}_m(t)|}{\sqrt{s^2 - t^2}} dt \le K_0 \frac{\pi}{2} \int_0^s \sum_{j=1}^{n} \left| \tilde{v}_j(\tau) \right| d\tau, \; m = 1, 2, \ldots, \; n; \quad s \in [0, T]. \qquad (12)$$

Using (11) and (12) then we find

$$|\tilde{v}_m(t)| \le K_0 \int_0^t \frac{t}{\sqrt{t^2 - \tau^2}} \sum_{j=1}^{n} \left| \tilde{v}_j(\tau) \right| d\tau \le \frac{T K_0{}^2 \pi}{2} \int_0^t \sum_{j=1}^{n} \left| \tilde{v}_j(\tau) \right| d\tau,$$

or

$$|\tilde{v}_m(t)| \le T K_0{}^2 \frac{\pi}{2} \int_0^t \sum_{j=1}^{n} \left| \tilde{v}_j(\tau) \right| d\tau, \quad m = 1, 2, \ldots, \; n; \quad s \in [0, T]. \qquad (13)$$

Summing the last relation we have

$$\tilde{y}(t) \le K_1 \int_0^t \tilde{y}(\tau) d\tau, \; t \in [0, T]. \qquad (14)$$

Using (13) and Gronwall's inequality (Evans, 1998, pp. 625), [1]. We deduce

$$\tilde{y}(t) \equiv 0, \quad t \in [0, T]. \qquad (15)$$

This means that

$$|\tilde{v}_m(t)| = 0, \qquad m = 1, 2, 3, \ldots, n; \; t \in [0, T],$$

or

$$\bar{v}_1(t) = \bar{v}_2(t); \; t \in [0, T].$$

This proves the Theorem 2.

Example 2. Consider the integral

$$I = -\int_0^u \left(\frac{u - s}{s}\right)^{\alpha} \frac{ds}{(u - s)(s - t)}, \qquad 0 < \alpha < 1, \qquad (16)$$

and set $u - s = vs$ in it. Then we have

$$I = \int_0^\infty \frac{v^{\alpha - 1} dv}{vt - (u - t)}.$$

Next set

$$\xi = \begin{cases} \dfrac{tv}{u - t}, & 0 < t < u \\ \dfrac{tv}{t - u}, & u < t \end{cases}.$$

The result is

$$-\int_0^u \left(\frac{u - s}{s}\right)^{\alpha} \frac{ds}{(u - s)(s - t)}$$

$$= \begin{cases} \frac{1}{t}\left(\frac{u-t}{t}\right)^{\alpha-1} \int_0^\infty \frac{\xi^{\alpha-1}d\xi}{\xi-1} = -\frac{(u-t)^{\alpha-1}}{t^\alpha}\pi\cot(\alpha\pi), & 0<t<u \\ \frac{1}{t}\left(\frac{t-u}{t}\right)^{\alpha-1} \int_0^\infty \frac{\xi^{\alpha-1}d\xi}{\xi+1} = -\frac{(t-u)^{\alpha-1}}{t^\alpha}\frac{\pi}{\sin(\alpha\pi)}, & u<t \end{cases}. \quad (17)$$

For the special case $a = \frac{1}{2}$, (17) reduces to

$$-\int_0^u \frac{ds}{\sqrt{s(u-s)(s-t)}} = \begin{cases} 0, & 0<t<u \\ \frac{\pi}{\sqrt{t((t-u)}}, & u<t \end{cases}. \quad (18)$$

Another interesting result that we need is

$$\int_{\max(s,t)}^1 \frac{du}{\sqrt{(u-s)}\sqrt{(u-t)}} = ln\left(\frac{\sqrt{1-s}+\sqrt{1+t}}{\sqrt{1-s}-\sqrt{1+t}}\right), \quad (19)$$

which is proved as follows.

For $s < t$, the left side of equation (19) is

$$\int_t^1 \frac{du}{\sqrt{(u-s)(u-t)}} = \int_t^1 \frac{du}{\sqrt{u^2-(s+t)u+st}} . \quad (20)$$

Using the formula

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} ln\left(\sqrt{ax^2+bx+c} + \sqrt{a}x + \frac{b}{\sqrt{a}}\right).$$

Equation (20) yields

$$\int_t^1 \frac{du}{\sqrt{(u-s)(u-t)}} = \left[ln\left(\sqrt{u^2-(s+t)+st} + u - \frac{s+t}{2}\right)\right]_t^1$$

$$= ln\left[\sqrt{(1-s)(1-t)} + (1-t) + \left(\frac{t-s}{2}\right)\right] - ln\left(\frac{t-s}{2}\right)$$

$$= ln\left(\frac{\sqrt{1-s}+\sqrt{1+t)}}{\sqrt{1-s}-\sqrt{1+t}}\right), \quad (21)$$

Which proves (19) for $s < t$. The same steps are needed for case $s > t$.

3. CONCLUSION

The main result of this study is that linear system of the 3-D Volterra integral equations with a polar kernel were solved by the successive approximations, and existence and uniqueness theorems for the solution of an integral equation and a system of integral equations with polar kernel were proved.

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