Asian Journal of Fuzzy and Applied Mathematics (ISSN: 2321 – 564X)
Volume 03 – Issue 02, April 2015

The Extended Riesz Theorem and its Results

Atefeh Jaberi, Hamidreza Goudarzi[2*]

[1] Department of Mathematics, Faculty of Science, Yasouj University, Yasouj, Iran.

[2] Department of Mathematics, Faculty of Science, Yasouj University, Yasouj, Iran.

[*] *Corresponding author's email: goudarzi [AT] mail.yu.ac.ir*

ABSTRACT— ***The main purpose of this paper is to extended the Riesz theorem in fuzzy anti n-normed linear spaces as a generalization of linear n-normed space. Also we study some properties of fuzzy anti n-normed linear spaces.***

Keywords— Riesz theorem, Fuzzy n-compact sets, Fuzzy anti n-norms, α n-norms.

---

1. INTRODUCTION

A satisfactory theory of 2 norms of a linear space has been introduced and developed by Gahler to n-norm on a linear space [6]. In following H. Gunawan and M. Mashadi [7], S. S. Kim and Y. J. Cho [11], R. Malceski [17] and A. Misiak [18] developed the theory of n-normed space [18]. The more detailes about the theory of fuzzy normed linear space can be found in [1, 2, 5, 21]. The concept of fuzzy sets was introduced by L. A. Zadeh in 1965 [26] and thereafter several authors applied it in different branches of pure and applied Mathematics. The concept of fuzzy norms was introduced by A. K. Katsaras in 1984 [9]. In 1992, C. Felbin introduced the concept of Fuzzy normed linear space[5]. The notion of Fuzzy 2 normed linear spaces introduced by A.R. Meenakshi and R. Gokilavani in 2001. B. Sundander Reddy introduced the idea of Fuzzy anti 2-normed linear spaces [25]. AL. Narayanan and S. Vijayabalaji introduced the definition of fuzzy n-norm on a linear space and Also, Vijayabalaji [19] and Thillaigovindan introduced study of the complete fuzzy n-normed linear spaces [27]. I. H. Jebril and S. K. Samanta gave the definition of a Fuzzy anti normed linear space in 2011 [16]. F. Riesz obtained the Riesz theorem in a normed space[22]. Park and Chu have extended the Riesz theorem in a normed space to n-normed linear space [20].

Following Kavikumar, Yang Bae Jun and Azme Khamis [10], in this paper extend the Riesz theorem in n-normed linear spaces to fuzzy Anti n-normed linear spaces. Also, we establish some basic results.

2. PRIMILINARIES

The main purpose of this article is the extension of Riesz theorem to fuzzy anti n-normed linear spaces. In the first part, we try to establish some basic theorems and by aimes of this result, we do our main goal.

Definition 2.1 [8] If *W* is a linear subspace of a finite-dimentional vector space *V*, then the codimension of *W* in *V* is the difference between the dimensions,

$$co\dim(W) = \dim(V) - \dim(W)$$

Definition 2.2 [10] Let $n \in \mathbb{N}$ and *X* be a real linear space of dimension $d \geq n$. (Here we allow *d* to be infinite). A real valued function $\|\bullet,\ldots,\bullet\|$ on $X \times \ldots \times X$ (*n times* = X^n) satisfying four properties:

(*N1*) $\|x_1,\ldots,x_n\| = 0$ iff $x_1,\ldots,x_n$ are linearly dependent,

(*N2*) $\|x_1,\ldots,x_n\|$ is invariant under any permutation of $x_1,\ldots,x_n$,

(*N3*) $\|x_1,\ldots,cx_n\| = |c|\ \|x_1,\ldots,x_n\|$, for any real *c*,

(*N4*) $\|x_1,\ldots,x_{n-1},y+z\| \leq \|x_1,\ldots,x_{n-1},y\| + \|x_1,\ldots,x_{n-1},z\|$,

is called a n-normed on *X* and the pair $(X,\|\bullet,\ldots,\bullet\|)$ is called a n-normed linear space.

Definition 2.3 [10] A sequence $\{x_n\}$ in a linear n-normed space $(X, \|\bullet,...,\bullet\|)$ is said to be n- convergent to $x \in X$ and denote by $x_k \to x$ as $k \to \infty$ if

$$\lim_{k\to\infty} \| x_1,..., x_{n-1}, x_n - x \| = 0.$$

Definition 2.4 [15] A subset of a linear n-normed space $(X, \|\bullet,...,\bullet\|)$ is called a n-compact subset if for every sequence $\{x_n\}$ in Y, there exists a subsequence of $\{x_{n_k}\}$ which converges to an element $x \in X$.

From this view point, Park and Chu [20] obtained the following theorem in n-normed spaces:

Theorem 2.1 [10] *Let Y and Z be two subspaces of a linear n- normed space X, and Y be a n-compact proper subset of Z with codimension greater than* $n-1$. *For each* $\theta \in (0,1)$, *there exists an element* $(z_1,..., z_n) \in Z_n$ *such that*

$$\| z_1,..., z_n \| = 1, \qquad \| z_1 - y,..., z_n - y \| \geq \theta,$$

for all $y \in Y$.

Definition 2.5 [3] A binary operation $\lozenge : [0,1]\times[0,1] \to [0,1]$ is a continuous t - conorm if $\lozenge$ satisfies the following conditions:

(i) $\lozenge$ is commutative and associative,
(ii) $\lozenge$ is continuous,
(iii) $a \lozenge 0 = a, \quad \forall a \in [0,1],$
(iv) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c, b \leq d$ and $a,b,c,d \in [0,1]$

A few examples of continuous t - conorm are $a \lozenge b = a + b - ab, a \lozenge b = \max\{a,b\}$ and $a \lozenge b = \min\{a+b,1\}$.

Remark 2.1 *[1] For any* $a,b \in (0,1)$ *with a > b there exists* $c \in (0,1)$ *such that a > c◊b.*

Definition 2.6 [27] Let *X* be a linear space over a real field F. A fuzzy subset *N* of $X^n \times [0,\infty)$ is called a fuzzy anti n-norm on *X* if and only if:

(FAN1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1,..., x_n, t) = 1$,
(FAN2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1,..., x_n, t) = 1$, $x_1,..., x_n$ are linearly dependent,
(FAN3) $N(x_1,..., x_n, t)$ is invariant under any permutation of $x_1,..., x_n$,
(FAN4) $N(x_1,..., cx_n, t) = N(x_1,..., x_n, t/|c|)$ if $c \neq 0, c \in F$,
(FAN5) $N(x_1,..., x_n + x_n', s+t) \leq N(x_1,..., x_n, s) \lozenge N(x_1,..., x_n', t)$ for all $s,t \in \mathbb{R}$,
(FAN6) $N(x_1,..., x_n, .)$ is a continuous and non-increasing function of $\mathbb{R}$ such that

$$\lim_{t\to\infty} N(x_1,..., x_n, t) = 0.$$

Then (X, N) is called a fuzzy anti n-normed linear space.

Definition 2.7 [27] A sequence $\{x_n\}$ in a fuzzy anti n-normed space (X, N) is said to converge to *x* if for given $r > 0, t > 0$ and $0 < r < 1$, there exists an integer $n_0 \in \mathbb{N}$ such that $N(x_1,..., x_{n-1}, x_n - x, t) < r$, for all $n \geq n_0$.

Example 2.1 [27] Let $(X, \|\bullet,...,\bullet\|)$ be a n-normed linear space. Define,

$$N(x_1,..., x_n, t) = \begin{cases} 1 - \dfrac{t}{t + \| x_1,..., x_n \|} & t > 0, \forall x \in X, \\ 1 & t \leq 0, \forall x \in X. \end{cases}$$

Then (X, N) is a fuzzy anti n-normed linear space.

Theorem 2.2 *[27] Let* (X, N) *be a fuzzy anti n- normed space. Assume that condition that*

$$(FAN7) \qquad N(x_1,..., x_n, t) > 0, \forall t > 0,$$

implies $x_1,..., x_n$ *are linearly dependent. Define* $\| x_1,..., x_n \|_\alpha = \sup\{t : N(x_1,..., x_n, t) \leq 1 - \alpha\}, \alpha \in (0,1).$ *Then* $\{\|\bullet,...,\bullet\|_\alpha : \alpha \in (0,1)\}$ *is a descending family of n-normes on X. These n-norms are called α-n- norms on X corre-spending to the fuzzy anti n-norm on X.*

Definition 2.8 [2] The fuzzy normed space (X,N) is said to be a fuzzy anti n-normed Banach space whenever X is complete with respect to the fuzzy metric induced by fuzzy anti n-norm.

3. FUZZY RIESZ THEOREM

Riesz [22] obtained the following theorem in a normed space.

Theorem 3.1 *[22] Let Y and Z be subspaces of a normed space X, and Y a closed proper subset of Z. For each* $\theta \in (0,1)$*, there exists an element* $z \in Z$ *such that*

$$\| z \| = 1, \qquad \| z - y \| \geq \theta,$$

for all $y \in Y$.

Now we try to extend Riesz theorem to fuzzy anti n-normed linear spaces. Also, we prove some corollaries of this theorem at the end of this section.

Definition 3.1 A subset Y of a fuzzy anti n-normed linear space (X,N) is called a fuzzy n-compact subset if for every sequence $\{y_n\}$ in Y, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges to an element $y \in Y$. In other words, given $t > 0$ and $0 < r < 1$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$N(y_1,\ldots,y_{n-1},y_{n_k}\text{-}y,\ t/k) < r,$$

for all $n,k \geq n_0$ and $n_k \geq n_0$.

Lemma 3.1 *Let* (X,N) *be a fuzzy anti n-normed linear space. Assume that* $x_i \in X$ *for each* $i \in \{1,\ldots,n\}$ *and* $c \in F$*. Then*

$$N(x_1,\ldots,x_j + cx_i,\ldots,x_n,\ t) = N(x_1,\ldots x_i,\ldots,x_j,\ldots,x_n,\ t).$$

Proof.

$$\begin{aligned} N(x_1,\ldots,x_j + cx_i,\ldots,x_n,\ t) &= N(x_1,\ldots,x_j + cx_i,\ldots,x_n,\ t/2 + t/2) \\ &\leq \max\{N(x_1,\ldots x_i,\ldots,x_j,\ldots,x_n,\ t/2), N(x_1,\ldots x_i,\ldots,x_j,\ldots,x_n,\ t/2)\} \\ &= \max\{N(x_1,\ldots x_i,\ldots,x_j,\ldots,x_n,\ t/2), N(x_1,\ldots x_i,\ldots,x_j,\ldots,x_n,\ t/|c|2)\}, (|c|=1) \\ &= \max\{N(x_1,\ldots x_i,\ldots,x_j,\ldots,x_n,\ t/2), N(x_1,\ldots x_i,\ldots,x_j,\ldots,x_n,\ t/2)\} \\ &\geq N(x_1,\ldots x_i,\ldots,x_j,\ldots,x_n,\ t). \end{aligned}$$

□

Theorem 3.2 *Let* (X,N) *be a fuzzy anti n- normed linear space. If the*

$$\sup_{y\in Y}\{t > 0 : N(x_1 - y,\ldots,x_n - y,t)\} = 0,$$

for $(x_1,\ldots,x_n) \in X_n$ *and Y is a fuzzy n-compact subset of X, then there exists an element* $y_0 \in Y$ *such that*

$$\{t > 0 : N(x_1 - y_0,\ldots,x_n - y_0,t)\} = 0,$$

Proof. *Let t >* 0 *and* $\varepsilon \in (0,1)$. *Choose* $r \in (0,1)$ *such that* $r \Diamond r < \varepsilon$ *(remark 2.1). Since Y is a fuzzy n-compact subset of X, there exists an integer* $n_0 \in \mathbb{N}$ *such that*

$$N(x_1 - y_k,\ldots,x_n - y_k,ct) < r,$$

for all $n,k \geq n_0$ *and a constant c. Since* $\{y_k\}$ *is a sequence in a fuzzy n- compact subset Y of X. Without loss of generality assume that* $\{y_k\}$ *converges to* $y_0 \in Y$*, as* $k \to \infty$*. Then for given,* $0 < \lambda < 1$*, there exists an integer* $n_1 \in \mathbb{N}$ *such that*

$$N(y_k - y_0,\omega_2\ldots,\omega_n,t) < \lambda,$$

for all $\omega_i \in X\,(i = 1,\ldots,n)$ *and* $n_0 > n_1$*. For every* $r \in (0,1)$*, there exists* $\lambda \in (0,1)$ *such that (remark 2.1)*

$$\overbrace{\lambda\Diamond\lambda\Diamond\ldots\lambda}^{n} < r,$$

by lemma 3.1, if $n_0 > n_1$, *then we have*

$$\begin{aligned}
N(x_1-y_0,x_2-y_0,\ldots,x_n-y_0,t) &\le N(y_k-y_0,x_2-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_0,\ldots,x_n-y_0,(k-1)t/k)\\
&\le N(y_k-y_0,x_2-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,y_k-y_0,x_3-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_k,x_3-y_0,\ldots,x_n-y_0,(k-2)t/k)\\
&\le N(y_k-y_0,x_2-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,y_k-y_0,x_3-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_k,y_k-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_k,x_3-y_k,\ldots,x_n-y_0,(k-3)t/k)\\
&\le N(y_k-y_0,x_2-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,y_k-y_0,x_3-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_k,y_k-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond \cdots\\
&\Diamond N(x_1-y_k,x_2-y_k,x_3-y_k,\ldots,y_k-y_0,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_k,x_3-y_k,\ldots,x_{n-1}-y_k,x_n-y_0,(k-(n-1))t/k)
\end{aligned}$$

Therefore

$$\begin{aligned}
N(x_1-y_0,x_2-y_0,\ldots,x_n-y_0,t) &\le N(y_k-y_0,x_2-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,y_k-y_0,x_3-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_k,y_k-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond \cdots\\
&\Diamond N(x_1-y_k,x_2-y_k,x_3-y_k,\ldots,y_k-y_0,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_k,x_3-y_k,\ldots,x_{n-1}-y_k,y_k-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_k,x_3-y_k,\ldots,x_{n-1}-y_k,x_n-y_k,(k-n)t/k)\\
&= N(y_k-y_0,x_2-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_0,y_k-y_0,x_3-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_0,x_2-y_0,y_k-y_0,\ldots,x_n-y_0,t/k)\\
&\Diamond \cdots\\
&\Diamond N(x_1-y_0,x_2-y_0,x_3-y_0,\ldots,y_k-y_0,x_n-y_0,t/k)\\
&\Diamond N(x_1-y_0,y_k-y_0,x_3-y_0,\ldots,x_{n-1}-y_0,y_k-y_0,t/k)\\
&\Diamond N(x_1-y_k,x_2-y_k,x_3-y_k,\ldots,x_{n-1}-y_k,x_n-y_k,ct)\\
&< \overbrace{\lambda\Diamond\lambda\Diamond\ldots\lambda\Diamond r}^{n}
\end{aligned}$$

$$< r\Diamond r < \varepsilon.$$

Since ε is arbitrary,

$$\sup\{t > 0 : N(x_1 - y_0, x_2 - y_0, ..., x_n - y_0, t)\} = 0$$

□

Now, we reperesent Riesz Theorem for fuzzy anti n-normed linear spaces.

Theorem 3.3 ***Riesz Theorem***. *Let (X, N) be a fuzzy anti n-normed linear space satisfying condition (FAN7) and $\{\| \bullet,...,\bullet \|_\alpha : \alpha \in (0,1)\}$ be a descending family of α- n- norms corresponding to (X, N). Let Y and Z be subspaces of X and Y be a fuzzy n-compact proper subset of Z with $\dim Z \geq n$. For each $k \in (0,1)$, there exists an element $(z_1,...,z_n) \in Z_n$ such that*

$$\| z_1,...,z_n \|_\alpha = 0, \qquad N(z_1 - y,...,z_n - y,t) \geq \alpha,$$

for all $y \in Y$.

Proof. *Let $\alpha \in (0,1)$, $(v_1,...,v_n) \in Z - Y$ with $v_1,...,v_n$ are linearly independent. Let*

$$\sup_{y \in Y} \| v_1 - y,...,v_n - y \|_\alpha = k.$$

We follow the proof in two cases:

Case (i): Assume that k = 0. By theorem 3.2, there is an element $y_0 \in Y$ such that $N(v_1 - y_0,...,v_n - y_0) = 0$.

If $y_0 = 0$, then $v_1,...,v_n$ are linearly independent, which is a contradiction.

If $y_0 \neq 0$, then $v_1,...,v_n$ are linearly independent.

Case (ii): Let k > 0, where

$$k = \| v_1 - y,...,v_n - y \|_\alpha = \sup\{s : N(v_1 - y,...,v_n - y, s) \leq \alpha\},$$

Since $N(v_1 - y,...,v_n - y,s)$ is continuous (definition 2.6), now we have (by theorem 4.4, in [19]),

$$N(v_1 - y,...,v_n - y,s) \leq 1 - \alpha,$$

So for each $k_1 \in (0,1)$, there exists an element $y_0 \in Y$ such that

$$k \geq \| v_1 - y_0,...,v_n - y_0 \|_\alpha \geq \frac{k}{k_1}.$$

For each j=1,…,n, let

$$z_j = \frac{v_j - y_0}{\| v_1 - y_0,...,v_n - y_0 \|_\alpha^{\frac{1}{n}}}.$$

Then it is obvious that $\| z_1,...,z_n \|_\alpha = 0.$

Now,

$$\| z_1 - y_0,...,z_n - y_0 \|_\alpha = \left\| \frac{v_1 - y_0}{\| v_1 - y_0,...,v_n - y_0 \|_\alpha^{\frac{1}{n}}} - y,...,\frac{v_n - y_0}{\| v_1 - y_0,...,v_n - y_0 \|_\alpha^{\frac{1}{n}}} - y \right\|_\alpha$$

$$= \left\| \frac{1}{\| v_1 - y_0,...,v_n - y_0 \|_\alpha} \right\| v_1 - (y_0 + y \| v_1 - y_0,...,v_n - y_0 \|_\alpha^{\frac{1}{n}},...,$$

$$v_n - (y_0 + y \| v_1 - y_0,...,v_n - y_0 \|_\alpha^{\frac{1}{n}} \|$$

$$\leq \frac{1}{\| v_1 - y_0, ..., v_n - y_0 \|_\alpha}$$

$$\leq \frac{k}{k / k_1}$$

$$= k_1,$$

By (FAN7), there exists $\alpha \in (0,1)$ *such that*

$$\sup\{k > 0 : N(z_1 - y, ..., z_n - y, k) \leq 1 - \alpha\} \leq k_1.$$

Then there exists $\alpha_0 \in (0,1)$ *such that*

$$N(z_1 - y, ..., z_n - y, k_1) > \alpha_0 \geq 1 - \alpha,$$

for all $y \in Y$.

□

Corollary 3.1 *Given a strictly nested sequence of closed subspaces*

$$\{0\} \text{Ø } N_1 \text{ Ø } N_2 \text{ Ø } N_3 \text{ Ø } N_4 \text{ Ø} ...$$

of a fuzzy Banach space X, one can find a sequence of vectors $x_1, ..., x_n \in N_n$ *with* $\| x_1, ..., x_n \|_\alpha = 0$ *and* $N(x_1 - N_{n-1}, ..., x_n - N_{n-1}) \geq \frac{1}{2}.$ *Similarly, for a sequence of closed subspaces nested in the opposite direction* $\{0\} \text{ Ù } R_1 \text{ Ù } R_2 \text{ Ù } R_3 \text{ Ù } R_4 \text{ Ù} ...,$ *there are unit vectors* $x_n \in R_n$ *with* $N(x_1 - R_{n+1}, ..., x_n - R_{n+1}) > \frac{1}{2}.$

Proof. *Pick any* x_1 *of norm* N_1. *Let* F_1 *be the linear span of* x_1. *Then* F_1 *is finite dimensional and, hence, closed. By Riesz's Lema, there is an* x_2 *of norm* N_1 *such that* $N(x_1 - N_{n-1}, x_2 - N_{n-1}) \geq \frac{1}{2}.$ *Let* F_2 *be the linear span of* x_1 *and* x_2. *Then* F_2 *is finite dimensional and, hence, closed. By Riesz's Lemma, there is an* x_3 *of norm* N_1 *such that* $N(x_1 - N_{n-1}, x_2 - N_{n-1}) \geq \frac{1}{2}.$ *Continue ...* □

The same corollary has been achieved in linear normed space as follows:

Corollary 3.2 *Given a strictly nested sequence of closed subspaces*

$$\{0\} \text{Ø } N_1 \text{ Ø } N_2 \text{ Ø } N_3 \text{ Ø } N_4 \text{ Ø} ...$$

of a Banach space X, one can find a sequence of vectors $x_n \in N_n$ *with* $\| x_n \| = 1$ *and* $dist(x_n, N_{n-1}) \geq \frac{1}{2}$. *Similarly, for a sequence of closed subspaces nested in the opposite direction,* $\{0\} \text{ Ù } R_1 \text{ Ù } R_2 \text{ Ù} ...,$ *there are unit vectors* $x_n \in R_n$ *with* $dist(x_n, R_{n-1}) \geq \frac{1}{2}.$

4. REFERENCES

[1] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces , The Journal of Fuzzy Mathematics, 11 (2003), no. 3, 687 - 705.

[2] S. C. Cheng, J. N. Mordeson, Fuzzy linear operators and fuzzy normed spaces, Bull. Cal. Math. Soc. 86 (1994) 429-436.

[3] B. Dinda, T.K. Samanta and Iqbal, H. Jebril, Fuzzy anti-norm and fuzzy a- anti- convergence, arXiv preprint arXiv:1002.3818 (2010).

[4] B. Dinda, T.K. Samanta and Iqbal, H. Jebril, Fuzzy anti- bounded linear operator, Stud. Univ. Babes- Bolyen, Math, 56 (2011), No. 4, 123-137.

[5] C. Felbin, Finite dimensional fuzzy normed linear space, Fuzzy Sets and Syst, 48 (1992), 239-248.

[6] S. Gahler, Unter Suchyngen Uber Veralla gemeinerte m-metrische Raume I , Math. Nachr., 40 (1969), 165 - 189.

[7] H. Gunawan and M. Mashadi, On n-normed spaces, Int. J. Math and Math. Sci., 27 (2001), No. 10, 631 - 639.

[8] J. R. Smith Complement of codimention - two submanifolds - III- cobordism theory, pacific Journal of mathematical, Vol 94, No. 2, (1981), 423 - 484.

[9] A. K. Katsaras, Fuzzy topological vector spaces II, Fuzzy Sets and Syst. 12 (1984), 143-154.

[10] J. Kavikumar, Y. B. jun and A. khamis, The Riesz theorem in fuzzy n-normed linear spaces, J. Appl. Math. And Informatics , 27 (2009), No. 3 - 4, pp. 541 - 555.

[11] S. S. Kim and Y. J. Cho,Strict convexity in linear n-normed spaces , DemonstratioMath., 29 (1996), No. 4, 739 -744.

[12] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, Am. Math. Monthly, 72 (1965), 1004-1006.

[13] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica. 11 (1975), 326-334.

[14] S. V. Krishna and K. K. M. Sarma,Separation of fuzzy normed linear spaces , Fuzzy Sets Syst, 40 (1969), 165 -189.

[15] Kuldip Raj, Sunil K. Sharma and Ajay K. Sharma, Difference sequence spaces in n-normed spaces defined by Musielak-Orlicz function, Armenian Journal of mathematics, Vol 3, No 3, (2010), 127-141.

[16] I. H. Jebril and S. K. Samanta,Fuzzy anti-normed linear space , J. math. and Tech, February, pp. 66-77, 2010.

[17] R. Malceski,Strong n-convex n-normed spaces , Mat. Bilten, 21 (1997), no. 47, 81 - 102.

[18] A. Misiak, n-inner product spaces, Math. Nachr., 140 (1989), 299 - 319.

[19] AL. Narayanan and S. Vijayabalaji,Fuzzy n-normed linear space , Int. J. Math. and Math. Sci., (2005), no. 24, 3963 - 3977.

[20] C. Park and H. Y. Chu,The Riesz theorem in linear n-normed spaces , Journal of the Chungcheong Math. Soc., 19 (2006), no. 2, 207 - 211.

[21] G. S. Rhie, B. M. Choi and S. K. Dong,On the completeness of fuzzy normed linear spaces , Math. Japonica, 45 (1997), no. 1, 33 - 37.

[22] F. Riesz,Uber lineare Funktionalgleichungen , ActaMath, 41 (1918), 71 - 98.

[23] J. Schauder, Der Fixpunktsatz in Funktionalraumen, Studia Math. 2 (1930) 171-180.

[24] B. Schweizer and A. Sklar,Statistical metric spaces , Pacific J. Math., 10 (1960), 314 - 334.

[25] B. Surender Reddy ,Fuzzy Anti-2-Normed Linear Space, Journal of Mathematics Research, Vol. 3, No. 2, May (2011).

[26] L. A. Zadeh, Fuzzy set, Information and Control, 8 (1965), pp. 338-353.

[27] S. Vijayabalaji and N. Thillaigovindan,Complete fuzzy n-normed linear space , Journal of Fund. Sci., 3 (2007), 119 – 126.