Intuitionistic Fuzzy Representations of Intuitionistic Fuzzy Groups

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ABSTRACT— ***In this paper, we study the intuitionistic fuzzy representations of intuitionistic fuzzy groups. A fundamental theorem of intuitionistic fuzzy representations of quotient groups has been derived. It is shown that every intuitionistic fuzzy representation gives rise to an intuitionistic fuzzy representation on the factor group. Moreover, an intuitionistic fuzzy version of the famous Cayley's theorem has also been derived.***

Keywords— Intuitionistic fuzzy group, intuitionistic fuzzy homomorphism (isomorphism), intuitionistic fuzzy representation .

1. INTRODUCTION

The theory of representations has been a powerful tool used in the study of groups. It is concern with the classification of homomorphisms of abstract finite groups into groups of matrices or linear transformations. Frobenius developed the group representation theory at the end of the 19^{th} century. The works of Burnside on representation theory mainly focus on the group theoretical calculations which are easier to carry out in the group of matrices than in abstract groups. The representation theory was developed using the notion of embedding a group G into a general linear group GL(V). The theory has important applications to physics, especially in quantum mechanics.

In this paper, we study the intuitionistic fuzzy representations of intuitionistic fuzzy groups. A fundamental theorem of intuitionistic fuzzy representations of quotient groups has been introduced. Moreover, an intuitionistic fuzzy version of the famous Cayley's theorem has also been derived.

2. PRELIMINARIES

In this section, we list some basic concepts and well known results on intuitionistic fuzzy groups for the sake of completeness of the topic under study.

Definition (2.1)[2, 3] Let X be a non-empty fixed set. An intuitionistic fuzzy set (IFS) A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.

Remark (2.2)(i) When $\mu_A(x) + \nu_A(x) = 1$, i.e. when $\nu_A(x) = 1 - \mu_A(x) = \mu_A^c(x)$. Then A is called a **fuzzy set.**

(ii) We denote the IFS $A=\{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ by $A = (\mu_A, \nu_A)$

Definition (2.3) [2] Let A and B be IFS's of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in X\}$. Then

(i) $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$.

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(iii) $A^c = \{ \langle x, \nu_A(x), \mu_A(x)\rangle / x \in X \}$.

(iv) $A \cap B = \{ \langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle / x \in X \}$.

(v) $A \cup B = \{ \langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle / x \in X \}$.

Definition (2.4)[7] An IFS $A = (\mu_A, \nu_A)$ of a group G is said to **be intuitionistic fuzzy group** (IFG) or **intuitionistic fuzzy subgroup of G** (IFSG) of G if

(i) $\mu_A(xy) \ge \mu_A(x) \wedge \mu_A(y)$ (ii) $\mu_A(x^{-1}) = \mu_A(x)$

(iii) $\nu_A(xy) \le \nu_A(x) \vee \nu_A(y)$ (iv) $\nu_A(x^{-1}) = \nu_A(x)$, for all $x, y \in G$

Equivalently, an IFS A of a group G is IFSG of G if

$\mu_A(xy^{-1}) \ge \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \le \nu_A(x) \vee \nu_A(y)$ holds for all $x, y \in G$.

Definition (2.5)[7] An IFSG $A = (\mu_A, \nu_A)$ of a group G is said to be **intuitionistic fuzzy normal subgroup** of G (In short IFNSG) of G if

(i) $\mu_A(xy) = \mu_A(yx)$ (ii) $\nu_A(xy) = \nu_A(yx)$, for all $x, y \in G$

Or Equivalently A is an IFNSG of a group G is **normal** if and only if
$\mu_A(y^{-1} x y) = \mu_A(x)$ and $\nu_A(y^{-1} x y) = \nu_A(x)$, for all $x, y \in G$

Definition(2.6)[7] Let A be intuitionistic fuzzy set of a universe set X . Then (α, β)-cut of A is a crisp subset $C_{\alpha,\beta}(A)$ of the IFS A is given by

$C_{\alpha,\beta}(A) = \{ x : x \in X$ such that $\mu_A(x) \ge \alpha, \nu_A(x) \le \beta \}$, where $\alpha, \beta \in [0,1]$ with $\alpha + \beta \le 1$.

Theorem (2.7) [7] If A is IFS of a group G . Then A is IFSG (IFNSG) of G if and only if $C_{\alpha,\beta}(A)$ is a subgroup (normal) of group G , for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \le 1$.

Definition (2.8) [8**]** Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Let A and B be IFSs of X and Y respectively. Then the image of A under the map f is denoted by $f(A)$ and is defined as

$$\mu_{f(A)}(y) = \begin{cases} \vee\{ \mu_A(x) : x \in f^{-1}(y)\} \\ 0 \quad ; \quad \text{otherwise} \end{cases} \text{ and } \nu_{f(A)}(y) = \begin{cases} \wedge\{ \nu_A(x) : x \in f^{-1}(y)\} \\ 1 \quad ; \quad \text{otherwise} \end{cases}.$$

Also the pre-image of B under f is denoted by $f^{-1}(B)$ and is defined as

$$f^{-1}(B)(x) = B(f(x)) \;;\; \forall\, x \in X$$

Remark(2.9) Note that $\mu_A(x) \le \mu_{f(A)}(f(x))$ and $\nu_A(x) \ge \nu_{f(A)}(f(x))$; $\forall\, x \in X$, and equality hold when f is bijective.

Theorem (2.10)[8] Let $f : G_1 \to G_2$ be surjective homomorphism and A be IFSG (IFNSG) of group G_1. Then $f(A)$ is IFSG (IFNSG) of group G_2.

Theorem (2.11)[8] Let $f : G_1 \to G_2$ be homomorphism of group G_1 into a group G_2. Let B be IFSG (IFNSG) of group G_2. Then $f^{-1}(B)$ is IFSG (IFNSG) of group G_1.

3. INTUITIONISTIC FUZZY REPRESENTATIONS OF INTUITIONISTIC FUZZY GROUPS

Definition (3.1) [4] Let G be a group and M be a vector space over a field K. A ***linear representation of G*** with representation space M is a homomorphism of G into GL(M), where GL(M) is the group of units in $Hom_K(M,M)$ called the general linear group GL(M).

Definition(3.2) Let G and G_1 be groups. Let A be a intuitionistic fuzzy group on G and B be a intuitionistic fuzzy group on G_1. Let *f* be a group homomorphism of G onto G_1. Then *f* is called a ***weak intuitionistic fuzzy homomorphism*** of A into B if *f* (A) $\subseteq$ B. The homomorphism *f* is an intuitionistic fuzzy homomorphism of A onto B if *f* (A) = B. We say that A is an intuitionistic fuzzy homomorphic to B and we write A ≈ B.

Let *f* : G → G_1 be an isomorphism. Then *f* is called a ***weak intuitionistic fuzzy isomorphism*** *f* (A) $\subseteq$ B and *f* is an ***intuitionistic fuzzy isomorphism*** if *f* (A) = B.

Definition (3.3) *Let* G be a group M be a vector space over K and T: G→GL(M) be a representation of G in M. Let A be an intuitionistic fuzzy group on G and B be a intuitionistic fuzzy group on the range of T. Then the representation T is a *intuitionistic fuzzy representation* if T is an intuitionistic fuzzy homomorphism of A onto B .

Example (3.4) Let G = (Z, +) and M be a vector space over R. Let T: G→ GL(M) be defined as T(*x*) = T_x , where T_x : M → M, such that $T_x(m) = xm$, for $x \in$ M. Then T is a representation.

Now, we defined the IFS A on G by

$$\mu_A(x) = \begin{cases} 1 & ;\text{ when } x \text{ is even} \\ 0.5 & ;\text{ when } x \text{ is odd} \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & ;\text{ when } x \text{ is even} \\ 0.2 & ;\text{ when } x \text{ is odd} \end{cases}, \quad \forall\, x \in G.$$

Then A is IFSG of G. Let B be IFSG on the range of T defined by

$$\mu_B(T_x) = \begin{cases} 1 & ;\text{ when } x \text{ is even} \\ 0.5 & ;\text{ when } x \text{ is odd} \end{cases} \quad \text{and} \quad \nu_B(T_x) = \begin{cases} 0 & ;\text{ when } x \text{ is even} \\ 0.2 & ;\text{ when } x \text{ is odd} \end{cases}, \quad \forall\, T_x \in T(\text{G}).$$

Then, we have

$$T(A)(T_x) = \left(\mu_{T(A)}(T_x),\ \nu_{T(A)}(T_x)\right), \text{ where}$$

$$\mu_{T(A)}(T_x) = \vee\left\{\mu_A(z):\ z \in T^{-1}(T_x)\right\} \quad \text{and} \quad \nu_{T(A)}(T_x) = \wedge\left\{\nu_A(z):\ z \in T^{-1}(T_x)\right\}$$

$$Now,\ \mu_{T(A)}(T_{even}) = \vee\left\{\mu_A(z):\ z \in T^{-1}(T_{even})\right\} = 1;\ \nu_{T(A)}(T_{even}) = \wedge\left\{\nu_A(z):\ z \in T^{-1}(T_{even})\right\} = 0,$$

$$and\ \ \mu_{T(A)}(T_{odd}) = \vee\left\{\mu_A(z):\ z \in T^{-1}(T_{odd})\right\} = 0.5;\ \nu_{T(A)}(T_{odd}) = \wedge\left\{\nu_A(z):\ z \in T^{-1}(T_{odd})\right\} = 0.2.$$

∴ T(A) = B. Hence T is an intuitionistic fuzzy representation of A onto B.

Theorem (3.5) Let A be an IFSG of G and N be a normal subgroup of G.

Let $A_N = (\mu_{A_N}, \nu_{A_N})$, where μ_{A_N} , ν_{A_N} : G/N → [0,1] defined by

μ_{A_N} (*x*N) = ∨{$\mu_A(xn)$: $n \in$N} and ν_{A_N} (*x*N) = ∧{$\nu_A(xn)$: $n \in$N}, $\forall x \in$G.

Then A_N is an IFSG of G/N.

Proof. Let A= (μ_A, ν_A) be a IFSG of G and N be a normal subgroup of G.

Define $A_N = (\mu_{A_N}, \nu_{A_N})$, where μ_{A_N} , ν_{A_N} : G/N → [0,1] by

μ_{A_N} (*x*N) = ∨{$\mu_A(xn)$: $n \in$N} and ν_{A_N} (*x*N) = ∧{$\nu_A(xn)$: $n \in$ N}, $\forall x \in$G

Let xN, yN $\in$ G/N, where $x, y\in$G, we have

μ_{A_N} {(xN)(yN)} = μ_{A_N} {(xy)N}

$= \vee\{ \mu_A(\{xy\}n) : n \in N\}$

$= \vee\{ \mu_A(\{xy\}n_1n_2) : n_1, n_2 \in N \}$, where $n = n_1n_2$, for some $n_1, n_2 \in N$

$\geq \vee \{ \mu_A(xn_1) \wedge \mu_A(yn_2) : n_1, n_2 \in N \}$

$\geq [\vee\{\mu_A(xn_1) : n_1 \in N \}] \wedge [\vee\{ \mu_A(yn_2) : n_2 \in N \}]$

$= \mu_A(xN) \wedge \mu_A(yN)$

Thus, $\mu_{A_N}[(xN)(yN)] \geq \mu_{A_N}(xN) \wedge \mu_{A_N}(yN)$.

Similarly, we can show that $\nu_{A_N}[(xN)(yN)] \leq \nu_{A_N}(xN) \vee \nu_{A_N}(yN)$.

Also, $\mu_{A_N}\{(xN)^{-1}\} = \mu_{A_N}(x^{-1}N) = \vee\{\mu_A(x^{-1} n) : n\in N\}= \vee\{\mu_A(xn) : n\in N\}= \mu_{A_N}(xN)$

Similarly, we can show that $\nu_{A_N}\{(xN)^{-1}\} = \nu_{A_N}(xN)$.

Hence A_N is an intuitionistic fuzzy subgroup of G/N.

Theorem (3.6) (A Fundamental Theorem of Intuitionistic fuzzy representation)

Let G be a group, M be a vector space over K and T:G→GL(M) be a representation of G, then ψ: G/N→GL(M) defined by $\psi(xN) = T(x) = T_x$, $\forall$ $x\in$G, is an intuitionistic fuzzy representation of G/N, where N is a normal subgroup of G.

Proof. Let A= (μ_A, ν_A) be a IFSG of G. Since T is an intuitionistic fuzzy representation, there exist an IFSG B on T(G) such that T(A) = B. We have to prove that ψ is an intuitionistic fuzzy representation on G/N.

$$\begin{array}{ccc} G & \xrightarrow{T} & GL(M) \\ (\mu_A,\ \nu_A)\downarrow & \square & \uparrow \psi \\ I\times I & \xleftarrow[(\mu_{A_N},\ \nu_{A_N})]{} & G/N \end{array}$$

Given ψ: G/N→GL(M) defined by $\psi(xN) = T(x) = T_x$, $\forall$ $x \in$G. Then ψ is a homomorphism of G/N into GL(M), for xN, yN $\in$ G/N, where $x, y\in$G, we have $\psi((xN)(yN)) = \psi((xyN)) = T_{xy}$. Then for $m\in$M , we have

$T_{xy}(m) = (xy)(m) = x(ym) = T_x(ym) = T_x(T_y(m)) = (T_xT_y)(m)$ $\therefore$ $T_{xy} = T_xT_y$

Thus, we have $\psi((xN)(yN)) = \psi(xN)\, \psi(yN)$.

Also, $T_{\alpha x}(m) = (\alpha x)(m) = \alpha(xm) = \alpha T_x(m) = (\alpha T_x)(m)$, for $m\in$M and $\alpha\in$K

$\therefore$ $T_{\alpha x} = \alpha T_x$. Thus, we have $\psi\{(\alpha x)N\} = \alpha\, \psi(xN)$

Hence ψ: G/N→GL(M) is a representation .

Let $A_N = (\mu_{A_N}, \nu_{A_N})$, where μ_{A_N}, ν_{A_N} : G/N →[0,1] defined by $\mu_{A_N}(xN) = \vee\{\mu_A(xn): n\in N\}$

and $\nu_{A_N}(xN) = \wedge\{\nu_A(xn): n\in N\}$, $\forall x\in$G be intuitionistic fuzzy subgroup of G/N. Then

$\psi(A_N)(T_x) = \left(\mu_{\psi(A_N)}(T_x),\ \nu_{\psi(A_N)}(T_x)\right)$, where $\mu_{\psi(A_N)}(T_x) = \vee\left\{\mu_{A_N}(xN) :\ xN \in \psi^{-1}(T_x),\ T_x \in \psi(G/N)\right\}$

and $\nu_{\psi(A_N)}(T_x) = \wedge\left\{\nu_{A_N}(xN) :\ xN \in \psi^{-1}(T_x),\ T_x \in \psi(G/N)\right\}$

$$
\begin{aligned}
Now,\ \mu_{\psi(A_N)}(T_x) &= \vee\left\{\mu_{A_N}(xN) : xN \in \psi^{-1}(T_x),\ T_x \in \psi(G/N)\right\} \\
&= \vee\left\{\vee\left\{\mu_A(z) : z \in xN,\ T_x \in T(G)\right\}\right\} \\
&= \vee\left\{\mu_A(z) : z \in xN,\ T_x \in T(G)\right\} \\
&= \mu_{T(A)}(T_x).
\end{aligned}
$$

Similarly, we can show that $\nu_{\psi(A_N)}(T_x) = \nu_{T(A)}(T_x)$.

Thus, $\psi(A_N) = T(A) = B$. Hence ψ is an intuitionistic fuzzy representation of A_N onto B.

Example (3.7) Let G = {1, -1, *i*, - *i* }, a group under multiplication and M be a vector space over R. Let N = {1, -1}. Then N is a normal subgroup of G. Let T : G → GL(M) be defined by T(x) = T_x , where $T_x(m) = xm$, $\forall\ x \in G$ and $m \in M$. Define the IFS A on G by

$$
\mu_A(x) = \begin{cases} 1 & ;\ \text{when } x = 1 \text{ or } -1 \\ 0.6 & ;\ \text{when } x = i \text{ or } -i \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & ;\ \text{when } x = 1 \text{ or } -1 \\ 0.25 & ;\ \text{when } x = i \text{ or } -i \end{cases},\ \forall\ x \in G .
$$

Then A is IFSG of G. Let B be a IFSG on T(G), defined by

$$
\mu_B(T_x) = \begin{cases} 1 & ;\ \text{when } x = 1 \text{ or } -1 \\ 0.6 & ;\ \text{when } x = i \text{ or } -i \end{cases} \quad \text{and} \quad \nu_B(T_x) = \begin{cases} 0 & ;\ \text{when } x = 1 \text{ or } -1 \\ 0.25 & ;\ \text{when } x = i \text{ or } -i \end{cases},\ \forall\ T_x \in T(G).
$$

Then $\mathrm{T(A)(T}_x) = \left(\mu_{\mathrm{T(A)}}(\mathrm{T}_x),\ \nu_{\mathrm{T(A)}}(\mathrm{T}_x)\right)$, where

$\mu_{\mathrm{T(A)}}(\mathrm{T}_x) = \vee\left\{\mu_{\mathrm{A}}(x) : x \in T^{-1}(\mathrm{T}_x)\right\}$ and $\nu_{\mathrm{T(A)}}(\mathrm{T}_x) = \wedge\left\{\nu_{\mathrm{A}}(x) : x \in T^{-1}(\mathrm{T}_x)\right\}$.

Thus, we have $\mu_{\mathrm{T(A)}}(\mathrm{T}_1) = 1 = \mu_{\mathrm{T(A)}}(\mathrm{T}_{-1})$ and $\nu_{\mathrm{T(A)}}(\mathrm{T}_1) = 0 = \nu_{\mathrm{T(A)}}(\mathrm{T}_{-1})$

also, $\mu_{\mathrm{T(A)}}(\mathrm{T}_i) = 0.6 = \mu_{\mathrm{T(A)}}(\mathrm{T}_{-i})$ and $\nu_{\mathrm{T(A)}}(\mathrm{T}_i) = 0.25 = \nu_{\mathrm{T(A)}}(\mathrm{T}_{-i})$

Therefore, T is a intuitionistic fuzzy representation of A onto B.

Now, G/N = {N, *i*N}. Let $A_N = (\mu_{A_N}, \nu_{A_N})$, where μ_{A_N} , ν_{A_N} : G/N → [0,1] defined by

μ_{A_N} (xN) = $\vee\{\mu_A(xn) : n \in N\}$ and ν_{A_N} (xN) = $\wedge\{\nu_A(xn) : n \in N\}$, $\forall x \in G$

be intuitionistic fuzzy subgroup of G/N.

Clearly, μ_{A_N} (N) = 1, ν_{A_N} (N) = 0 and μ_{A_N} (*i*N) = 0.6, ν_{A_N} (*i*N) = 0.25

Let ψ: G/N→GL(M) defined by $\psi(x\mathrm{N}) = \mathrm{T}(x) = \mathrm{T}_x$, $\forall\ x \in G$. Then we have

$\psi(A_N)(T_x) = \left(\mu_{\psi(A_N)}(T_x),\ \nu_{\psi(A_N)}(T_x)\right)$, where $\mu_{\psi(A_N)}(T_x) = \vee\left\{\mu_{A_N}(xN) : xN \in \psi^{-1}(T_x),\ T_x \in \psi(G/N)\right\}$

and $\nu_{\psi(A_N)}(T_x) = \wedge\left\{\nu_{A_N}(xN) : xN \in \psi^{-1}(T_x),\ T_x \in \psi(G/N)\right\}$

Now, $\mu_{\psi(A_N)}(T_1) = \vee\left\{\mu_{A_N}(N) : N \in \psi^{-1}(T_1),\ T_1 \in \psi(G/N)\right\} = 1$

and $\nu_{\psi(A_N)}(T_1) = \wedge\left\{\nu_{A_N}(N) : N \in \psi^{-1}(T_1),\ T_1 \in \psi(G/N)\right\} = 0$

Similarly, we can show that $\mu_{\psi(A_N)}(T_{-1}) = 1,\ \nu_{\psi(A_N)}(T_{-1}) = 0,\ \mu_{\psi(A_N)}(T_i) = 0.6,\ \nu_{\psi(A_N)}(T_i) = 0.25,$

$\mu_{\psi(A_N)}(T_{-i}) = 0.6,\ \nu_{\psi(A_N)}(T_{-i}) = 0.25$

Thus, we see that $\psi(\mathrm{A}_N)$ = B. Hence ψ is an intuitionistic fuzzy representation of G/N onto B.

Remark(3.8) when the general linear space, in short GL(M) is replaced by a group G′ , we get the following corollary, which may be called as a “ **fundamental theorem of intuitionistic fuzzy homomorphisms”**.

Corollary (3.9)(i) Let T be an intuitionistic fuzzy homomorphism of A onto B where A is IFSG on G and B is IFSG on T(G). Then ψ : G/N → G′, defined by $\psi(x\mathrm{N}) = \mathrm{T}(x)$, $\forall\ x \in \mathrm{G}$, is a intuitionistic fuzzy homomorphism of A_N onto B, where A_N is an IFSG on G/N, N being a normal subgroup of G.

Proof. Straight forward.

(ii) Let T be an intuitionistic fuzzy homomorphism of G onto G′ and K_T be the kernel of T. If T is an intuitionistic fuzzy homomorphism of A onto B where A is an IFSG on G and B is an IFSG on G′, then ψ : G/K_T → G′ is an intuitionistic fuzzy isomorphism of A_{K_T} onto B where A_{K_T} is an IFSG on G/K_T.

Proof. Straight forward.

Example (3.10) Let G = (Z, +), N = set of all even integers and G′ = {1, -1}, the group under multiplication. N is a normal subgroup of G.

Define T : G → G′ by $T(x) = \begin{cases} 1 & ;\ \text{when } x \text{ is even} \\ -1 & ;\ \text{when } x \text{ is odd} \end{cases}$, $\forall\ x \in G$.

Then T is a homomorphism of G onto G′.

Let A be the IFSG on G defined by

$$\mu_A(x) = \begin{cases} t & ;\ \text{when } x \text{ is even} \\ t_0 & ;\ \text{when } x \text{ is odd} \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s & ;\ \text{when } x \text{ is even} \\ s_0 & ;\ \text{when } x \text{ is odd} \end{cases},$$

$$\textit{where} \quad \mathrm{t} > \mathrm{t}_0 \quad ; \quad \mathrm{s} < \mathrm{s}_0 \quad \text{and} \quad \mathrm{t} + \mathrm{s} \le 1 \quad ; \quad \mathrm{t}_0 + \mathrm{s}_0 \le 1$$

Let B be IFSG on G′ defined by

$$\mu_B(y) = \begin{cases} t & ;\ \text{when y} = 1 \\ t_0 & ;\ \text{when y} = -1 \end{cases} \quad \text{and} \quad \nu_B(y) = \begin{cases} s & ;\ \text{when y} = 1 \\ s_0 & ;\ \text{when y} = -1 \end{cases}, \ \forall\ \mathrm{y} \in \mathrm{G}'.$$

$$\textit{Then}, \quad \mathrm{T(A)(y)} = \left(\mu_{\mathrm{T(A)}}(y),\ \nu_{\mathrm{T(A)}}(y)\right), \ \text{where}$$

$$\mu_{\mathrm{T(A)}}(y) = \vee\left\{\mu_{\mathrm{A}}(x) :\ x \in \mathrm{T}^{-1}(y)\right\} \quad \text{and} \quad \nu_{\mathrm{T(A)}}(y) = \wedge\left\{\nu_{\mathrm{A}}(x) :\ x \in \mathrm{T}^{-1}(y)\right\}.$$

$$\textit{Now}, \ \mu_{\mathrm{T(A)}}(1) = \vee\left\{\mu_{\mathrm{A}}(x) :\ x \in \mathrm{T}^{-1}(1)\right\} = \vee\left\{\mu_{\mathrm{A}}(x) :\ x \text{ is even integer}\right\} = t \quad \text{and}$$

$$\nu_{\mathrm{T(A)}}(1) = \wedge\left\{\nu_{\mathrm{A}}(x) :\ x \in \mathrm{T}^{-1}(1)\right\} = \wedge\left\{\nu_{\mathrm{A}}(x) :\ x \text{ is even integer}\right\} = s$$

$$\textit{Also}, \ \mu_{\mathrm{T(A)}}(-1) = \vee\left\{\mu_{\mathrm{A}}(x) :\ x \in \mathrm{T}^{-1}(-1)\right\} = \vee\left\{\mu_{\mathrm{A}}(x) :\ x \text{ is odd integer}\right\} = t_0 \quad \text{and}$$

$$\nu_{\mathrm{T(A)}}(-1) = \wedge\left\{\nu_{\mathrm{A}}(x) :\ x \in \mathrm{T}^{-1}(-1)\right\} = \wedge\left\{\nu_{\mathrm{A}}(x) :\ x \text{ is odd integer}\right\} = s_0$$

∴ T(A) = B , Therefore, T is an intuitionistic fuzzy homomorphism of A into B.

Now, G/N = { x + N : $x \in$ G }. Define $A_N = (\mu_{A_N}, \nu_{A_N})$, where μ_{A_N} , ν_{A_N} : G/N → [0,1] by

$\mu_{A_N}(x + \mathrm{N}) = \vee\{\mu_A(x + n) : n \in \mathrm{N}\}$ and $\nu_{A_N}(x + \mathrm{N}) = \wedge\{\nu_A(x + n) : n \in \mathrm{N}\}$, $\forall x \in$ G.

$$\textit{Clearly}, \quad \mu_{A_N}\left(x + N\right) = \begin{cases} t & ;\ x \text{ is even} \\ t_0 & ;\ x \text{ is odd} \end{cases} \quad \text{and} \quad \nu_{A_N}\left(x + N\right) = \begin{cases} s & ;\ x \text{ is even} \\ s_0 & ;\ x \text{ is odd} \end{cases}.$$

Then $\psi : G/N \rightarrow G'$ *defined by* $\psi(x+N) = T(x), \ \forall \ x \in G.$

Now, $\psi(A_N)(y) = \left(\mu_{\psi(A_N)}(y), \ \nu_{\psi(A_N)}(y)\right)$, where

$\mu_{\psi(A_N)}(y) = \vee\left\{\mu_{A_N}(x+N): \ x+N \in \psi^{-1}(y), y \in G'\right\}$ and

$\nu_{\psi(A_N)}(y) = \wedge\left\{\nu_{A_N}(x+N): \ x+N \in \psi^{-1}(y), y \in G'\right\}$

Thus, $\mu_{\psi(A_N)}(1) = \vee\left\{\mu_{A_N}(x+N): \ x+N \in \psi^{-1}(1), 1 \in G'\right\} = t$ and

$\nu_{\psi(A_N)}(1) = \wedge\left\{\nu_{A_N}(x+N): \ x+N \in \psi^{-1}(1), 1 \in G'\right\} = s.$

Similarly, we get, $\mu_{\psi(A_N)}(-1) = t_0$ and $\nu_{\psi(A_N)}(-1) = s_0$.

Therefore, $\psi(A_N) = B$. Hence ψ is an intuitionistic fuzzy homomorphism of A_N *onto B.*

Remark (3.11) The fundamental theorem illustrates that every intuitionistic fuzzy representation or intuitionistic fuzzy homomorphism on G gives rise to an intuitionistic fuzzy representation or intuitionistic fuzzy homomorphism on the factor group G/N where N is a normal subgroup of G.

Proposition (3.12) Let A and A_N be IFSG of G and G/N respectively where N is a normal subgroup of G. If π is the natural homomorphism from G onto G/N, then π is an intuitionistic fuzzy homomorphism of A onto A_N.

Proof. Since π is the canonical homomorphism of G onto G/N, π(*g*) = *g*N, *g*∈G. We have to show that π is an intuitionistic fuzzy homomorphism of A onto B.

$$G \xrightarrow{\ \pi\ } G/N$$

$$(\mu_A, \ \nu_A) \quad\quad\quad (\mu_{A_N}, \ \nu_{A_N})$$

$$I \times I$$

For every $gN \in G/N$, *we have* $\pi(A)(gN) = \left(\mu_{\pi(A)}(gN), \ \nu_{\pi(A)}(gN)\right)$, where

$$\begin{aligned}
\mu_{\pi(A)}(gN) &= \vee\left\{\mu_A(z): \ z \in \pi^{-1}(gN)\right\} \\
&= \vee\left\{\mu_A(z): \ \pi(z) = gN\right\} \\
&= \vee\left\{\mu_A(z): \ zN = gN\right\} \\
&= \vee\left\{\mu_A(z): \ z = gn, \ n \in N\right\} \\
&= \vee\left\{\mu_A(gn): \ n \in N\right\} \\
&= \mu_{A_N}(gN)
\end{aligned}$$

Similarly, we can show that $\nu_{\pi(A)}(gN) = \nu_{A_N}(gN).$

Therefore, $\pi(A) = A_N$.

Hence π is an intuitionistic fuzzy homomorphism of A onto A_N.

Example (3.13) Let G = {1, -1, *i*, - *i* }; N ={1, -1}; groups under multiplication. Then G/N ={ N , *i*N}. Consider the IFSG A on G defined by

$$\mu_A(x) = \begin{cases} 1 & ; \text{ when } x = 1 \text{ or } -1 \\ 0.6 & ; \text{ when } x = i \text{ or } -i \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & ; \text{ when } x = 1 \text{ or } -1 \\ 0.25 & ; \text{ when } x = i \text{ or } -i \end{cases}, \ \forall \ x \in G$$

Consider that π : G → G/N is the canonical homomorphism, so that π(*g*) = *g*N , *g*∈G.

Define $A_N = (\mu_{A_N}, \nu_{A_N})$, where μ_{A_N}, ν_{A_N} : G/N →[0,1] be the IFSG on G/N by

$$\mu_{A_N}(xN) = \begin{cases} 1 & ;\ if\ x = 1 \\ 0.6 & ;\ if\ x = i \end{cases} \quad \text{and} \quad \nu_{A_N}(xN) = \begin{cases} 0 & ;\ if\ x = 1 \\ 0.25 & ;\ if\ x = i \end{cases}$$

$\pi(A)(xN) = \left(\mu_{\pi(A)}(xN),\ \nu_{\pi(A)}(xN)\right)$, where

$\mu_{\pi(A)}(xN) = \vee\left\{\mu_A(z):\ z \in \pi^{-1}(xN)\right\}$ and $\nu_{\pi(A)}(xN) = \wedge\left\{\nu_A(z):\ z \in \pi^{-1}(xN)\right\}$

Now, $\mu_{\pi(A)}(N) = \vee\left\{\mu_A(z):\ z \in \pi^{-1}(N)\right\} = \vee\left\{\mu_A(z):\ z \in \{1,-1\}\right\} = 1;$

$\nu_{\pi(A)}(N) = \wedge\left\{\nu_A(z):\ z \in \pi^{-1}(N)\right\} = \wedge\left\{\nu_A(z):\ z \in \{1,-1\}\right\} = 0;$

Also, $\mu_{\pi(A)}(iN) = \vee\left\{\mu_A(z):\ z \in \pi^{-1}(iN)\right\} = \vee\left\{\mu_A(z):\ z \in \{i,-i\}\right\} = 0.6;$

$\nu_{\pi(A)}(iN) = \wedge\left\{\nu_A(z):\ z \in \pi^{-1}(iN)\right\} = \wedge\left\{\nu_A(z):\ z \in \{i,-i\}\right\} = 0.25.$

$\therefore$ $\pi(A) = A_N$. Hence π is an intuitionistic fuzzy homomorphism of A onto A_N.

Now, we proceed to prove a proposition which gives us more insight into the relation between two intuitionistic fuzzy groups which are intuitionistic fuzzy homomorphic.

Proposition (3.14) Let T be a homomorphism from G onto $\overline{G}$ and let A and B be IFSG of G and $\overline{G}$ respectively such that T(A) = B. Let π be the natural homomorphism from $\overline{G}$ onto $\overline{G}/\overline{N}$ where $\overline{N}$ is a normal subgroup of $\overline{G}$ such that N = { $x \in$G : T(x) $\in \overline{N}$ }. Then there $\exists$ a homomorphism ρ from G / N to $\overline{G}/\overline{N}$ and ρ is an intuitionistic fuzzy homomorphism of A_N onto $B_{\overline{N}}$ on G / N and $\overline{G}/\overline{N}$ respectively.

Proof. Let T: G→ $\overline{G}$ be the homomorphism of G onto $\overline{G}$, defined by T(g) = $\overline{g}$, $g \in$G, $\overline{g} \in \overline{G}$ Let $\pi : \overline{G} \to \overline{G}/\overline{N}$ be the natural homomorphism. Then π o T = ψ is a homomorphism of G onto $\overline{G}/\overline{N}$ and $\psi(g)$ = (π o T) (g) = π(T(g)) = $\pi(\overline{g}) = \overline{g}\,\overline{N}$.

The IFSG A_N *on* G/N *is defined by* $A_N(xN) = (\mu_{A_N}(xN),\ \nu_{A_N}(xN))$, *where*

$\mu_{A_N}(xN) = \vee\left\{\mu_A(z):\ z \in xN\right\}$ and $\nu_{A_N}(xN) = \wedge\left\{\nu_A(z):\ z \in xN\right\}$

Now, $\psi(N)$ = (π o T) (N) = π(T(N)) = $\pi(\overline{N}) = \overline{N}$. $\therefore$ N is the kernel of ψ.

$\therefore$ ψ is a homomorphism of G onto $\overline{G}/\overline{N}$ with kernel N. Hence $\exists$ a homomorphism ρ of G / N onto $\overline{G}/\overline{N}$, defined by ρ (gN) = $\overline{g}\,\overline{N}$. We have to show that ρ is an intuitionistic fuzzy homomorphism of A_N onto $B_{\overline{N}}$.

G	$\xrightarrow{T}$	$\overline{G}$
(μ_A, ν_A) □	(μ_B, ν_B) □	
	I × I	$\downarrow \pi$
(μ_{A_N}, ν_{A_N}) □	$(\mu_{B_{\overline{N}}}, \nu_{B_{\overline{N}}})$ □	
G/N	$\xrightarrow[\rho]{}$	$\overline{G}/\overline{N}$

For $\overline{g}\,\overline{N} \in \overline{G}/\overline{N}$, we have

$\rho(A_N)(\overline{g}\,\overline{N}) = \left(\mu_{\rho(A_N)}(\overline{g}\,\overline{N}),\ \nu_{\rho(A_N)}(\overline{g}\,\overline{N})\right)$, where

$\mu_{\rho(A_N)}(\overline{g}\,\overline{N}) = \vee\left\{\mu_{A_N}(gN):\ gN \in \rho^{-1}(\overline{g}\,\overline{N})\right\}$ and $\nu_{\rho(A_N)}(\overline{g}\,\overline{N}) = \wedge\left\{\nu_{A_N}(gN):\ gN \in \rho^{-1}(\overline{g}\,\overline{N})\right\}$

$$
\begin{aligned}
Now,\ \mu_{\rho(A_N)}(\overline{g}\overline{N}) &= \vee\left\{\mu_{A_N}(gN):\ gN \in \rho^{-1}(\overline{g}\overline{N})\right\} \\
&= \vee\left\{\vee\{\mu_A(gn):\ n \in N\},\ g \in G,\ T(g) = \overline{g},\ \rho(gN) = T(g)\overline{N})\right\} \\
&= \vee\left\{\mu_A(gn):\ n \in N,\ g \in G,\ T(g) = \overline{g}\right\} \\
&= \vee\left\{\mu_A(z):\ \mathrm{z} \in T^{-1}(\overline{g})\right\} \\
&= \mu_{\mathrm{T(A)}}(\overline{g}) \\
&= \mu_B(\overline{g}) \quad , \quad \overline{g} \in \overline{G} \\
&= \vee\left\{\mu_B(\overline{g}):\ \pi(\overline{g}) = \overline{g}\overline{N}\right\} \\
&= \vee\left\{\mu_B(\overline{g}):\ \overline{g} \in \pi^{-1}\left(\overline{g}\overline{N}\right)\right\} \\
&= \mu_{\pi(\mathrm{B})}(\overline{g}\overline{N}) \\
&= \mu_{B_{\overline{N}}}(\overline{g}\overline{N}) \quad \left[\begin{array}{l} \because \text{ By Proposition (3.11), } \pi \text{ is intuitionistic} \\ \quad \text{fuzzy homomorphism of B onto } \mathrm{B}_{\overline{N}} \end{array}\right]
\end{aligned}
$$

Similarly, we can show that $\nu_{\rho(A_N)}(\overline{g}\overline{N}) = \nu_{B_{\overline{N}}}(\overline{g}\overline{N})$

$\therefore\ \ \rho\ (\mathrm{A}_N) = B_{\overline{N}}$. Hence ρ is an intuitionistic fuzzy homorphism of A_N onto $B_{\overline{N}}$.

Example (3.15) Let G = (Z, +), $\overline{G}$ ={1, -1}, a group under multiplication.

Let $f : G \to \overline{G}$ be defined by $f(x) = \begin{cases} 1 & ;\ \text{when } x \text{ is even} \\ -1 & ;\ \text{when } x \text{ is odd.} \end{cases}$

Then f is a homomorphism of G onto $\overline{G}$. We have N = { x∈G : $f(x) \in \overline{N}$ }.

Let $\overline{N}$ = {1}. Then $\overline{N}$ is a normal subgroup of $\overline{G}$ ∴ N = { x ∈G : $f(x)$ = 1}= 2Z.

We know that N is a normal subgroup of G. Define the IFS A on G by

$\mu_A(x) = \begin{cases} 1 & ;\ \text{when } x \text{ is even} \\ 0.5 & ;\ \text{when } x \text{ is odd} \end{cases}$ and $\nu_A(x) = \begin{cases} 0 & ;\ \text{when } x \text{ is even} \\ 0.2 & ;\ \text{when } x \text{ is odd} \end{cases}$, $\forall\ x \in \mathrm{G}$. Then A is IFSG of G. Define the IFS B on $\overline{G}$ by

$\mu_B(y) = \begin{cases} 1 & ;\ if\ \ \mathrm{y} = 1 \\ 0.5 & ;\ if\ \ \mathrm{y} = -1 \end{cases}$ and $\nu_B(y) = \begin{cases} 0 & ;\ if\ \ \mathrm{y} = 1 \\ 0.2 & ;\ if\ \ \mathrm{y} = -1 \end{cases}$, we have

G / N = { N , 1 + N }, where N = set of even integers and 1 + N = set of odd integers.

Define $A_N = (\mu_{A_N}, \nu_{A_N})$, where μ_{A_N}, ν_{A_N} : G/N → [0,1] be the IFS on G/N by

$\mu_{A_N}(x+N) = \begin{cases} 1 & ;\ \text{when } x = 0 \\ 0.5 & ;\ \text{when } x = 1 \end{cases}$ and $\nu_{A_N}(x+N) = \begin{cases} 0 & ;\ \text{when } x = 0 \\ 0.2 & ;\ \text{when } x = 1 \end{cases}$ Then A_N is an IFSG of G / N. Now, $\overline{G}/\overline{N}$ = { 1. $\overline{N}$, (-1) $\overline{N}$ } = { $\overline{N}$, - $\overline{N}$ }.

Define IFS $B_{\overline{N}}$ on $\overline{G}/\overline{N}$ by

$\mu_{B_{\overline{N}}}(y\overline{N}) = \begin{cases} 1 & ;\ \text{when } \mathrm{y} = 1 \\ 0.5 & ;\ \text{when } \mathrm{y} = -1 \end{cases}$ and $\nu_{B_{\overline{N}}}(y\overline{N}) = \begin{cases} 0 & ;\ \text{when } \mathrm{y} = 1 \\ 0.2 & ;\ \text{when } \mathrm{y} = -1 \end{cases}$

Clearly, $B_{\overline{N}}$ is IFSG on $\overline{G}/\overline{N}$.

Define ρ : G / N → $\overline{G}/\overline{N}$ by ρ (x + N) = $f(x)\,\overline{N}$ = y $\overline{N}$, where y = $f(x)$.

Then, ρ(N) = $f(0)\,\overline{N}$ = 1. $\overline{N}$ = $\overline{N}$, ρ(1 + N) = $f(1)\,\overline{N}$ = (-1) $\overline{N}$.

Thus, we have $\rho(A_N)(y\overline{N}) = \left(\mu_{\rho(A_N)}(y\overline{N}),\ \nu_{\rho(A_N)}(y\overline{N})\ \right)$, where

$\mu_{\rho(A_N)}(y\overline{N}) = \vee\left\{\mu_{A_N}(x+N):\ y = f(x),\ x+N \in \rho^{-1}(y\overline{N})\right\}$ and

$\nu_{\rho(A_N)}(y\overline{N}) = \wedge\left\{\nu_{A_N}(x+N):\ y = f(x),\ x+N \in \rho^{-1}(y\overline{N})\right\}$

$\therefore\quad \mu_{\rho(A_N)}(1.\overline{N}) = \vee\left\{\mu_{A_N}(0+N):\ 1 = f(0),\ N \in \rho^{-1}(\overline{N})\right\} = \vee\left\{\mu_{A_N}(N):\ \rho(N) = \overline{N}\right\} = 1$

and $\nu_{\rho(A_N)}(1.\overline{N}) = \wedge\left\{\nu_{A_N}(0+N):\ 1 = f(0),\ N \in \rho^{-1}(\overline{N})\right\} = \wedge\left\{\nu_{A_N}(N):\ \rho(N) = \overline{N}\right\} = 0$

Similarly, we can show that $\mu_{\rho(A_N)}(-\overline{N}) = 0.5$ and $\nu_{\rho(A_N)}(-\overline{N}) = 0.2$.

$\therefore$ $\rho(A_N) = B_{\overline{N}}$. Hence ρ is an intuitionistic fuzzy homomorphism of A_N onto $B_{\overline{N}}$.

Proposition (3.16) Let ϕ be an intuitionistic fuzzy homomorphism of A onto B where A and B are IFSGs of group G and $\overline{G}$ respectively. Let N be a normal subgroup of G and $\overline{N} = \phi(N)$. Then the canonical homomorphism $\pi : \overline{G} \to \overline{G}/\overline{N}$ is an intuitionistic fuzzy homomorphism of B onto $B_{\overline{N}}$ where $B_{\overline{N}}$ is an IFSG of $\overline{G}/\overline{N}$ and $\psi = \pi \circ \phi$ is an intuitionistic fuzzy homomorphism of A onto $B_{\overline{N}}$.

Proof.

$$\begin{array}{ccc} G & \xrightarrow{\phi} & \overline{G} \\ (\mu_A, \nu_A) & (\mu_B, \nu_B) & \\ & I\times I & \downarrow \pi \\ & & (\mu_{B_{\overline{N}}}, \nu_{B_{\overline{N}}}) \\ & & \overline{G}/\overline{N} \end{array}$$

We know that $\psi = \pi \circ \phi$ is a homomorphism of G onto $\overline{G}/\overline{N}$ where π is the canonical homomorphism from $\overline{G}$ onto $\overline{G}/\overline{N}$. We have to show that $\pi \circ \phi$ is an intuitionistic fuzzy homomorphism of A onto $B_{\overline{N}}$. i.e., to show that $(\pi \circ \phi)(A) = B_{\overline{N}}$.

Consider $\overline{g}\overline{N} \in \overline{G}/\overline{N}$, $\quad (\pi\circ\phi)(A)(\overline{g}\overline{N}) = \left(\mu_{(\pi\circ\phi)(A)}(\overline{g}\overline{N}),\ \nu_{(\pi\circ\phi)(A)}(\overline{g}\overline{N})\right)$, where

$\mu_{(\pi\circ\phi)(A)}(\overline{g}\overline{N}) = \vee\left\{\mu_A(z):\ z \in (\pi\circ\phi)^{-1}(\overline{g}\overline{N})\right\}$ and $\nu_{(\pi\circ\phi)(A)}(\overline{g}\overline{N}) = \wedge\left\{\nu_A(z):\ z \in (\pi\circ\phi)^{-1}(\overline{g}\overline{N})\right\}$

Now, $\mu_{(\pi\circ\phi)(A)}(\overline{g}\overline{N}) = \vee\left\{\mu_A(z):\ z \in (\pi\circ\phi)^{-1}(\overline{g}\overline{N})\right\}$

$$= \vee\left\{\mu_A(z):\ z \in \phi^{-1}\left(\pi^{-1}(\overline{g}\overline{N})\right)\right\}$$

$$= \mu_{\phi(A)}\left(\pi^{-1}(\overline{g}\overline{N})\right)$$

$$= \mu_B\left(\pi^{-1}(\overline{g}\overline{N})\right)\ \left[\because\ \phi \text{ is an intuitionistic fuzzy homomorphism, } \phi(A) = B\right]$$

$$= \mu_{\left(\pi^{-1}\right)^{-1}(B)}(\overline{g}\overline{N})$$

$$= \mu_{\pi(B)}(\overline{g}\overline{N})$$

$$= \mu_{B_{\overline{N}}}(\overline{g}\overline{N})\ \left[\because\ \pi \text{ is an intuitionistic fuzzy homomorphism, } \pi(B) = B_{\overline{N}}\right].$$

Similarly, we can show that $\nu_{(\pi\circ\phi)(A)}(\overline{g\overline{N}}) = \nu_{B_{\overline{N}}}(\overline{g\overline{N}})$

$\therefore$ $\pi\circ\phi$ is an intuitionistic fuzzy homomorphism of A onto $B_{\overline{N}}$.

Example (3.17) Let G = R – {0}, $\overline{G}$ = {1, -1}, groups under multiplication.

Define f : G → $\overline{G}$ by $f(x) = \begin{cases} 1 & ;\text{ when } x \text{ is positive real number} \\ -1 & ;\text{ when } x \text{ is negative real number} \end{cases}$, $\forall\, x \in G$.

Then f is a homomorphism of G onto $\overline{G}$. Define the IFS A on G by

$$\mu_A(x) = \begin{cases} 1 & ;\text{ when } x \text{ is rational} \\ 0.5 & ;\text{ when } x \text{ is irrational} \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & ;\text{ when } x \text{ is s rational} \\ 0.2 & ;\text{ when } x \text{ is irrational} \end{cases}, \forall\, x \in \text{G}.$$

Then A is IFSG of G. Define the IFS B on $\overline{G}$ by

$$\mu_B(y) = \begin{cases} 1 & ;\ if\ \ y = 1 \\ 0.5 & ;\ if\ \ y = -1 \end{cases} \quad \text{and} \quad \nu_B(y) = \begin{cases} 0 & ;\ if\ \ y = 1 \\ 0.2 & ;\ if\ \ y = -1 \end{cases}, \text{ we have}$$

B is IFSG of $\overline{G}$.

Let $\overline{N}$ = {1}. Then N ={ x∈G : $f(x)$ = 1}= (0, ∞). N and $\overline{N}$ are normal subgroup of G and $\overline{G}$ respectively.

Now, G/N = { xN , yN : x∈Q^+ , $y \in R^+ - Q^+$ }.

Define $A_N = (\mu_{A_N}, \nu_{A_N})$, where μ_{A_N}, ν_{A_N} : G/N →[0,1] be the IFS on G/N by

$$\mu_{A_N}(zN) = \begin{cases} 1 & ;\text{ when } z = x \\ 0.5 & ;\text{ when } z = y \end{cases} \quad \text{and} \quad \nu_{A_N}(zN) = \begin{cases} 0 & ;\text{ when } z = x \\ 0.2 & ;\text{ when } z = y \end{cases}$$

Then A_N is an IFSG of G / N. Now, $\overline{G}/\overline{N}$ = { 1. $\overline{N}$, (-1) $\overline{N}$ } = { $\overline{N}$, - $\overline{N}$ }.

Define IFS $B_{\overline{N}}$ on $\overline{G}/\overline{N}$ by

$$\mu_{B_{\overline{N}}}(t\overline{N}) = \begin{cases} 1 & ;\text{ when } t = 1 \\ 0.5 & ;\text{ when } t = -1 \end{cases} \quad \text{and} \quad \nu_{B_{\overline{N}}}(t\overline{N}) = \begin{cases} 0 & ;\text{ when } t = 1 \\ 0.2 & ;\text{ when } t = -1 \end{cases}$$

Clearly, $B_{\overline{N}}$ is IFSG on $\overline{G}/\overline{N}$.

Define ρ : G / N → $\overline{G}/\overline{N}$ by ρ (zN) = f(z) $\overline{N}$ = t $\overline{N}$, where t = f(z).

Then, ρ(xN) = $f(x)$ $\overline{N}$ = 1. $\overline{N}$ = $\overline{N}$, ρ(yN) = f(y) $\overline{N}$ = (-1) $\overline{N}$.

Thus, we have $\rho(A_N)(t\overline{N}) = \left(\mu_{\rho(A_N)}(t\overline{N}),\ \nu_{\rho(A_N)}(t\overline{N})\right)$, where

$$\mu_{\rho(A_N)}(t\overline{N}) = \vee\left\{\mu_{A_N}(zN) :\ t = f(z),\ zN \in \rho^{-1}(t\overline{N})\right\} \quad \text{and}$$

$$\nu_{\rho(A_N)}(t\overline{N}) = \wedge\left\{\nu_{A_N}(zN) :\ t = f(z),\ zN \in \rho^{-1}(t\overline{N})\right\}$$

$$\therefore \quad \mu_{\rho(A_N)}(1.\overline{N}) = \vee\left\{\mu_{A_N}(1.N) :\ 1 = f(x),\ xN \in \rho^{-1}(\overline{N})\right\} = \vee\left\{\mu_{A_N}(N) :\ \rho(xN) = \overline{N}\right\} = 1$$

$$\text{and} \quad \nu_{\rho(A_N)}(1.\overline{N}) = \wedge\left\{\nu_{A_N}(1.N) :\ 1 = f(x),\ xN \in \rho^{-1}(\overline{N})\right\} = \wedge\left\{\nu_{A_N}(N) :\ \rho(xN) = \overline{N}\right\} = 0.$$

Similarly, we can show that $\mu_{\rho(A_N)}(-\overline{N}) = 0.5$ and $\nu_{\rho(A_N)}(-\overline{N}) = 0.2$.

$\therefore$ ρ (A_N) = $B_{\overline{N}}$. Hence $A_N \approx B_{\overline{N}}$.

Proposition (3.18)[5] **(Cayley's Theorem)**

Every group is isomorphic to a subgroup of A(S) for some appropriate set S.

It may be recall that the A(S) is the set of all one to one mapping of a set S onto itself.

Theorem (3.19) Any isomorphism ψ : G ≈ A(G) gives rise to a intuitionistic fuzzy isomorphism between a pair of intuitionistic fuzzy subgroups of G and ψ(G).

$$G \xrightarrow{\psi} A(G)$$

Proof. $(\mu_A,\ \nu_A)$ ↘ ↙ $(\mu_B,\ \nu_B)$

$$I \times I$$

Let A be IFSG of the group G. By Cayley's theorem, ∃ an isomorphism ψ : G → A(G).

Let $\psi(g) = t_g$, $\forall g \in G$, where $t_g \in A(G)$ defined as $t_g(h) = gh$, $\forall\ h \in G$.

Define IFS $B = (\mu_B, \nu_B) : A(G) \to I \times I$ defined by

$$\mu_B(t_g) = \mu_A(g) \text{ and } \nu_B(t_g) = \nu_A(g),\ \forall\ t_g \in A(G)$$

Then it is easy to verify that B is IFSG on A(G) and $B(t_g) = A(g)$, $\forall\ t_g \in A(G)$, $g \in G$.

$$Now,\quad \psi(A)(t_g) = \left(\mu_{\psi(A)}(t_g),\ \nu_{\psi(A)}(t_g)\right),\ \forall\ t_g \in \psi(G), \text{ where}$$

$$\mu_{\psi(A)}(t_g) = \vee\left\{\mu_A(x):\ x \in \psi^{-1}(t_g)\right\} = \mu_A(g) \text{ and } \nu_{\psi(A)}(t_g) = \wedge\left\{\nu_A(x):\ x \in \psi^{-1}(t_g)\right\} = \nu_A(g)$$

$$For\ t_g, t_h \in \psi(G), \text{ we have}$$

$$\begin{aligned}\mu_{\psi(A)}(t_g t_h) &= \mu_{\psi(A)}(t_{gh}) \qquad [\because\ t_g t_h = t_{gh}\ ]\\ &= \mu_A(gh)\\ &\geq \mu_A(g) \wedge \mu_A(h)\\ &= \mu_{\psi(A)}(t_g) \wedge \mu_{\psi(A)}(t_h)\end{aligned}$$

$$Similarly, \text{ we can show that } \nu_{\psi(A)}(t_g t_h) \leq \nu_{\psi(A)}(t_g) \vee \nu_{\psi(A)}(t_h)$$

$$\mu_{\psi(A)}\{(t_g)^{-1}\} = \mu_{\psi(A)}(t_{g^{-1}}) = \mu_A(g^{-1}) = \mu_A(g) = \mu_{\psi(A)}(t_g).$$

$$Similarly, \text{ we can show that } \nu_{\psi(A)}\{(t_g)^{-1}\} = \nu_{\psi(A)}(t_g).$$

$$\therefore\quad \psi(A) \text{ is an IFSG of } \psi(G).$$

$$Also,\ \psi(A)(t_g) = A(g) = B(t_g),\ \forall\ t_g \in \psi(G), \text{ and hence } \psi(A) = B.$$

$$Hence\ \psi \text{ is an intuitionistic fuzzy isomorphism of A onto B.}$$

Example(3.20) Let G = {1, -1}. Define isomorphism ψ : G→ A(G) by $\psi(g) = t_g$, $\forall g \in G$, where t_g is an automorphism on G defined by $t_g(h) = gh$, $\forall\ h \in G$. Define the IFS A on G by

$$\mu_A(x) = \begin{cases} 1 & if\ x = 1 \\ 0.5 & if\ x = -1 \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & if\ x = 1 \\ 0.25 & if\ x = -1 \end{cases}$$. Then A is IFSG of G.

$$Let \text{ B be IFS on } \psi(G) \text{ defined by}$$

$$\mu_B(t_g) = \begin{cases} 1 & if\ g = 1 \\ 0.5 & if\ g = -1 \end{cases} \quad \text{and} \quad \nu_B(t_g) = \begin{cases} 0 & if\ g = 1 \\ 0.25 & if\ g = -1 \end{cases}$$

Then, $\psi(\mathrm{A})(\mathrm{t}_g) = \left(\mu_{\psi(\mathrm{A})}(\mathrm{t}_g),\ \nu_{\psi(\mathrm{A})}(\mathrm{t}_g)\right),\ \forall\ \mathrm{t}_g \in \psi(G),$ where

$\mu_{\psi(\mathrm{A})}(\mathrm{t}_g) = \vee\left\{\mu_{\mathrm{A}}(x):\ x \in \psi^{-1}(\mathrm{t}_g)\right\}$ and $\nu_{\psi(\mathrm{A})}(\mathrm{t}_g) = \wedge\left\{\nu_{\mathrm{A}}(x):\ x \in \psi^{-1}(\mathrm{t}_g)\right\}$

Now, $\mu_{\psi(\mathrm{A})}(\mathrm{t}_1) = \vee\left\{\mu_{\mathrm{A}}(x):\ x \in \psi^{-1}(\mathrm{t}_1)\right\} = 1$ and $\nu_{\psi(\mathrm{A})}(\mathrm{t}_1) = \wedge\left\{\nu_{\mathrm{A}}(x):\ x \in \psi^{-1}(\mathrm{t}_1)\right\} = 0$

Similarly, we can show that $\mu_{\psi(\mathrm{A})}(\mathrm{t}_{-1}) = 0.5$ and $\nu_{\psi(\mathrm{A})}(\mathrm{t}_1) = 0.25.$

$\therefore$ ψ(A) = B. Hence ψ is an intuitionistic fuzzy isomorphism of A onto B.

Definition (3.21) Let T be an intuitionistic fuzzy representation of a group G with representation space M and H be a subgroup of G. Then the restriction of T on *H* is defined as $(T|_H)(x) = T(x)$, $\forall\ x \in H$

Proposition (3.22) If T is an intuitionistic fuzzy representation of G with representation space M and H is a subgroup of G, $T|_H$ is intuitionistic fuzzy representation on H.

Proof: Since T is an intuitionistic fuzzy representation, there exists IFSGs A and B of G and T(G) such that T(A) = B. We have to show that $T|_H$ is a intuitionistic fuzzy representation.

For y $\in$ T (G), we have $(T|_H)(A)(y) = \left(\mu_{(T|_H)(A)}(y),\ \mu_{(T|_H)(A)}(y)\right),$ where

$\mu_{(T|_H)(A)}(y) = \vee\left\{\mu_A(x):\ x \in (T|_H)^{-1}(\mathrm{y})\right\}$ and $\nu_{(T|_H)(A)}(y) = \wedge\left\{\nu_A(x):\ x \in (T|_H)^{-1}(\mathrm{y})\right\}$

$$
\begin{aligned}
\textit{Now, }\ \mu_{(T|_H)(A)}(y) &= \vee\left\{\mu_A(x):\ x \in (T|_H)^{-1}(\mathrm{y})\right\} \\
&= \vee\left\{\mu_A(x):\ (T|_H)(x) = y\right\} \\
&= \vee\left\{\mu_A(x):\ T(x) = y,\ x \in H\right\} \\
&= \vee\left\{\mu_A(x):\ T(x) = y,\ \mathrm{y} \in T(H)\right\} \\
&= \vee\left\{\mu_A(x):\ x \in T^{-1}(T(H))\right\} \\
&= \mu_{T(A)}(y),\ \text{where } \mathrm{y} \in \mathrm{T(H)} \\
&= \mu_B(y),\ \text{where } \mathrm{y} \in \mathrm{T(H)} \\
&= \mu_{B|_{T(H)}}(y)
\end{aligned}
$$

Similarly, we can show that $\nu_{(T|_H)(A)}(y) = \nu_{B|_{T(H)}}(y)$

$\therefore$ $(T|_H)(A) = B|_{T(H)}$. *So,* $T|_H$ *is an intuitionistic fuzzy homomorphism of* $A|_H$ *onto* $B|_{T(H)}$.

Hence $T|_H$ is an intuitionistic fuzzy representation of $A|_H$ onto $B|_{T(H)}$.

Definition (3.23)[4] Two representations T and T′ with spaces M and M′ are said to be equivalent if $\exists$ a K-isomorphism S of M onto M′such that T′(*g*)(S) = ST(*g*), $\forall$ *g*$\in$G. i.e., T′(*g*)S(*m*) = ST(*g*)(*m*), $\forall$ *g*$\in$G and *m*$\in$M.

Definition (3.24) Let A be IFSG on G. Two intuitionistic fuzzy representations T and T′ of G with spaces M and M′ are said to be equivalent of $T^{-1}(B)(x) = T^{-1}(C)(x)$, $\forall$ *x*$\in$G,

i.e., B(T(*x*)) = C(T′(*x*)), $\forall$ *x*$\in$G, where B and C are IFSG on T(G) and T′(G) respectively.

G $\xrightarrow{T}$ GL(M)

T $\downarrow$ □ (μ_B, ν_A) $\downarrow (\mu_B, \nu_B)$

GL(M′) $\xrightarrow{(\mu_C, \nu_C)}$ $\mathrm{I} \times \mathrm{I}$

Remark (3.25) The equivalence of intuitionistic fuzzy representation is an equivalence relation

4. CONCLUSION

The theory of group representations has an applications in several branches of Mathematics and practical Physics. So the study of its intuitionistic fuzzy version is expected to have many practical applications.

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