ON BOUNDED LINEAR OPERATORS IN b-HILBERT SPACES AND THEIR NUMERICAL RANGES

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ABSTRACT. In this paper, we introduce the notions of b-bounded linear operator, b-numerical range and b-numerical radius in a b-Hilbert space and describe some of their properties. Then we will show that this new numerical range (radius) can be considered as a usual numerical range (radius) in a Hilbert space, so it shares many useful properties with numerical range (radius).

1. Introduction and Preliminaries

Quadratic forms and their applications appear in many parts of mathematics and the sciences. A natural extension of these ideas in finite- and infinite-dimensional spaces leads us to the numerical range [7]. The subject has been studied by great mathematicians like K. E. Gustafson, D. K. M. Rao, R. Bahatia, F. Kittaneh, S. S. Dragomir, M. S. Moslehian and others (cf. e.g. [2, 4, 7, 8, 10, 12] and also to the references cited therein), and they have contributed a lot for the extension of this branch of mathematics.

The concept of linear 2-normed spaces was investigated by S. Gähler in 1964 [5], and has been developed extensively in different subjects by many authors [6, 13, 15, 16]. A concept which is closely related to 2-normed space is 2-inner product space which has been intensively studied by many mathematicians in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [3].

In the following we provide some notations, definitions and auxiliary facts which will be used later in this paper.

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Definition 1.1. Let $\mathcal{X}$ be a linear space of dimension greater than 1 over the field $\mathbb{k}$, where $\mathbb{k}$ is the real or complex numbers field. Suppose that $\langle .,.|.\rangle$ is a $\mathbb{k}$-valued function defined on $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ satisfying the following conditions:
$(I1)$ $\langle x,x|z\rangle \geq 0$ and $\langle x,x|z\rangle = 0$ if and only if x and z are linearly dependent,
$(I2)$ $\langle x,x|z\rangle = \langle z,z|x\rangle$,
$(I3)$ $\langle x,y|z\rangle = \overline{\langle y,x|z\rangle}$,
$(I4)$ $\langle \alpha x,y|z\rangle = \alpha\langle x,y|z\rangle$ for all $\alpha \in \mathbb{k}$,
$(I5)$ $\langle x_1 + x_2,y|z\rangle = \langle x_1,y|z\rangle + \langle x_2,y|z\rangle$.
Then $\langle .,.|.\rangle$ is called a 2-inner product on $\mathcal{X}$ and $(\mathcal{X}, \langle .,.|.\rangle)$ is called a 2-inner product space (or 2-pre Hilbert space).

From the definition of 2-inner product it is easy to verify the following assertions:
(i) $\langle 0,y|z\rangle = \langle x,0|z\rangle = \langle x,y|0\rangle = 0$.
(ii) $\langle x,\alpha y|z\rangle = \overline{\alpha}\langle x,y|z\rangle$.
(iii) $\langle x,y|\alpha z\rangle = |\alpha|^2\langle x,y|z\rangle$, for all $x,y,z \in \mathcal{X}$ and $\alpha \in \mathbb{k}$.
Using the above properties, we can prove the Cauchy-Schwarz inequality

$$|\langle x,y|z\rangle|^2 \leq \langle x,x|z\rangle\langle y,y|z\rangle.$$

Example 1.2. (see [1, Example 1.1]) If $(\mathcal{X}, \langle .,.\rangle)$ is an inner product space, then the standard 2-inner product $\langle .,.|.\rangle$ is defined on $\mathcal{X}$ by

$$\langle x,y|z\rangle = \begin{vmatrix} \langle x,y\rangle & \langle x,z\rangle \\ \langle z,y\rangle & \langle z,z\rangle \end{vmatrix} = \langle x,y\rangle\langle z,z\rangle - \langle x,z\rangle\langle z,y\rangle,$$

for all $x,y,z \in \mathcal{X}$.

In any 2-inner product space $(\mathcal{X}, \langle .,.|.\rangle)$ we can define a function $\|.,.\|$ on $\mathcal{X} \times \mathcal{X}$ by

$$\|x,z\| = \langle x,x|z\rangle^{\frac{1}{2}}, \tag{1.1}$$

for all $x,z \in \mathcal{X}$. It is easy to see that, this functions satisfies the following conditions:
$(N1)$ $\|x,z\| \geq 0$ and $\|x,z\| = 0$ if and only if x and z are linearly dependent,
$(N2)$ $\|x,z\| = \|z,x\|$,
$(N3)$ $\|\alpha x,z\| = |\alpha|\|x,z\|$ for all $\alpha \in \mathbb{k}$,
$(N4)$ $\|x_1 + x_2,z\| \leq \|x_1,z\| + \|x_2,z\|$.
Any function $\|.,.\|$ defined on $\mathcal{X} \times \mathcal{X}$ and satisfying the conditions $(N1)$-$(N4)$ is called a

2-norm on $\mathcal{X}$ and $(\mathcal{X}, \|.,.\|)$ is called a linear 2-normed space. Whenever a 2-inner product space $(\mathcal{X}, \langle .,.|.\rangle)$ is given, we consider it as a linear 2-normed space $(\mathcal{X}, \|.,.\|)$ with the norm defined by (1.1).

Let $\mathcal{X}$ be a 2-inner product space. A sequence $\{x_n\}$ of $\mathcal{X}$ is said to be convergent if there exists an element $x \in \mathcal{X}$ such that $\lim_{n\to\infty} \|x_n - x, z\| = 0$, for all $z \in \mathcal{X}$. Similarly, we can define a Cauchy sequence in $\mathcal{X}$. A 2-inner product space $\mathcal{X}$ is called a 2-Hilbert space if it is complete. That is, every Cauchy sequence in $\mathcal{X}$ is convergent in this space [13]. Clearly, the limit of any convergent sequence is unique. Now suppose that b is a nonzero fixed vector in $\mathcal{X}$ and take $z = b$, then definition of Cauchy, convergent and 2-Hilbert space change to b-Cauchy, b-convergent and b-Hilbert space [9]. If a sequence $\{x_n\}$ is b-convergent to an element of b-Hilbert space $\mathcal{X}$ say x, then we denote it by $\lim_{n\to\infty}{}^{\|.,b\|} x_n = x$.
It is easily verified that in any b-Hilbert space $\mathcal{X}$, the mapping $\langle .,.|b\rangle$ is sequentially continuous with respect to semi-norm $\|.,b\|$.

Remark 1.3. (see [1, Pages 127-128]) Assume that $(\mathcal{X}, \langle .,.|.\rangle)$ is a 2-Hilbert space and L_ξ the subspace generated with ξ for a fix element ξ in $\mathcal{X}$. Denote by $\mathcal{M}_\xi$ the algebraic complement of L_ξ in $\mathcal{X}$. So $L_\xi \oplus \mathcal{M}_\xi = \mathcal{X}$. We first define the inner product $\langle .,.\rangle_\xi$ on $\mathcal{X}$ as following:

$$\langle x, z\rangle_\xi = \langle x, z|\xi\rangle.$$

A straightforward calculations shows that $\langle .,.\rangle_\xi$ is a semi-inner product on $\mathcal{X}$. It is well-known that this semi-inner product induces an inner product on the quotient space $\mathcal{X}/L_\xi$ as

$$\langle x + L_\xi, z + L_\xi\rangle_\xi = \langle x, z\rangle_\xi, \quad (x, z \in \mathcal{X}).$$

By identifying $\mathcal{X}/L_\xi$ with $\mathcal{M}_\xi$ in an obvious way, we obtain an inner product on $\mathcal{M}_\xi$. Define

$$\|x\|_\xi = \sqrt{\langle x, x\rangle_\xi} \quad (x \in \mathcal{M}_\xi).$$

Then $(\mathcal{M}_\xi, \|.\|_\xi)$ is a normed space. Let $\mathcal{X}_\xi$ be the completion of the inner product space $\mathcal{M}_\xi$. For each $b \in \mathcal{X}$, we denote by L_b the subspace generated by b. Let $x_1, x_2 \in \mathcal{X}$, then x_1 is said to b-congruent to x_2, if $x_1 - x_2 \in L_b$.

In the present work, we shall introduce the concept of b-bounded linear operator and describe some fundamental properties of it. Then we establish b-numerical range (radius) for

b-bounded linear operators. This numerical range (radius) can be considered as a usual numerical range (radius) in a Hilbert space, so it shares many useful properties with numerical range (radius).

Throughout this paper, unless otherwise specified, $\mathcal{X}, H$ and $L_b^{\perp}$ denote b-Hilbert space, Hilbert space with the inner product $\langle .,. \rangle$ chosen to be linear in the first entry, and the orthogonal complement of L_b in H, respectively.

2. Main Result

Definition 2.1. Let $\mathcal{X}$ be a b-Hilbert space. A linear operator $T : \mathcal{X} \to \mathcal{X}$ is called b-bounded if T invariants L_b and there is a non-negative real number M such that $\|T(x), b\| \leq M\|x, b\|$ for all $x \in \mathcal{X}$. We define $\|T\|_b$ infimum of such M. Obviously,

$$\|T\|_b = \sup\{\|T(x), b\| : \|x, b\| \leq 1\} = \sup\{\|T(x), b\| : \ \|x, b\| = 1\}.$$

We denote the set of all b-bounded linear operators on the b-Hilbert space $\mathcal{X}$, by $B_b(\mathcal{X})$. It is not hard to see that if $T \in B_b(\mathcal{X})$, then it (sequentially) continuous.

Let T and T' be b-bounded linear operators on the b-Hilbert space $\mathcal{X}$. They are called equal up to b-congruent if range$(T - T') \subseteq L_b$. Due to the fact $(B_b(\mathcal{X}), \|.\|_b)$ is a semi-normed space.

Similarly a linear functional $f : \mathcal{X} \to \mathbb{C}$ is called b-bounded if $f(L_b) = \{0\}$ and there is a non-negative real number M such that $|f(x)| \leq M\|x, b\|$ for all $x \in \mathcal{X}$. We define $\|f\|_b$ infimum of such M. We observe that $\|f\|_b = \sup\{|f(x)| : \ \|x, b\| \leq 1\}$ and it defines a norm on the set of all b-bounded linear functionals on $\mathcal{X}$ which is denoted by $(\mathcal{X}^*)_b$.

Example 2.2. Let $\mathcal{X} = l^2$ together with the standard 2-inner product. Then $\mathcal{X} = l^2$ is a $(1, 0, 0, ...)$-Hilbert space. Assume that $T : \mathcal{X} \to \mathcal{X}$ is a map which is defined by $T(a_1, a_2, ...) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, ...)$. It is readily verified that T is $(1, 0, 0, ...)$-bounded linear operator. Indeed, $\|T((a_1, a_2, ...)), (1, 0, 0, ...)\|^2 = \sum\limits_{n=2}^{\infty}\Big(\frac{|a_n|}{n}\Big)^2 \leq \sum\limits_{n=2}^{\infty}|a_n|^2 = \|(a_1, a_2, ...), (1, 0, 0, ...)\|^2$.

Example 2.3. Let $L^2([-\pi, \pi]) = \{f : [-\pi, \pi] \to \mathbb{R}, \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty\}$ and let $\mathcal{X} = \{f \in L^2([-\pi, \pi]) : f^{(k)} \in L^2([-\pi, \pi]), \ k = 1, 2, \ldots\}$. Then $\mathcal{X}$ with the standard 2-inner product is an e^x-Hilbert space. Define the operator $T : \mathcal{X} \to \mathcal{X}$ by $T(f) = f'$. An easy computation shows that T invariants L_{e^x} but it is not e^x-bounded. Since $\|T(\sin nx), e^x\|^2 = n^2(\frac{\pi}{2}(e^{2\pi} -$

$e^{-2\pi}) - \frac{(e^{\pi}-e^{-\pi})^2}{(n+n^3)^2})$ and $\|\sin(nx), e^x\|^2 = \frac{\pi}{2}(e^{2\pi} - e^{-2\pi}) - \frac{n^2(e^{\pi}-e^{-\pi})^2}{(1+n^2)^2}$, then $\|T(\sin nx), e^x\|$ goes to infinity as $n \to \infty$.

Proposition 2.4. *Let $\langle ., .|.\rangle$ be the standard 2-inner product on the Hilbert space H, $b \in H$ and $T \in B(H)$ in which T reduces L_b, then $T : (H, \langle ., .|.\rangle) \to (H, \langle ., .|.\rangle)$ is a b-bounded linear operator.*

Proof. Clearly if range$(T) \subseteq L_b$, then $\|T\|_b = 0$. Otherwise, since $T \in B(H)$, so there is a constant $M > 0$ such that $\|T(x)\| \leq M\|x\|$ for all $x \in H$. On the other hand, we have $H = L_b \oplus L_b^{\perp}$, therefore every element x of H can be written uniquely as $y + z$ for some $y \in L_b$ and $z \in L_b^{\perp}$. Now since T reduces L_b, then by the definition of standard 2-inner product it follows that

$$
\begin{aligned}
\|T(x), b\| &= \|T(y+z), b\| \leq \|T(y), b\| + \|T(z), b\| \\
&= \|T(z), b\| = (\|T(z), b\|^2)^{\frac{1}{2}} = (\|T(z)\|^2\|b\|^2 - |\langle T(z), b\rangle|^2)^{\frac{1}{2}} \\
&= \|T(z)\|\|b\| \leq M\|z\|\|b\|.
\end{aligned} \tag{2.1}
$$

Cauchy-Schwarz inequality implies that $|\langle y, b\rangle| = \|y\|\|b\|$, thus we find that

$$
\begin{aligned}
\|x, b\|^2 &= \|y+z\|^2\|b\|^2 - |\langle y+z, b\rangle|^2 \\
&= (\|y\|^2 + \|z\|^2)\|b\|^2 - |\langle y, b\rangle|^2 \\
&= (\|y\|^2 + \|z\|^2)\|b\|^2 - \|y\|^2\|b\|^2 = \|z\|^2\|b\|^2.
\end{aligned} \tag{2.2}
$$

By (2.1) and (2.2), we get the desired result. □

Proposition 2.5. *Let $\langle ., .|.\rangle$ be the standard 2-inner product on the Hilbert space H, $b \in H$ and $T : (H, \langle ., .|.\rangle) \to (H, \langle ., .|.\rangle)$ be a b-bounded linear operator in which invariants $L_b^{\perp}$, then T is a bounded linear operator on $L_b^{\perp}$.*

Proof. First suppose that range$(T) \nsubseteq L_b$. Let $x \in L_b^{\perp}$. By virtue of the fact that T invariants $L_b^{\perp}$ and also definition of standard 2-inner product we deduce

$$
\begin{aligned}
\|T(x)\|^2\|b\|^2 &= \|T(x)\|^2\|b\|^2 - |\langle T(x), b\rangle|^2 = \|T(x), b\|^2 \\
&\leq \|T\|_b^2\|x, b\|^2 = \|T\|_b^2(\|x\|^2\|b\|^2 - |\langle x, b\rangle|^2) \\
&= \|T\|_b^2\|x\|^2\|b\|^2.
\end{aligned}
$$

Whence $\|T(x)\| \leq \|T\|_b \|x\|$, for each $x \in L_b^{\perp}$ and so $T|_{L_b^{\perp}}$ is bounded. Now if range$(T) \subseteq L_b$, then range$(T|_{L_b^{\perp}}) \subseteq L_b \cap L_b^{\perp} = \{0\}$. It forces that $T|_{L_b^{\perp}} = 0$. □

Let $\mathcal{X}$ be a b-Hilbert space. As Remark 1.3, denote by $\mathcal{M}_b$, the algebraic complement of L_b in $\mathcal{X}$ and identifying $\mathcal{M}_b$ by $\mathcal{X}/L_b$. Also let $\mathcal{X}_b$ be the completion of the inner product space $\mathcal{M}_b$. Let $T \in B_b(\mathcal{X})$, define the map $T_b : \mathcal{X}_b \to \mathcal{X}_b$ by setting $T_b(z) := \lim_{n\to\infty} T(x_n) + L_b$, where $z = \lim_{n\to\infty} x_n + L_b \in \mathcal{X}_b$. We observe that T_b is a well-defined linear operator. Clearly $T_b = 0$, if range$(T) \subseteq L_b$. Otherwise, the inequality

$$\|(T(x_n) + L_b) - (T(x_m) + L_b)\|_b \leq \|T\|_b \|(x_n + L_b) - (x_m + L_b)\|_b$$

implies that the sequence $\{T(x_n) + L_b\}$ is Cauchy and so convergent in $\mathcal{X}_b$.
It is rutin to verify that if T and S are in $B_b(\mathcal{X})$ and α is any scalar in $\mathbb{k}$, then $(\alpha T + S)_b = \alpha T_b + S_b$ and $(TS)_b = T_b S_b$.
According to Remark 1.3, one obtains that $z = (\lim_{m\to\infty}{}^{\|.,b\|} x_m) + L_b$, where $z = \lim_{n\to\infty}{}^{\|\cdot\|_b} x_n + L_b \in \mathcal{X}_b$. By virtue of that fact we get the following result.

Proposition 2.6. *Let $\mathcal{X}$ be a b-Hilbert space and T be a b-bounded linear operator on $\mathcal{X}$, then T_b is a bounded linear operator on the Hilbert space $\mathcal{X}_b$ and moreover $\|T_b\| = \|T\|_b$.*

P. K. Harikrishnan et al., [9] proved a version of Riesz representation theorem in framework of b-Hilbert spaces. By a slightly modification in the proof of [9, Theorem 3.5] we see that this theorem holds for a b-bounded linear functional defined on a b-Hilbert space.

Proposition 2.7. *Let $\mathcal{X}$ be a b-Hilbert space and f be a b-bounded linear functional on $\mathcal{X}$. Then there exists a unique $y \in \mathcal{X}$ up to b-congruent such that $f(x) = \langle x, y|b\rangle$ and $\|f\|_b = \|y, b\|$.*

Definition 2.8. Let $\mathcal{X}$ be a b-Hilbert space. A complex valued function B on $\mathcal{X} \times \mathcal{X}$ is called a conjugate-bilinear functional, if it is linear in the first variable and conjugate-linear in the second. Furthermore, it is called b-bounded, if $B(\mathcal{X} \times L_b) = B(L_b \times \mathcal{X}) = B(L_b \times L_b) = \{0\}$ and there is a nonnegative real number M such that $|B(x, y)| \leq M\|x, b\|\|y, b\|$ for all $x, y \in \mathcal{X}$. We denote by $\|B\|_b$ the infimum of such M. It is easy to verify that $\|B\|_b = \sup\{|B(x, y)| : x, y \in \mathcal{X},\ \|x, b\| \leq 1, \|y, b\| \leq 1\}$. Trivially $\|.\|_b$ defines a norm on the set of b-bounded conjugate-bilinear functionals on $\mathcal{X}$. Assume $S \in B_b(\mathcal{X})$, define $B_S(x, y) := \langle S(x), y|b\rangle$ for each $x, y \in \mathcal{X}$. It is easy to verify that B_S is a b-bounded conjugate-bilinear functional on $\mathcal{X}$ and $\|B_S\|_b = \|S\|_b$.

Now we are in a position to investigate existence of an adjoint ,which is named b-adjoint, for a b-bounded linear operator defined on a b-Hilbert space. Indeed, we will show that if $\mathcal{X}$ is a b-Hilbert space and $T \in B_b(\mathcal{X})$, then there exists a unique $T^* \in B_b(\mathcal{X})$ up to b-congruent in which $\langle T(x), y|b\rangle = \langle x, T^*(y)|b\rangle$ for each $x, y \in \mathcal{X}$. We use a similar method applied in [11, pp. 98-101] for Hilbert spaces in order to obtain a b-adjoint for a b-bounded linear operator in a b-Hilbert space.

Let $\mathcal{X}$ be a b-Hilbert space. Consider equivalence relation $\sim$ on $\mathcal{X}$, in which $x \sim y$, if $x, y \in L_b$ and $x \sim x$, if $x \in \mathcal{X} - L_b$. In this case equivalence class $\tilde{\mathcal{X}}$ is $\{L_b, \tilde{x} = \{x\} : x \in \mathcal{X} - L_b\}$. We observe that $(\tilde{\mathcal{X}}, \tilde{\|.\|})$ is a normed space, where

$$\begin{aligned} \tilde{x} + \tilde{y} &= \widetilde{x+y}, \\ \tilde{x} + L_b &= L_b + \tilde{x} = \tilde{x}, \quad L_b + L_b = L_b, \\ \alpha\tilde{x} &= \widetilde{\alpha x}, \quad \alpha L_b = L_b, \end{aligned}$$

$\tilde{\|L_b\|} = 0$ and $\tilde{\|\tilde{x}\|} = \|x, b\|$, for each $x, y \in \mathcal{X} - L_b$ and $\alpha \in \mathbb{k}$. Define $\tilde{J} : \tilde{\mathcal{X}} \to (\mathcal{X}^*)_b$ by $\tilde{J}(L_b) = 0$ and if $x \in \mathcal{X} - L_b$, then $\tilde{J}(\tilde{x}) = J_x$, where $J_x(y) = \langle y, x, b\rangle$ for each $y \in \mathcal{X}$. It is easily seen that, $\tilde{J}$ is a surjective isometric conjugate linear operator. Assume that $V : \tilde{\mathcal{X}} \to \mathcal{X}$ defined by $V(L_b) = 0$ and $V(\tilde{x}) = x$ for each $x \in \mathcal{X} - L_b$, clearly V is a linear operator and $\|V\|_b = \sup\{\|V(\tilde{x}), b\| : \tilde{\|\tilde{x}\|} \leq 1\} \leq 1$.

Let B be a b-bounded conjugate-bilinear functional on $\mathcal{X}$, $U : \mathcal{X} \to (\mathcal{X}^*)_b$ be defined by $(Ux)(y) := \overline{B(x, y)}$. Then U is a b-bounded conjugate linear operator and by Proposition 2.7, for each $x \in \mathcal{X}$, there exists a unique $z \in \mathcal{X}$ up to b-congruent in which $Ux = \phi_z$, where $\phi_z(y) = \langle y, z|b\rangle$. Set $S := V\tilde{J}^{-1}U$, it is a b-bounded linear operator on $\mathcal{X}$. Indeed we have

$$\|V\tilde{J}^{-1}Ux, b\| \leq \|V\|_b \tilde{\|\tilde{J}^{-1}Ux\|} = \|V\|_b\|Ux\| \leq \sup\{|Ux(y)| : \|y, b\| \leq 1\} < \|B\|_b\|x, b\|,$$

for each $x \in \mathcal{X}$. Now if $B_S(x, y) = \langle S(x), y|b\rangle$, then B_S is a b-bounded conjugate-bilinear functional on $\mathcal{X}\times\mathcal{X}$, $\|B_S\|_b = \|S\|_b$ and furthermore, $B_S(x, y) = \overline{\langle y, S(x)|b\rangle} = \overline{\langle y, V\tilde{J}^{-1}Ux|b\rangle} = \overline{\langle y, z|b\rangle} = \overline{\phi_z(y)} = \overline{Ux(y)} = B(x, y)$. Trivially if x or y are in L_b, then $B(x, y) = B_S(x, y) = 0$. Hence every b-bounded conjugate bilinear functional is of the form B_S for some $S \in B_b(\mathcal{X})$.

Theorem 2.9. *Let T be a b-bounded linear operator on a b-Hilbert space $\mathcal{X}$, then there exists a unique b-bounded linear operator $T^* \in B_b(\mathcal{X})$ up to b-congruent such that $\langle T(x), y|b\rangle = \langle x, T^*(y)|b\rangle$ for each $x, y \in \mathcal{X}$. In addition, if S and S' are two b-adjoints of T, then $S_b = S'_b$.*

Proof. Define $B(x,y) = \langle x, T(y)|b\rangle$. It is easily verified that B is a b-bounded conjugate-bilinear functional on $\mathcal{X} \times \mathcal{X}$. So

$$B(x,y) = B_S(x,y) = \langle S(x), y|b\rangle,$$

for some b-bounded linear operator S on $\mathcal{X}$. Put $T^* := S$, then T^* is a b-adjoint of T.

Using the same reasoning as [11, Theorem 2.4.1] b-adjoint of T is unique up to b-congruent. It remains to show that $S_b = S'_b$, for b-adjoints S and S' of T. For, let $z_1, z_2 \in \mathcal{X}_b$, then $z_1 = \lim_{n\to\infty} x_n + L_b$ and $z_2 = \lim_{m\to\infty} y_m + L_b$ for some sequences $\{x_n\}$ and $\{y_m\}$ in $\mathcal{X}$. Since $S = S'$ up to b-congruent, so for each $n \in \mathbb{N}$, there is a scalar μ_n in which $S(x_n) = S'(x_n) + \mu_n b$. Thus we have

$$\begin{aligned}
\langle S_b(z_1), z_2\rangle_b &= \langle S_b(\lim_{n\to\infty} x_n + L_b), \lim_{m\to\infty} y_m + L_b\rangle_b \\
&= \langle \lim_{n\to\infty} S(x_n) + L_b, \lim_{m\to\infty} y_m + L_b\rangle_b \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \langle S'(x_n) + \mu_n b, y_m|b\rangle \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \langle S'(x_n) + L_b, y_m + L_b\rangle_b \\
&= \langle S'_b(z_1), z_2\rangle_b.
\end{aligned}$$

It follows that $S_b = S'_b$. □

As an immediate consequence of the above theorem we have $T = T^{**}$ up to b-congruent.

Let $\mathcal{X}$ be a b-Hilbert space and $T \in B_b(\mathcal{X})$, then T is called b-selfadjoint if $T = T^*$ up to b-congruent or equivalently $\langle T(x), y|b\rangle = \langle x, T(y)|b\rangle$ for each $x, y \in \mathcal{X}$ and it is called b-unitary, if $TT^* = T^*T = I$ (identity operator on $\mathcal{X}$) up to b-congruent. Note that if T is b-unitary, then range$(T) \nsubseteq L_b$.

Now we are ready to establishing b-numerical range (radius) for a b-bounded linear operator in b-Hilbert spaces. To extend a well-known result in Hilbert spaces to b-Hilbert spaces.

Definition 2.10. Let $T : \mathcal{X} \to \mathcal{X}$ be a b-bounded linear operator on a b-Hilbert space $\mathcal{X}$. Then b-numerical range of T which is denoted by $W_b(T)$ is $\{\langle T(x), x|b\rangle : x \in \mathcal{X}, \ \|x, b\| = 1\}$. Also, b-numerical radius of T which is denoted by $\omega_b(T)$ is $\sup\{|\langle T(x), x|b\rangle| : x \in \mathcal{X}, \ \|x, b\| = 1\}$.

A remarkable fact about b-numerical range (radius) is its close relation with numerical range (radius) in the usual sense. Indeed, we have $W_b(T) = W(T_b)$ and $\omega_b(T) = \omega(T_b)$.

By virtue of this fact every question about b-numerical range (radius) in a b-Hilbert space can be solved as a question about numerical range (radius) in a Hilbert space.
It is easy to verify that $\omega_b(.)$ is a semi-norm on $B_b(\mathcal{X})$. Furthermore, using Proposition 2.6 and [7, Theorem 1.3.1], we have $\omega_b(T) \leq \|T\|_b \leq 2\omega_b(T)$, for each $T \in B_b(\mathcal{X})$.
In the following we extend [7, Theorem 1.2.2] in the framework of b-Hilbert spaces.

Theorem 2.11. *Let T be a b-bounded linear operator on a b-Hilbert space $\mathcal{X}$. Then T is b-selfadjoint if and only if $W_b(T) \subseteq \mathbb{R}$.*

Proof. Let $z_1 = \lim_{n\to\infty} x_n + L_b$ and $z_2 = \lim_{n\to\infty} y_n + L_b$ be arbitrary elements in $\mathcal{X}_b$. We get

$$\begin{aligned}
\langle z_1, (T_b)^*(z_2)\rangle_b &= \langle T_b(\lim_{n\to\infty} x_n + L_b), \lim_{n\to\infty} y_n + L_b\rangle_b \\
&= \langle \overset{\|.,b\|}{\lim_{n\to\infty}} T(x_n), \overset{\|.,b\|}{\lim_{n\to\infty}} y_n|b\rangle \\
&= \langle \overset{\|.,b\|}{\lim_{n\to\infty}} x_n, T^*(\overset{\|.,b\|}{\lim_{n\to\infty}} y_n)|b\rangle \\
&= \langle (\overset{\|.,b\|}{\lim_{n\to\infty}} x_n) + L_b, (\overset{\|.,b\|}{\lim_{n\to\infty}} T^*(y_n)) + L_b\rangle_b \\
&= \langle \lim_{n\to\infty} x_n + L_b, (T^*)_b(\lim_{n\to\infty} y_n + L_b)\rangle_b \\
&= \langle z_1, (T^*)_b(z_2)\rangle_b.
\end{aligned}$$

Therefore $(T_b)^* = (T^*)_b$. Now if T is b-selfadjoint, then $(T^*)_b = T_b$ and so $(T_b)^* = T_b$. Applying [7, Theorem 1.2.2] we deduce $W_b(T) = W(T_b) \subseteq \mathbb{R}$. Conversely, if $W_b(T) \subseteq \mathbb{R}$, then T_b is a selfadjoint linear operator on the Hilbert space $\mathcal{X}_b$. That is, $(T_b)^* = T_b$. Consequently for each $x, y \in \mathcal{X}$, $\langle T(x), y|b\rangle = \langle T_b(x+L_b), y+L_b\rangle_b = \langle x+L_b, T_b^*(y+L_b)\rangle_b = \langle x+L_b, T_b(y+L_b)\rangle_b = \langle x, T(y)|b\rangle$. Hence T is b-selfadjoint and so the proof is completed. □

In the light of the above discussions we have the following statement.

Suppose that U and I are b-unitary and identity operators on a b-Hilbert space $\mathcal{X}$, respectively and $T \in B_b(\mathcal{X})$. Then we have
(i) $W_b(\alpha + \beta T) = \alpha + \beta W_b(T)$, for each α and β in $\mathbb{k}$.
(ii) $W_b(T^*) = \{\overline{\lambda} : \lambda \in W_b(T)\}$.
(iii) $W_b(U^*TU) = W_b(T)$.

References

[1] A. Arefijamaal and Gh. Sadeghi, *Frames in 2-inner product spaces*, Iranian J. Math. Sci. and informatics, **8**(2013), no. 2, 123–130.

[2] R. Bahatia, *Matrix analysis*, Grad. Texts in Math. 169, Springer-Verlag, New York, 1997.

[3] Y. J. Cho, P. C. S. Lin, S. S. Kim and A. Misiak, *Theory of 2-inner product spaces*, Nova Science Publishers, Inc. New York, 2001.

[4] S. S. Dragomir, *Some numerical radius inequalities for power series of operators in Hilbert spaces*, J. Inequal. Appl. **298**(2013) 1–12.

[5] S. Gähler, *Linear 2-normierte Räume*, Math. Nachr. **28**(1964) 1–43.

[6] I. Goleţ, *On probabilistic 2-normed spaces*, Novi. Sad J. Math. **35**(2005) no. 1, 95–102.

[7] K. E. Gustafson and D. K. M. Rao, *Numerical range*, Springer-Verlag, New York, 1997.

[8] P. R. Halmos, *A Hilbert space problem book*(2nd ed.), Springer-Verlag, New York, 1982.

[9] P. K. Harikrishnan, P. Riyas and K. T. Ravindran, *Riesz theorems in 2-inner product spaces*, Novi. Sad J. Math. **41**(2011) no. 2, 57–61.

[10] O. Hirzallah, F. Kittaneh and K. Shebrawi, *Numerical radius inequalities for commutators of Hilbert space operators*, Numer. Funct. Anal. Optim. **32**(7), (2011) 739–749.

[11] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, Acadmic Press. Inc. New York, 1983.

[12] F. Kittaneh, M. S. Moslehian and T. Yamazaki, *Cartesian decomposition and numerical radius inequalities*, Linear Algebra Appl. **471**(2015) 46–53.

[13] Z. Lewandowska, *Bounded 2-linear operators on 2-normed sets*, Glas. Math. Ser. III, **39**(59), (2004) 303–314.

[14] H. Mazaheri and R. Kazemi, *Some results on 2-inner product spaces*, Novi. Sad J. Math. **37**(2007) no. 2, 35–40.

[15] Sh. Rezapour and I. Kupka, *1-Type Lipschitz selections in generalized 2-Normed Spaces*, Anal. Theory Appl. **24**(2008) no. 3, 205–210.

[16] N. Srivastava, S. Bhattacharya and S. N. Lal, *2-Normed algebras-I*, Publ. de L'institut math. Nouvelle srie, tome, **88**(102) (2010), 111–121.

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