A New Super Convergent Implicit Runge-Kutta Method for First Order Ordinary Differential Equations

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ABSTRACT--- ***We present a new efficient super convergent implicit Runge-kutta method (RKM) for solving differential equations (ODEs). Chybechev's polynomial is used as basis function. Collocation and Matrix inversion method is used to derive our continuous schemes. The continuous formula is evaluated at zeros of the first Chybechev's polynomial to give us Runge-kutta evaluation functions for the direct iteration of our solutions. Experimental examples used show that the method is A stable, highly efficient, has simple coefficients, less implementation cost when compared with similar methods in the literature.***

Keywords--- Super Convergence RKM, Chybechev's polynomial, Collocation and Matrix inversion method, Zeros of Chybechev's polynomial, A stable)

1. INTRODUCTION

Linear multi-step methods (LMM) for solving differential equations have been widely implored to solve ordinary differential equations by Lambert [1], Butcher [2] etc. These methods have often been modified to obtain better results, for examples, multi-step using Chybechev's polynomials. (see Adeniyi et al [3], [4]), Fox [5] etc. Lie and Norsett [6] also used multi-step collocation method to develop Super convergence multi-step method for fist order ordinary differential equations (ODEs).

Onumanyi et al [7] also developed new linear multi-step method with continuous coefficients for solving ODEs. This method was modified to linear multi-step implicit Runge-kutta method by Yakubu ([8],[9]), Chollom et al [10] etc. These methods mentioned above are good but they are of low order, having bigger coefficients and high computational cost when compare with our proposed method. In our paper we have addressed those problems and developed a superior Runge-kutta method with with simple coefficients, low implementation cost and more efficient than Lie et al [6].

2. METHODOLOGY

We shall find s-stage implicit Runge-kutta method of solution of first order differential equation of the form $y' = f(x,y),\ y(x_0) = y_0,\ a \le x \le b$ (2.01)

We use a polynomial of the form

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_n + h \sum_{j=1}^{m-1} \beta_j(x) f\left(c_j y(c_j)\right) \quad (2.02)$$

Where t denotes the number of interpolation points $x_{n+j}\ (j = 0,1,..t-1)$, m denotes the number of collocation points,$c_j, (j = 0, ...m-1)$ are the collocation points and $f(x,y)$ is are smooth function.

The constants coefficients α_j, β_j are elements of $(t+m)X(t+m)$ square matrix. They are selected so that high accurate approximation of the solution (2.01) is obtained, h is a constant step size.

The functions $\alpha_j(x),\ \beta_j(x)$ in (2.02) can be represented by polynomial of the form

$$\alpha_j(x) = \sum_{j=0}^{t-1} \alpha_j, i+1 x^i, \quad j = (0,1,\ldots t-1$$

$$h\beta_j(x) = \sum_{j=0}^{m-1} h\beta_j, i+1 x^i \ , j = (0,1,\ldots m-1) \qquad (2.03)$$

The coefficients $\alpha_j, i+1$ and $\beta_j, i+1$ are to be determined.
Substituting (2.03) in (2.02), we have

$$y(x) = \sum_{i=0}^{m+t-1} \left\{ \sum_{j=0}^{t-1} \alpha_j, i+1 \ y_{n+i} + h \sum_{j=0}^{m-1} h\beta_j, i+1 \ f_{n+j} \right\} x^i = \sum_{i=0}^{m+t-1} \alpha_j, x^i \qquad (2.04)$$

where

$$a_j = \left(\sum_{j=0}^{t-1} \alpha_j \ \ i+1 \ y_{n+j} + h \sum_{j=0}^{m-1} h\beta_j, i+1 \ f_{n+j} \right)$$

$a_j \in R^j, j \in [0,1,2 \ldots t+m-1], y \in C^m[a,b]$.
This can be expressed in matrix form as

$$y(x) = (y_n, \ldots y_{n+t-1}, f_n, \ldots . f_{n+m-1}) \ C^T (1, x_n, \ldots . x_n^{t+m-1})^T$$

where

$$C = \begin{bmatrix} \alpha_{01} & . & . & \alpha_{t-1,1} & h\beta_{0,1} & . & . & . & h\beta_{m-1,1} \\ \alpha_{02} & . & . & \alpha_{t-1,2} & h\beta_{0,2} & . & . & . & h\beta_{m-1,2} \\ \alpha_{03} & . & . & \alpha_{t-1,3} & h\beta_{0,3} & . & . & . & h\beta_{m+1,1} \\ \alpha_{04} & . & . & \alpha_{t-1,4} & h\beta_{0,4} & . & . & . & h\beta_{m-1,4} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ \alpha_{0,t+m} & . & . & \alpha_{t-1,t+m} & h\beta_{0,t+m} & . & . & . & h\beta_{m-1,t+m} \end{bmatrix} \qquad (2.05)$$

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & . & . & x_n^{t+m-1} \\ 0 & 1 & 2x_{n+q_1} & . & . & (t+m-1)x_{n+q_1}^{t+m-2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 1 & 2x_{n+q_{m-1}} & . & . & (t+m-1)x_{n+q_{m-1}}^{t+m-2} \end{bmatrix} \qquad (2.06)$$

q_i are the collocation points

THEOREM 1.0
Let I denote the identity Matrix of dimension $(m+t)$x $(m+t)$ and Matrices C and D defined by (2.05) and (2.06) satisfies
i) $DC = I$

$$y(x) = \sum_{i=0}^{m+t-1} \left\{ \sum_{j=0}^{t-1} \alpha_{j,i+1}, y_{n+j} + \sum_{j=0}^{m-1} h \ \beta_{j,i+1} f_{n+j} \right\} x^i \qquad (2.07)$$

(see proof (Onumanyi e tal [7])

Now we assume a power series solution of degree 5 of the form

$$y(x) = \sum_{j=0}^{5} a_j x^j \ , \quad y'(x) = \sum_{j=0}^{5} j a_j x^{j-1} \qquad (2.08)$$

we interpolate at $x = x_n$, and collocate at $x = x_{n+q_i}, i = 0,1,..,5$

equation (2.08) yields system of simultaneous equation of the form

$$\begin{aligned} a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + \cdots a_5x_n^5 &= y_n \\ a_1 + 2a_2x_{n+q_1} + 3a_3x_{n+q_1}^2 + \quad 5a_5x_{n+q_1}^4 &= f_{n+q_1} \\ a_1 + 2a_2x_{n+q_2} + 3a_3x_{n+q_2}^2 + \quad 5a_5x_{n+q_2}^4 &= f_{n+q_2} \\ \ldots \quad \ldots \qquad \ldots \qquad \ldots. \qquad \ldots. \qquad \ldots. \qquad \ldots \qquad \ldots. & \\ a_1 + 2a_2x_{n+q_5} + 3a_3x_{n+q_5}^2 + \quad 5a_5x_{n+q_5}^4 &= f_{n+q_5} \end{aligned} \tag{2.09}$$

where a_j are parameters to be determined.

When (2.09) is rewritten in matrix form, we have

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 0 & 1 & 2x_{n+q_1} & 3x_{n+q_1}^2 & 4x_{n+q_1}^3 & 5x_{n+q_1}^4 \\ 0 & 1 & 2x_{n+q_2} & 3x_{n+q_2}^2 & 4x_{n+q_2}^3 & 5x_{n+q_2}^4 \\ 0 & 1 & 2x_{n+q_3} & 3x_{n+q_3}^2 & 4x_{n+q_3}^3 & 5x_{n+q_3}^4 \\ 0 & 1 & 2x_{n+q_4} & 3x_{n+q_4}^2 & 4x_{n+q_4}^3 & 5x_{n+q_4}^4 \\ 0 & 1 & 2x_{n+q_5} & 3x_{n+q_5}^2 & 4x_{n+q_5}^3 & 5x_{n+q_5}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ f_{n+q_1} \\ f_{n+q_2} \\ f_{n+q_3} \\ f_{n+q_4} \\ f_{n+q_5} \end{bmatrix}$$

$ie\ DA = Y$, where

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 0 & 1 & 2x_{n+q_1} & 3x_{n+q_1}^2 & 4x_{n+q_1}^3 & 5x_{n+q_1}^4 \\ 0 & 1 & 2x_{n+q_2} & 3x_{n+q_2}^2 & 4x_{n+q_2}^3 & 5x_{n+q_2}^4 \\ 0 & 1 & 2x_{n+q_3} & 3x_{n+q_3}^2 & 4x_{n+q_3}^3 & 5x_{n+q_3}^4 \\ 0 & 1 & 2x_{n+q_4} & 3x_{n+q_4}^2 & 4x_{n+q_4}^3 & 5x_{n+q_4}^4 \\ 0 & 1 & 2x_{n+q_5} & 3x_{n+q_5}^2 & 4x_{n+q_5}^3 & 5x_{n+q_5}^4 \end{pmatrix} = D - matrix \tag{2.10}$$

The D-matrix is non-singular and has inverse $D^{-1} = C$

Using Maple Mathematical software, we obtain the coefficients $a_j\ j = (0,1, \ldots,5)$ and from (2.04) we have

$$y(x) = \alpha_0(x)y_n + h\{\beta_{n+q_1}(x)f_{n+q_1} + \beta_{n+q_2}(x)f_{n+q_2} + \beta_{n+q_3}(x)f_{n+q_3} + \beta_{n+q_4}(x)f_{n+q_4} + \beta_{n+q_5}(x)f_{n+q_5}\} \tag{2.10a}$$

Solving for a_j we obtain a continuous scheme for $y(x)$.

When substituting the values of $\alpha_j and\ \beta_{n+q_i}\ i = 1,2,..5\ in\ (2.10a)$ ie$q_1 = \frac{1}{2} - \frac{\sqrt{3}}{4}, q_2 = \frac{1}{4}, q_3 = \frac{1}{2},\ q_4 = \frac{3}{4}, q_5 = \frac{1}{2} + \frac{\sqrt{3}}{4}$, we have the following discrete schemes.

$$y_{n+q_1} = y_n + \left(\frac{7}{90} + \frac{3\sqrt{3}}{320}\right)hf_{n+q_1} + \left(\frac{27}{320} - \frac{3\sqrt{3}}{40}\right)hf_{n+q_2} + \left(\frac{13}{90} - \frac{\sqrt{3}}{15}\right)hf_{n+q_3} + \left(\frac{37}{320} - \frac{3\sqrt{3}}{40}\right)hf_{n+q_4} + \left(\frac{7}{90} - \frac{41\sqrt{3}}{960}\right)hf_{n+q_5}$$

$$y_{n+q_2} = y_n + \left(\frac{29}{360} + \frac{3\sqrt{3}}{64}\right)hf_{n+q_1} + \frac{31}{320}hf_{n+q_2} - \frac{1}{90}hf_{n+q_3} + \frac{1}{320}hf_{n+q_4}$$

$$+\left(\frac{29}{360}-\frac{3\sqrt{3}}{64}\right)hf_{n+q_5}$$

$$y_{n+q_3}=y_n+\left(\frac{7}{90}+\frac{\sqrt{3}}{24}\right)hf_{n+q_1}+\frac{9}{40}hf_{n+q_2}+\frac{13}{90}hf_{n+q_3}-\frac{1}{40}hf_{n+q_4}+\left(\frac{7}{90}-\frac{\sqrt{3}}{24}\right)hf_{n+q_5}$$

$$y_{n+q_4}=y_n+\left(\frac{3}{40}+\frac{3\sqrt{3}}{64}\right)hf_{n+q_1}+\frac{63}{320}hf_{n+q_2}+\frac{3}{10}hf_{n+q_3}+\frac{33}{320}hf_{n+q_4}+\left(\frac{3}{40}-\frac{3\sqrt{3}}{64}\right)hf_{n+q_5}$$

$$y_{n+q_5}=y_n+\left(\frac{7}{90}+\frac{41\sqrt{3}}{960}\right)hf_{n+q_1}+\left(\frac{27}{320}+\frac{3\sqrt{3}}{40}\right)hf_{n+q_2}+\left(\frac{13}{90}+\frac{\sqrt{3}}{15}\right)hf_{n+q_3}$$

$$+\left(\frac{37}{320}+\frac{3\sqrt{3}}{40}\right)hf_{n+q_4}+\left(\frac{7}{90}-\frac{3\sqrt{3}}{320}\right)hf_{n+q_5} \qquad (2.11)$$

To convert to Runge-kutta, the discrete schemes must satisfy the differential equation (2.01), that is

$$y'_{n+q_1}=f(x_{n+q_1},y_{n+q_1})=f[x_n,y_n+\left(\frac{7}{90}+\frac{3\sqrt{3}}{320}\right)hf_{n+q_1}+\left(\frac{27}{320}-\frac{3\sqrt{3}}{40}\right)hf_{n+q_2}$$

$$+\left(\frac{13}{90}-\frac{\sqrt{3}}{15}\right)hf_{n+q_3}+\left(\frac{37}{320}-\frac{3\sqrt{3}}{40}\right)hf_{n+q_4}+\left(\frac{7}{90}-\frac{41\sqrt{3}}{960}\right)hf_{n+q_5}]$$

$$y'_{n+q_2}=f(x_{n+q_2},y_{n+q_2})=f[x_n,y_n+\left(\frac{29}{360}+\frac{3\sqrt{3}}{64}\right)hf_{n+q_1}+\frac{31}{320}hf_{n+q_2}-\frac{1}{90}hf_{n+q_3}$$

$$+\frac{1}{320}hf_{n+q_4}+\left(\frac{29}{360}-\frac{3\sqrt{3}}{64}\right)hf_{n+q_5}]$$

$$y'_{n+q_3}=f(x_{n+q_3},y_{n+q_3})=f[x_n,y_n+\left(\frac{7}{90}+\frac{\sqrt{3}}{24}\right)hf_{n+q_1}+\frac{9}{40}hf_{n+q_2}+\frac{13}{90}hf_{n+q_3}$$

$$-\frac{1}{40}hf_{n+q_4}+\left(\frac{7}{90}-\frac{\sqrt{3}}{24}\right)hf_{n+q_5}]$$

$$y'_{n+q_4}=f(x_{n+q_4},y_{n+q_4})=f[x_n,y_n+\left(\frac{3}{40}+\frac{3\sqrt{3}}{64}\right)hf_{n+q_1}+\frac{63}{320}hf_{n+q_2}+\frac{3}{10}hf_{n+q_3}$$

$$+\frac{33}{320}hf_{n+q_4}+\left(\frac{3}{40}-\frac{3\sqrt{3}}{64}\right)hf_{n+q_5}]$$

$$y'_{n+q_5}=f(x_{n+q_5},y_{n+q_5})=f[x_n,y_n+\left(\frac{7}{90}+\frac{41\sqrt{3}}{960}\right)hf_{n+q_1}+\left(\frac{27}{320}+\frac{3\sqrt{3}}{40}\right)hf_{n+q_2}$$

$$+\left(\frac{13}{90}+\frac{\sqrt{3}}{15}\right)hf_{n+q_3}+\left(\frac{37}{320}+\frac{3\sqrt{3}}{40}\right)hf_{n+q_4}+\left(\frac{7}{90}-\frac{3\sqrt{3}}{320}\right)hf_{n+q_5}] \qquad (2.12)$$

substituting, $k_1=f(x_n,y_{n+q_1})=f_{n+q_1}$, $k_2=f(x_n,y_{n+q_2})=f_{n+q_2}$ etc.
we obtain the following function evaluations:

$$k_1=f[x_n,y_n+\left(\frac{7}{90}+\frac{3\sqrt{3}}{320}\right)hk_1+\left(\frac{27}{320}-\frac{3\sqrt{3}}{40}\right)hk_2+\left(\frac{13}{90}-\frac{\sqrt{3}}{15}\right)hk_3+\left(\frac{37}{320}-\frac{3\sqrt{3}}{40}\right)hk_4$$

$$+\left(\frac{7}{90}-\frac{41\sqrt{3}}{960}\right)hk_5]$$

$$k_2=f[x_n,y_n+\left(\frac{29}{360}+\frac{3\sqrt{3}}{64}\right)hk_1+\frac{31}{320}hk_2-\frac{1}{90}hk_3+\frac{1}{320}hk_4+\left(\frac{29}{360}-\frac{3\sqrt{3}}{64}\right)hk_5]$$

$$k_3=f[x_n,y_n+\left(\frac{7}{90}+\frac{\sqrt{3}}{24}\right)hk_1+\frac{9}{40}hk_2+\frac{13}{90}hk_3-\frac{1}{40}hk_4+\left(\frac{7}{90}-\frac{\sqrt{3}}{24}\right)hk_5]$$

$$k_4=f[x_n,y_n+\left(\frac{3}{40}+\frac{3\sqrt{3}}{64}\right)hk_1+\frac{63}{320}hk_2+\frac{3}{10}hk_3+\frac{33}{320}hk_4+\left(\frac{3}{40}-\frac{3\sqrt{3}}{64}\right)hk_5]$$

$$k_5 = f[x_n, y_n + \left(\frac{7}{90} + \frac{41\sqrt{3}}{960}\right)hk_1 + \left(\frac{27}{320} + \frac{3\sqrt{3}}{40}\right)hk_2 + \left(\frac{13}{90} + \frac{\sqrt{3}}{15}\right)hk_3 + \left(\frac{37}{320} + \frac{3\sqrt{3}}{40}\right)hk_4 + \left(\frac{7}{90} - \frac{3\sqrt{3}}{320}\right)hk_5\Big] \quad (2.13)$$

The Butcher coefficient tableau for the above function evaluations can be written in the form below

C	(A)					(U)
$\left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)$	$\left(\frac{7}{90} + \frac{3\sqrt{3}}{320}\right)$	$\left(\frac{27}{320} - \frac{3\sqrt{3}}{40}\right)$	$\left(\frac{13}{90} - \frac{\sqrt{3}}{40}\right)$	$\left(\frac{37}{320} - \frac{3\sqrt{3}}{40}\right)$	$\left(\frac{7}{90} - \frac{41\sqrt{3}}{960}\right)$	1
$\frac{1}{4}$	$\left(\frac{29}{360} + \frac{3\sqrt{3}}{64}\right)$	$\frac{31}{320}$	$-\frac{1}{90}$	$\frac{1}{320}$	$\left(\frac{29}{360} - \frac{3\sqrt{3}}{64}\right)$	1
$\frac{1}{2}$	$\left(\frac{7}{90} + \frac{\sqrt{3}}{24}\right)$	$\frac{9}{40}$	$\frac{13}{90}$	$-\frac{1}{40}$	$\left(\frac{7}{90} - \frac{\sqrt{3}}{24}\right)$	1
$\frac{3}{4}$	$\left(\frac{3}{40} + \frac{3\sqrt{3}}{64}\right)$	$\frac{63}{320}$	$\frac{3}{10}$	$\frac{33}{320}$	$\left(\frac{3}{40} - \frac{3\sqrt{3}}{64}\right)$	1
$\left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right)$	$\left(\frac{7}{90} + \frac{41\sqrt{3}}{960}\right)$	$\left(\frac{27}{320} + \frac{3\sqrt{3}}{40}\right)$	$\left(\frac{13}{90} + \frac{\sqrt{3}}{15}\right)$	$\left(\frac{37}{320} + \frac{3\sqrt{3}}{40}\right)$	$\left(\frac{7}{90} - \frac{3\sqrt{3}}{320}\right)$	1
b_i	$\frac{7}{45}$	$\frac{1}{5}$	$\frac{13}{45}$	$\frac{1}{5}$	$\frac{7}{45}$	1
	(B)					(V)

(2.14)

The general R-K for this method is defined as

$y_{n+1} = y_n + b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5$

where

$b_1 = \frac{7}{45}, b_2 = \frac{1}{5}, b_3 = \frac{13}{45}, b_4 = \frac{1}{5}, b_5 = \frac{7}{45}$ *and* k_i *defined as* (2.13)

Note that the values of $b_i, i = 1,2,..5$ are obtained by evaluating the continuous scheme (2.10a) at $x = x_{n+1}$

The above table can be summarized as

A	U	
B	V	

where

$U = (1,1,1,1,1)^T$, $V = (1)$, $A = (a_{ij})$, a_{ij} are the coefficient of the Butcher table and $B = b_i\ i = 1, \ldots,5$.

3. ANALYSIS OF THE SCHEME

(i) The exact solution is $y(x_{n+i})$ and Runge-kutta solution is (y_{n+i}) and the error of the method is $E_r = y(x_{n+i}) - y_{n+i}$ and absolute error is AE_r is

$AE_r = |y(x_{n+i}) - y_{n+i}|$

The Order of the method is p , where

$y(x_{n+i}) - y_{n+i} = C_{p+1}y^{(n+1)}(x) + C_{p+2}y^{(n+2)}(x) \ldots. \quad = 0h^{(p+1)}$,

$c_1 = c_1 = c_1 = c_1 = \cdots . = c_p = 0$ and $c_{p+1} \neq 0$, c_{p+1} is the error constant see Butcher [2].

(ii) **Consistency**

The Runge-kutta method is consistent because

$$\sum_{j=0}^{s} a_{i,j} = c_i, \sum_{i=0}^{s} b_i = 1$$

see Butcher Table of (214).

(iii) **Stability**

The stability function $\Re(z)$ is defined as

$R(z) = I + ZB\ (I - ZA)^{-1}e$

$e = (1,1,1,1,1)$, $Z = \lambda h$, A is the coefficient matrix, I identity

$B = (b_1, b_2, b_3, b_4, b_5)$, weights.

The A-stability Region is the set of points satisfying

$\big(R(z)\big) = \{z:\ |R_e(z)| < 1\}$ and the A-stable is the region
$\big(R(z)\big) = \{z:\ R(z) \leq 0\ and\ |R(z)| \leq 1\}$

The characteristic polynomial is
$P(z) = \det(R(z) - \lambda I)$. The region of convergence can be plotted using mat lap etc

(iv) The exact solution at $x = x_i$ is defined as $y(x)$ and the approximate solution is $x = x_i$ is $y_{i,}$

Using Taylor's series expansion

$$y(x_{n+i}) = y_0 + \frac{n}{1!}hy'(x_i) + \frac{n(n-1)h^2}{2!}y''(x_i) + \cdots + \frac{n(n-1)(n-2)h^{n-1}}{n!}y^{n-1}(x_i)$$

and

$$y_{n+i} = y_n + \frac{7}{45}hk_1 + \frac{1}{5}hk_2 + \frac{13}{45}hk_3 + \frac{1}{5}hk_4 + \frac{7}{45}hk_5$$

The order of our scheme is $p = 6$, since $c_0 = c_1 = \cdots = c_6 = 0$ and $c_{p+1} = c_7 = \frac{1}{8601600}$ error constant.

4. NUMERICAL EXPERIMENTS

We use similar methods developed by lie S Norsett[11] and Adegboye and Yahaya [12] to observe the level of performances and efficiency of our new method in the following problems:

Problem 1

$y' = 8(x - y) + 1\ , y(0) = 2, h = 0.02$

Exact solution: $y(x) = x + 2e^{-8x}$

Problem 2

$y' = Sinx - 2y\,, \qquad y\left(\frac{\pi}{2}\right) = 3, h = \frac{\pi}{50}$

Exact solution: $y(x) = \frac{1}{10}\ Cosx + \frac{3}{10}Sinx + \frac{27}{10}\left(\frac{e^{-3x}}{e^{-\frac{3\pi}{2}}}\right)$

Problem 3

$y' + 20y = 20x^2 + 2x\ , y(0) = \frac{1}{3}, h = 0.025$

Exact solution: $y(x) = x^2 + \frac{1}{3}\ e^{-20x}$

The following are solution tables for the given problems above.

Table 1: Comparison of Numerical solution of Problem 1

x	Exact solution	Present method of (2.14)	Error of method [10]	Error of Present method (2.14)
0.02	1.724287577932420	1.724287577929230	2.56 E -11	3.19 E -12
0.04	1.492298074147380	1.492298074141940	4.35 E -11	5.44 E -12
0.06	1.297566783612280	1.297566783556630	5.57 E -11	6.96 E -12
0.08	1.134584848086100	1.134584848078190	6.32 E -11	7.91 E -12
0.10	0.998657928234444	0.998657928226020	6.74 E -11	8.42 E -12

Table 2: Comparison of Numerical solution of Problem 2

x	Exact solution	Present method of (2.14)	Error of method [10]	Error of Present method (2.14)
$\frac{13\pi}{25}$	2.5418383600099	2.541838359996723	1.05 E -10	1.32 E -11
$\frac{27\pi}{50}$	2.162157581773023	2.162157581751139	1.75 E -10	2.19 E -11
$\frac{14\pi}{25}$	1.847250042547062	1.847250042519876	2.17 E -10	2.72 E -11
$\frac{29\pi}{50}$	1.585764824898583	1.585764824898583	2.40 E -10	3.00 E -11
$\frac{3\pi}{5}$	1.368303725239477	1.368303725208400	2.48 E -10	3.10 E -11

Table 3: Comparison of Numerical solution of Problem 3

x	Exact solution	Present method of (2.14)	Error of method [11]	Error of Present method (2.14)
0.5	0.125126480390481	0.12512647903	1.97 E -09	1.36 E -09
0.1	0.055111761078871	0.0551117600813	1.30 E -08	9.98 E -10
0.15	0.039095689455955	0.0390956889055	6.61 E -09	5.50 E -10
0.2	0.046105212962911	0.0461052126929	3.93 E -09	2.70 E -10

Note: The Method [11] has 5 iterations before obtaining the next solution while present method [2.14] has only 2 iterations.

5. CONCLUSION

We developed a highly super convergent Runge-kutta method for solution of first order order ODEs. The method is more superior than Lie and Norsett methods (see numerical Examples). It has lesser implementation cost.(see example3). For similar method mentioned in this research work you need to iterate five times before getting to the next solution (see Yahaya and Adegboye [11]). In our new method only two iterations is needed before getting the next solution and the results obtained are more efficient and stable than methods ([10], [11]). Finally we have successfully developed a new super convergent implicit RK method which converges perfectly more than the existing methods.

6. REFERENCES

1. Lambert J.D (1973): Computational Methods in Ordinary Differential Equations .John Wiley and Sons, New York. 278
2. Butcher JC (1996) general linear methods, *Computational Mathematics and Application, volume (31)4,5 pages 105-112*
3. Adeniyi RB, Adeyefa EO and Alabi MO (2006): A continuous formulation of some classical initial value solvers by non-perturbed multi-step collocation approach using chebysheve basis function. *Journal of Nigerian of Association of mathematical Physics volume 10, pages 261-274*
4. Adeniyi RB and Alabi MO (2006). Derivation of continuous multi step method using Chebyshev polynomial basis functions. *ABACUS, volume 23, N0 2, pages 72-83*
5. Fox L and Parker IB (1968): "Chebyshev polynomial in Numerical Analysis" Addison-Wesley Phillipines
6. Onumanyi P and Awoyemi D.O, Jator S.N. and Siriseria U.W.(1994) " New Linear Multistep Methods with continuous coefficients for first order ivps" *Journal of Nigeria mathematics society* 13: 37- 51.

7. Yakubu DG, Garba EJD and Adamu MS (2002) New Continuous Implicit Runge-kutta method from collocation for Stiff differential systems. *Nigerian journal of Basic and Applied Science, volume 11, pages 119-128*

8. Chollom J and Jackiewicz Z (2003) "Construction of two step Runge-Kutta methods with large region of absolute stability" *Journal of Computer and Applied maths Volume 159 pp 125-137*

9. Yakubu DG , Hamza A, Markus S, Kwami AM and Tumba P (2010) Uniformly Accurate order five Radau-Runge-kutta collocation methods. Journal of the Mathematical Association of Nigerian-Abacus pages 75-94

10. Lie .I. and Norsetts (1981) " Super convergence for multistep collocation" *mathematics of computations*, volume 52, No 185, pp65-79

11. Yahaya YA and Adegboye ZA (2011): Reformulation of Quade's Type four-step Block Hybrid Multistep method into Runge-kutta method for solution of first and second order ODEs. Journal of Mathematical Association of Nigeria (Abacus) volume 38 , No 2 pages 114-124