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Solution of Dirac Equation in Tensor Formalism in Standard Representation

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ABSTRACT---- ***In previous works, Dirac equation for half-spin particle has been written in tensor form, through two isotropic vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F'} = \vec{E'} - i\vec{H'}$, by using both the spinor and the standard representation of Dirac matrices ($\gamma^{\mu} - matrices$). When, we use the spinor representation of $\gamma^{\mu} - matrices$, Dirac equation in tensor form takes the form of non-linear Maxwell's like equations for two electromagnetic fields $(\vec{E}, \vec{H})$ and $(\vec{E'}, \vec{H'})$. By using the standard representation of $\gamma^{\mu} - matrices$, Dirac equation takes another form different from that obtained by using the spinor representation of $\gamma^{\mu} - matrices$. The solution of Dirac equation in tensor formalism for free particle, with the use of the spinor representation of $\gamma^{\mu} - matrices$ has been obtained in the previous work. In this work, we found the solution of Dirac equation in tensor formalism for free particle with the use of the standard representation of $\gamma^{\mu} - matrices$.***

Keywords--- Dirac equation, tensor form, standard representation of $\gamma^{\mu} -$ matrices, solution.

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1. INTRODUCTION

In previous works, Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell's like equations for two electromagnetic fields $(\vec{E}, \vec{H})$ and $(\vec{E'}, \vec{H'})$. These non-linear equations have been obtained by using the spinor representation of Dirac matrices ($\gamma^{\mu} -$ matrices). These tensor equations have been studied in details. Especially, their solution for free particle has been obtained. In the work that followed, Dirac equation has been written in tensor form by using the standard representation of Dirac matrices ($\gamma^{\mu} -$ matrices). Here, the result proved that the form of Dirac equation in tensor form is different for various representations of $\gamma^{\mu} -$ matrices.

In this work, we shall solve and we shall find the solution for free particle of Dirac equation in tensor formalism with the use of the standard representation of $\gamma^{\mu} -$ matrices.

2. RESEARCH METHOD

In this work, we shall solve Dirac equation in tensor formalism. The investigated here Dirac equation in tensor form has been obtained in the form of non-linear equations for two complex isotropic vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F'} = \vec{E'} - i\vec{H'}$, but with the use of the standard representation of Dirac matrices ($\gamma^{\mu} -$ matrices). To derive these non-linear tensor equations, Cartan map has been used. In this work, using the same method, based on Cartan map, we shall solve these non-linear equations, representing Dirac equation in tensor formalism.

3. SPINOR FORMULATION OF DIRAC EQUATION

Relativistic particle with spin $1/2$ and different from zero rest mass is described by the wave equation proposed by Dirac in 1928. This equation, written in symmetric form is

$$(\gamma^{\mu}\partial_{\mu} - m)\psi = 0. \quad (1)$$

Here γ^{μ} are square matrices of 4^{th} rank, satisfying the relations (Klifford-Dirac algebra)

$$\gamma_{\mu}\gamma_{\nu} + \gamma_{\nu}\gamma_{\mu} = 2\delta_{\mu\nu}, \quad (2)$$

Where $\mu, \nu = 0,1,2,3$.

It is natural to emphasize, that in general, Dirac matrices γ^{μ} are defined with accuracy to correspondence transformation. Thus, the representation of these matrices can be chosen in various forms. Ordinary, it is commonly used the representation of Dirac matrices in which γ^0 is diagonal:

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \vec{\gamma}=\begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}. \tag{3}$$

Here $\vec{\sigma}$ are second rank Pauli spin matrices, having the form

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \sigma_2=\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

This representation is often called the standard representation.

In this representation, Dirac bispinor ψ is written as

$$\Psi=\begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \tag{5}$$

Hereφ, χ are tridimensional (but two components) Pauli spinors.

Using formulae (3) and (5), equation (1) can be written in the form of a system of two equations:

$$\begin{cases} p_0\varphi - (\vec{p}\vec{\sigma})\chi = -m\,\varphi \\ p_0\chi - (\vec{p}\vec{\sigma})\varphi = m\chi \end{cases}. \tag{6}$$

Another representation of Dirac matrices is the spinor representation. In this representation γ^{μ}-matrices and Dirac bispinor ψ are written in the form

$$\gamma^0=\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \; \vec{\gamma}=\begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \tag{7}$$

$$\Psi=\begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{8}$$

With the help of formulae (7) and (8), Dirac equation (1) can be written in the form of a system of two equations

$$\begin{cases} \left(p_0 + (\vec{p}\vec{\sigma})\right) \eta = m\xi \\ \left(p_0 - (\vec{p}\vec{\sigma})\right)\xi = m\eta. \end{cases} \tag{9}$$

It follows from equation (1), that each component of the wave function ψ satisfies the Klein-Gordon equation

$$(\Box - m^2)\psi_i = 0. \tag{10}$$

Where i=1,2,3,4; $\Box=\frac{\partial}{\partial t^2}-\vec{\nabla}^2 -$ D´Alembert operator.

4. CARTAN MAP

Definition and Algebraic Properties

We shall denote by C^n, the complex vector space of dimension "n". We shall consider only C^2, $C^3$and C^4.

Elements of C^2 will be denoted by Geek syllables

$$\xi=\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{11}$$

and will be called spinors.

Elements of C^3 will be denoted by Latin syllables

$$\vec{F}=\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}, \qquad (12)$$

and will be called vectors.

Finally, elements of C^4 will be denoted by Latin syllables

$$j_\mu=\begin{bmatrix} j_0 \\ j_x \\ j_y \\ j_z \end{bmatrix}, \qquad (13)$$

and will be called four vectors.

Definition1: Cartan map is a bilinear transformation b from space $C^2 \times C^2$ into space C^4, defined as follows:

$$b^0(\xi,\tau) = -(\xi_1\tau_2 - \xi_2\tau_1) \qquad (14)$$

$$\vec{b}(\xi,\tau)=\begin{bmatrix} \xi_1\tau_1 - \xi_2\tau_2 \\ i(\xi_1\tau_1 + \xi_2\tau_2) \\ -(\xi_1\tau_2 + \xi_2\tau_1) \end{bmatrix}. \qquad (15)$$

From the definitions (14) and (15) follows that b^0 is antisymmetric and $\vec{b}$ is symmetric relative to the change ξ by τ, i.e.,

$$b^0(\xi,\tau) = -b^0(\tau,\xi) \qquad (16)$$

$$\vec{b}(\xi,\tau)= \vec{b}(\tau,\xi). \qquad (17)$$

In particular, for any spinor ξ

$$b^0(\xi,\xi)=0 \qquad (18)$$

Using the definitions (14)-(15), one can prove the following properties of Cartan map:

Lemma1: For any spinors ρ, ξ, τ of space C^2, the following identities are verified

$$\vec{b}(\rho,\xi)\,\vec{b}(\tau,\tau) = -2\, b^0(\rho,\tau)\, b^0(\xi,\xi) \qquad (19)$$

$$\vec{b}(\rho,\xi)\,\vec{b}(\xi,\tau) = -2\, b^0(\rho,\xi)\, b^0(\xi,\tau) \qquad (20)$$

$$\vec{b}(\rho,\tau)\,\vec{b}(\xi,\tau) = \quad b^0(\rho,\tau)\, b^0(\xi,\tau) \qquad (21)$$

$$\vec{b}(\xi,\xi)\,\vec{b}(\tau,\tau) = -2b^0(\xi,\tau)^{\,2} \qquad (22)$$

$$\vec{b}(\xi,\tau)\,\vec{b}(\tau,\xi) = \quad b^0(\xi,\tau)^{\,2} \qquad (23)$$

$$\vec{b}(\xi,\xi)\,\vec{b}(\tau,\xi) = \;0 \qquad (24)$$

Lemma2 : For any two spinors ξ and τ of space C^2, the following identity is verified

$$\vec{b}(\xi,\xi) \times \vec{b}(\tau,\tau) = 2i\, b^0(\xi,\tau)\, \vec{b}(\xi,\tau) \qquad (25)$$

Definition2: If

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in C^2 \qquad (26)$$

is a spinor, then the conjugate spinor ξ^* of the spinor ξ is defined as follows

$$\xi^* = \begin{bmatrix} -\bar{\xi}_2 \\ \bar{\xi}_1 \end{bmatrix} \in C^2 , \tag{27}$$

Where $\bar{\xi}_1, \bar{\xi}_2$ are complex conjugates of spinor components ξ_1 and ξ_2.

Lemma3 : For any two spinors ξ and τ of space C^2, the following identities are verified

$$b^0(\xi,\tau^*)=\overline{b^0(\tau,\xi^*)} , \tag{28}$$

$$\vec{b}(\xi,\tau^*) =\overline{\vec{b}(\tau,\xi^*)} , \tag{29}$$

$$b^0(\xi^*,\tau^*)=\overline{b^0(\xi,\tau)}, \tag{30}$$

$$\vec{b}(\xi^*,\tau^*) =-\overline{\vec{b}(\xi,\tau)}. \tag{31}$$

Let us introduce vectors $\vec{F} \in C^3$ and $j_\mu \in C^4$ as follows:

$$\vec{F}=\vec{E}+i\vec{H}=i\vec{b}(\xi,\xi), \tag{32}$$

$$j_\mu=b_\mu(\xi,\xi^*) . \tag{33}$$

Here $\vec{E}$ and $\vec{H}$ are real vectors.

From formula (31) follows, that

$$\vec{F^*}=\vec{E}-i\vec{H}=\overline{i\vec{b}(\xi,\xi)} = i\,\vec{b}(\xi^*,\xi^*). \tag{34}$$

Lemma4: From formulas (15) and (32) follows identity

$$\vec{F}^2= \vec{F}.\vec{F}=0. \tag{35}$$

i.e., $\vec{F}$ is isotropic vector.

Formula (35) is equivalent to two conditions, obtained by equating to zero separately real and imaginary parts of equality $\vec{F}^2=0$:

$$\vec{E}^2=\vec{H}^2, \tag{36}$$

$$\vec{E}.\vec{H}=0. \tag{37}$$

One can also prove, that

$$j_0=\left[\frac{\vec{F}.\vec{F}^*}{2}\right]^{1/2}=|\vec{E}|,$$

$$\vec{j} =i\,\frac{\vec{F}\times\vec{F}^*}{2j_0} = \frac{\vec{E}\times\vec{H}}{|\vec{E}|} . \tag{38}$$

Lemma5: For any spinor $\xi \in C^2$, the following identities are verified

$$j_0=|\vec{E}| = |\xi|^2, \tag{39}$$

$$\vec{j} = \frac{\vec{E}\times\vec{H}}{|\vec{E}|} = \bar{\xi}^T\vec{\sigma}\xi. \tag{40}$$

Where $\bar{\xi}^T$ is the transposed conjugate of the spinor ξ and $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$are Pauli spin matrices.

From formulas (39)-(40) follows, that under Lorentz relativistic transformations, $j_\mu$transforms as a four vector. Vectors $\vec{E}$ and $\vec{H}$ transform as components of electromagnetic field, i.e., form a second rank tensor $F_{\mu\nu}$.

Lemma6: For any pair of spinors ξ and τ of space C^2 and any vector $\vec{v}$ the following identities are verified

$$b^0(\vec{v}.\vec{\sigma}\xi,\tau) = \vec{v}.\vec{b}(\xi,\tau), \qquad (41)$$

$$\vec{b}(\vec{v}.\vec{\sigma}\xi,\tau) = \vec{v}b^0(\xi,\tau) + i\,\vec{v}\times\vec{b}(\xi,\tau), \qquad (42)$$

$$\vec{b}(\vec{v}.\vec{\sigma}\xi,\xi) = (\vec{v}.\vec{s})\,\vec{b}(\xi,\xi). \qquad (43)$$

Here $\vec{s} = (s_x, s_y, s_z)$ are Proka spin matrices, with $s_i = i(\varepsilon_i)_{jk}$, where ε_{ijk} is the tridimensional antisymmetric tensor Levi-Cevita.

From formula (43) follows, that if ξ is eigenvector of operator $(\vec{v}.\vec{\sigma})$ with eigenvalue λ, then $\vec{b}(\xi,\xi)$ is eigenvector of operator $(\vec{v}.\vec{s})$ with the same eigenvalue λ.

Definition3: Let ξ be a spinor field and $\widetilde{A}$, an operator acting on ξ. Let $\vec{b}$ maps spinor ξ on isotropic vector $\vec{F}=i\vec{b}(\xi,\xi)$. We shall say, that the operator $\widetilde{A}$ commutes with Cartan map and becomes $\widehat{A}$, acting on $\vec{F}$, if:

$$\widehat{A}\,\vec{F} = i\widehat{A}\,\vec{b}(\xi,\xi) = i\vec{b}(\widetilde{A}\xi,\xi). \qquad (44)$$

From formula (44) follows, that if ξ is eigenvector of operator $\widetilde{A}$ with eigenvalue λ, then $\vec{F}$ is eigenvector of operator $\widehat{A}$ with the same eigenvalue λ; i.e., Cartan map conserves eigenvectors and eigenvalues.

Lemma7 : For any spinor ξ of space C^2, the following identities are verified

$$b^0(\vec{p}\xi,\xi) = -i\{\vec{D}\vec{b}(\xi,\xi)\}.\vec{v}, \qquad (45)$$

$$\vec{b}(\vec{p}\xi,\xi) = \vec{D}\vec{b}(\xi,\xi), \qquad (46)$$

Where $\quad \vec{v} = \frac{\vec{j}}{j_0} = \frac{\vec{E}\times\vec{H}}{\vec{E}^2}$

$$\vec{D} = -i\frac{\hbar}{2}\vec{\nabla}.$$

5. SOLUTION OF DIRAC EQUATION IN TENSOR FORMALISM IN STANDARD REPRESENTATION

In previous work, by using Cartan map and the standard representation of γ^μ −matrices, Dirac equation has been written in tensor form as follows

$$\begin{cases} D_0\vec{F} - \frac{i}{\sqrt{2}}\vec{D}(\vec{F}\vec{F}')^{1/2} + \frac{i}{\sqrt{2}}\vec{D}\times\left[\frac{\vec{F}\times\vec{F}'}{(\vec{F}\vec{F}')^{1/2}}\right] = m\vec{F} \\ D_0\vec{F}' - \frac{i}{\sqrt{2}}\vec{D}(\vec{F}\vec{F}')^{1/2} - \frac{i}{\sqrt{2}}\vec{D}\times\left[\frac{\vec{F}\times\vec{F}'}{(\vec{F}\vec{F}')^{1/2}}\right] = -m\vec{F}'. \end{cases} \qquad (47)$$

Where

$$\vec{F} = \vec{E} + i\vec{H} = i\vec{b}(\varphi,\varphi),\ \ \vec{F}' = \vec{E}' - i\vec{H}' = i\vec{b}(\chi,\chi), \qquad (48)$$

$$D_0 = \frac{i}{2}\frac{\partial}{\partial t}, \qquad \vec{D} = -\frac{i}{2}\vec{\nabla}.$$

Here we use the natural system of units in which c=ħ=1.

The solution of Dirac equation (6) with consideration of positive and negative energies has the form

$$\psi = \binom{\varphi}{\chi} = be^{-i\varepsilon\mathcal{K}t + i\vec{k}\vec{r}}. \qquad (49)$$

Here

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, \tag{50}$$

where

$$\begin{aligned} b_1 &= \frac{1}{\sqrt{2}}\sqrt{1+\frac{\varepsilon m}{E}}\, s e^{-\frac{i}{2}\varphi}\sqrt{1+s\cos\theta}, \\ b_2 &= \frac{1}{\sqrt{2}}\sqrt{1+\frac{\varepsilon m}{E}}\, s e^{\frac{i}{2}\varphi}\sqrt{1-s\cos\theta}, \\ b_3 &= \frac{1}{\sqrt{2}}\sqrt{1-\frac{\varepsilon m}{E}}\, e^{-\frac{i}{2}\varphi}\sqrt{1+s\cos\theta}, \\ b_4 &= \frac{1}{\sqrt{2}}\varepsilon s\sqrt{1+\frac{\varepsilon m}{E}}\, e^{\frac{i}{2}\varphi}\sqrt{1-s\cos\theta}. \end{aligned} \tag{51}$$

Here $\mathcal{K}$=E=$\sqrt{\vec{k}^2+m^2}$ is energy of the particle, $\varepsilon=\pm1$ considers the sign of energy, $\varepsilon=+1$ for a particle and $\varepsilon=-1$ for an antiparticle. S=$\pm1$is the helicity of the particle; φ,θ are angles of direction of the wave vector $\vec{k}$, chosen so that $k_1+ik_2=k\sin\theta\, e^{i\varphi}$, $k_3=k\cos\theta$.

Replacing expression (51) in formulae (48), we find the solution of the system (47) in the form of complex isotropic vectors

$$\vec{F}=\vec{E}+i\vec{H}=\vec{F}^0 e^{-2i\varepsilon\mathcal{K}t+2i\vec{k}\vec{r}}, \tag{52}$$

$$\vec{F}'=\vec{E}'-i\vec{H}'=\vec{F}'^0 e^{-2i\varepsilon\mathcal{K}t+2i\vec{k}\vec{r}}, \tag{53}$$

where

$$\vec{F}^0=\left(1+\frac{\varepsilon m}{E}\right)\begin{bmatrix} \sin\varphi+is\cos\varphi\cos\theta \\ -\cos\varphi+is\sin\varphi\cos\theta \\ -is\sin\theta \end{bmatrix}, \tag{54}$$

$$\vec{F}'^0=\left(1-\frac{\varepsilon m}{E}\right)\begin{bmatrix} \sin\varphi+is\cos\varphi\cos\theta \\ -\cos\varphi+is\sin\varphi\cos\theta \\ -is\sin\theta \end{bmatrix}. \tag{55}$$

6. DISCUSSION AND CONCLUSION

In this work, we investigated the non-linear tensor equations, representing Dirac equation for half-spin particle in tensor formalism. These non-linear equations have been obtained by using Cartan map and with the use of the standard representation of Dirac matrices ($\gamma^\mu-$matrices). The study of these tensor equations proved that, the form of Dirac equation in tensor form varies with the used representation of $\gamma^\mu-$matrices. By using the same method based on Cartan map, we found the solution of these non-linear tensor equations for free particle in the form of plane waves. The result obtained proved that apart from the constant factor, this solution coincides with that obtained by using the spinor representation of $\gamma^\mu-$matrices. Thus, we can conclude that, in the difference of the equations, the solution of Dirac equation in tensor formalism does not depend on the used representation of $\gamma^\mu-$matrices as it should be.

7. REFERENCES

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