Explicit Solutions for Optimal Insurance Problems in Regime Switching Frameworks

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ABSTRACT – ***The present paper treats the generalized Merton-type optimal consumption investment problem for a financial market whose characterizing parameters depend on the regime of the economy. In particular, we consider an agent which controls both his consumption and investment, as well as an insurance contract, and whose objective is to maximize the total discounted utility of consumption over an infinite horizon. In the case of Hyperbolic Absolute Risk Aversion (HARA) utility functions it is possible to obtain explicit solutions to the optimal consumption, investment and insurance problems, showing that the optimal strategies depend on the state of the economy. Exploiting latter result we perform a novel financial analysis assuming that the economy is characterized by three volatility regimes, also studying the impact of adding an exogenous wage in the investor's wealth process.***

Keywords – Regime switching, insurance, stochastic optimization problem, Hamilton-Jacobi-Bellman equation, CRRA utility function, HARA utility function

1 INTRODUCTION

The optimal consumption and investment problem introduced by Merton in [12] is a key problem in both theory and practice of finance. Merton derived a closed-form solution to the optimization problem assuming *perfect market*, a Black & Scholes model for the dynamic of the risky asset, and a Constant Relative Risk Aversion (CRRA) utility function type. Such an approach has been then generalized along different lines as, e.g., in [8] and [17].

In the traditional models for consumption and investment problems there is only one source of risk given by the uncertainty of the stock price, whereas in real life an economic agent faces also risks such, e.g., the credit default, the property-liability risk or the *insurable risk.* Concerning the latter, the investor can buy insurance, which may reward him and offset capital losses if the risk event occurs. Obviously, the cost of insurance diminishes the investor's ability to consume and consequently his expected utility of consumption. The problem of determine the optimal insurance strategy of an economic agent under a given utility maximization criterion is called *optimal insurance problem.* In [1], Arrow gave first results with respect to the discrete time case, considering a static single-period model, and showing that under fairly general conditions the optimal insurance is deductible insurance.

In [13], Moore and Young analyzed for the first time the problem of combining the Merton's optimal consumption-investment problem and the Arrow's optimal insurance problem in a dynamic continuous-time framework. They allowed the horizon to be random also giving an impact measure for insurance on the investment and consumption strategies and finding solutions for different utility functions.

The introduction of an insurable risk in the optimization problem is an important extension, but not the only possible one. In particular it is well known that the market behavior is affected by macroeconomic conditions that may change dramatically as time evolves. The *Regime Switching Models* (RSM) approach takes into considerations such long-term factors by allowing the market parameters, e.g., risk-free interest rate, stock return and volatility, etc., to change in time according with the dynamic of an underlying continuous-time Markov chain whose states represent the different regimes of the economy.

Hamilton pioneered the econometric applications of RSM in [6], showing in particular that a Markov switching autoregressive time series model can represent the stock returns better than the usual model with deterministic coefficients. Thereafter, RSM have been applied, e.g., to option pricing, portfolio analysis, optimal consumption, investment and insurance problems. In [18] Sotomayor and Cadenillas consider a RSM where both the coefficients and the utility function depend on the regime, aiming at the study of the stochastic control problem of an investor who seeks to maximize his expected total discounted utility from consumption in the infinite horizon case. In particular they give an explicit solutions for four different utility functions, and defines the first version of the Hamilton-Jacobi-Bellman (HJB) equation for a stochastic control problem in infinite horizon with regime switching.

Results obtained in [18] and [13], have been recently combined by Zou and Cadenillas in [19] to provide for the first time a rigorous verification theorem to simultaneous optimal consumption, investment and insurance problem, in a RS framework. Following the general outline of the latter paper, we present in Sections 2 and 3 the structure of the model and two verification theorems, together with the HJB equation related to this particular optimization problem. In Section 4 we recall results concerning explicit solutions for optimal consumption, investment and insurance policies for the Hyperbolic Absolute Risk Aversion (HARA) utility functions case. The last section is devoted to a novel numerical analysis concerning the impact of regimes, market coefficients and investor's risk aversion on optimal policies. Such results extends those obtained by Zou and Cadenillas, who set their model in a two regimes framework, by assuming that there are three states in the economy. Moreover in Subsection 5.2 we presents new results concerning the impact of adding a non-zero, exogenous wage in investor's wealth equation and we outline a comparison with respect to the standard problem.

2 STRUCTURE OF THE MODEL

Let $W := \{W_t, t \geq 0\}$ be a standard Brownian motion and let $S := \{S_t, t \geq 0\}$ be an observable, continuous-time, stationary, finite-state Markov chain defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}_t = (\mathcal{F}_t)_{t\geq 0}$ indicates the $\mathbb{P}$-augmentation of the filtration $(\mathcal{F}_t^{W,S})_{t\geq 0}$, and, for every $t \in [0,\infty)$, $\mathcal{F}_t^{W,S} := \sigma\{W_s, S_s \mid 0 \leq s \leq t\}$. Note that, by definition of $\mathbb{P}$-augmentation, the filtration $\mathcal{F}_t$ is complete and right-continuous. Furthermore, we assume that W and S are independent. Let us denote by $\mathcal{M} := \{1, 2, \ldots, M\}$ the state space of the Markov chain, where $M \in \{2, 3, 4, \ldots\}$ corresponds to the number of regimes in the economy, hence, for every $t \in [0,\infty)$, $S_t \in \mathcal{M}$ represents the state of the economy at time t. Assume that the rates for the Markov chain transitions are described by a strongly irreducible generator matrix $Q = (q_{ij})_{i,j\in\mathcal{S}}$, where $\forall i \in \mathcal{M}$, $\sum_{j\in\mathcal{S}} q_{ij} = 0$, $q_{ij} > 0$ for $i \neq j$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$. Following [19], we consider a financial market with two assets, a bond (riskless asset) and a stock (risky asset), with price represented by the $\mathcal{F}_t$-adapted process $P^0 := \{P_t^0, t \geq 0\}$ and $P^1 := \{P_t^1, t \geq 0\}$, respectively. These price processes are assumed to be driven by the following dynamics:

$$\begin{aligned} dP_t^0 &= r_{S_t} P_t^0 dt \ , \\ dP_t^1 &= P_t^1(\mu_{S_t} dt + \sigma_{S_t} dW_t), \end{aligned}$$

with initial conditions $P_0^0 = 1$ and $P_0^1 > 0$. Note that the coefficients of the market depend on the state of the economy, and for all $i \in \mathcal{M}$ the rate of return r_i for the riskless asset, the expected rate of return μ_i, and the risky asset volatility σ_i, are all positive constants.

An investor chooses the *proportion of wealth invested in the stock*, $\pi := \{\pi_t, t \geq 0\}$, a *consumption rate process* $c := \{c_t, t \geq 0\}$ and he is subjected to an insurable loss $L(t, S_t, X_t)$ or L_t for short, X_t being his wealth at time t, characterized by a Poisson process $N = \{N_t, t \geq 0\}$ with intensity λ_{S_t}, where $\lambda_i > 0$ for every $i \in \mathcal{M}$. Suppose that the investor can also control the *payout amount* process $I := \{I_t, t \geq 0\}$, where $I_t : [0,\infty) \times \Omega \mapsto [0,\infty)$ and $I_t(\omega) := I_t(L(t, S_t(\omega), X_t(\omega)))$ or, in short, $I_t := I_t(L_t)$. In what follows we suppose that the premium is payable continuously by the investor, at a rate P_t which is proportional to the expected payout, namely $P_t := \lambda_{S_t}(1 + \theta_{S_t})\mathbb{E}[I_t(L_t)]$, where the positive constant θ_i, for $i \in \mathcal{M}$ is known as the *loading factor* and is due to the administrative costs, tax and profit that an insurance company has to sustain. Moreover we assume that W, S, and N are mutually independent, while the loss process L is independent of N and we denote with $\mathbb{F}$ the

$\mathbb{P}$-augmentation of the filtration generated by W, N, L and S. The *wealth process* associated to an investor with a triplet of strategies $u_t := (\pi_t, c_t, I_t)$ is given by the $\mathbb{F}$-adapted process $X = \{X_t, t \geq 0\}$, driven by the following dynamics:

$$dX_t = (r_{S_t}X_t + (\mu_{S_t} - r_{S_t})\pi_t X_t - c_t - P_t)dt + \sigma_{S_t}\pi_t X_t dW_t - (L_t - I_t(L_t))dN(t), \tag{1}$$

with initial conditions $X_0 = x > 0$ and $S_0 = i \in \mathcal{M}$. The investor's *utility of consumption* is given by a function $U : (0, \infty) \times \mathcal{M} \to \mathbb{R}$ such that, for each state $i \in \mathcal{M}$, $U(\cdot, i)$ is $C^2(0, +\infty)$, strictly increasing, strictly concave, satisfying a linear growth condition of the following type

$$\exists K > 0 \text{ s.t. } U(y, i) \leq K(1 + y), \ \forall y > 0, i \in \mathcal{M}. \tag{2}$$

The assumption that the utility function depends on the market regime is supported by many works on financial economics,see, e.g., [3, 9, 11]. Note that, for a fixed regime $i \in \mathcal{M}$, it is possible to define $U(0, i) := \lim_{y\to 0^+} U(y, i)$ and $U'(0, i) := \lim_{y\to 0^+} U'(y, i)$, extending the utility function to the domain $[0, \infty)$. A *criterion functional* of the optimal consumption-investment-insurance problem is defined as follows

Definition 2.1. Given a control process $u = u_t := (\pi_t, c_t, I_t)$ for $t \geq 0$ and initial conditions $X_0 = x > 0$, $S_0 = i \in \mathcal{M}$, the criterion functional J is defined as

$$J(x, i; u) := \mathbb{E}_{x,i}\left[\int_0^\infty e^{-\delta t} U(c_t, S_t)dt\right], \tag{3}$$

where $\delta > 0$ is the discount rate and $\mathbb{E}_{x,i}$ represents the expectation conditioned to $X_0 = x$ and $S_0 = i$.

Obviously, the investor will be able to consume and invest only if his wealth is positive, hence we define the *bankruptcy* stopping time as $\Theta := \inf\{t \geq 0 : X_t \leq 0\}$, and impose $X_t = 0$ for all $t \geq \Theta$. Moreover, it is necessary to characterize the admissibility of a control process.

Definition 2.2. A control $u := (\pi, c, I)$ is said to be *admissible* if $\{u_t\}_{t\geq 0}$ is predictable with respect to the filtration $\mathbb{F}$ and satisfies

$$\mathbb{E}_{x,i}\left[\int_0^t c_s ds\right] < +\infty \ ; \ \mathbb{E}_{x,i}\left[\int_0^t \sigma^2_{S_s}\pi^2_s ds\right] < +\infty \ ; \ \mathbb{E}_{x,i}\left[\int_0^\Theta e^{-\delta s} U^+(c_s, S_s)ds\right] < +\infty \ , \tag{4}$$

where, for all $t \in [0, \Theta]$, $U^+(c_t, S_t) := \max(0, U(c_t, S_t))$.

Moreover, $\forall t \geq 0$, $c_t \geq 0$ and $I_t \in \mathcal{I}_t := \{I : 0 \leq I(Y) \leq Y \ , \ Y \ , \ \mathcal{F}_t\text{-measurable process}\}$, i.e., the payout amount is never greater than the loss amount, so the investor cannot obtain a gain from his loss. The set of all admissible controls with initial conditions $X_0 = x$ and $S_0 = i$ will be denoted by $\mathcal{A}_{x,i}$.

The investor wants to solve the following optimization problem:

Problem 2.1. *Select an admissible control $\hat{u} = (\hat{\pi}, \hat{c}, \hat{I}) \in \mathcal{A}_{x,i}$ that maximizes the criterion functional* (3)*, and find the value function $V(x, i) := \sup_{u\in\mathcal{A}_{x,i}} J(x, i; u)$, then the control $\hat{u}$ is the optimal control (or optimal policy) of the optimization problem.*

3 VERIFICATION THEOREMS

Let $\psi : (0, \infty) \times \mathcal{M} \to \mathbb{R}$ be a function with $\psi(\cdot, i) \in C^2(0, \infty)$, $\forall i \in \mathcal{M}$ and define the operator $\mathcal{L}^u_i$, for all $i \in \mathcal{M}$, by $\mathcal{L}^u_i := (r_i x + (\mu_i - r_i)\pi x - c - \lambda_i(1 + \theta_i)\mathbb{E}[I(L)])\frac{\partial\psi}{\partial x} + \frac{1}{2}\sigma^2_i\pi^2 x^2 \frac{\partial^2\psi}{\partial x^2} - \delta\psi$, then, see [19], pp. 8-11. we have

Theorem 3.1. *Suppose that $U(0, i)$ is finite $\forall i \in \mathcal{M}$. Let $v(\cdot, i) \in C^2(0, \infty)$, $i \in \mathcal{M}$, be an increasing and concave function in $(0, \infty)$ such that $v(0, i) = \frac{U(0,i)}{\delta}$ $\forall i \in \mathcal{M}$. If the function $v(\cdot, i)$ satisfies the Hamilton-Jacobi-Bellman equation*

$$\sup_u\{\mathcal{L}^u_i v(x, i) + U(c, i) + \lambda_i\mathbb{E}[v(x - L + I(L), i) - v(x, i)]\} = -\sum_{j\in\mathcal{M}} q_{ij}\left(v(x, j) - \frac{U(0, j)}{\delta}\right), \tag{5}$$

for every $x > 0$, and the control $\hat{u} = (\hat{\pi}, \hat{c}, \hat{I})$ defined by

$$\hat{u}_t := \arg\sup_u \big(\mathcal{L}^u_{S_t} v(\hat{X}_t, S_t) + U(c, S_t) + \lambda_{S_t}\mathbb{E}[v(\hat{X}_t - L_t + I(L_t), S_t) - v(\hat{X}_t, S_t)]\big)1_{0\leq t<\theta} \ ,$$

is admissible, then $\hat{u}$ is an optimal solution to problem (2.1). Moreover, the value function is given by

$$V(x,i) := v(x,i) + \frac{1}{\delta}\mathbb{E}_{x,i}\left[\int_0^\infty e^{-\delta s} dU(0,S_s)\right],$$

where $dU(0,S_s) := \sum_{j\in\mathcal{M}} q_{S_s,j} U(0,j)ds$. Furthermore, if the utility function does not depend on the regime, i.e. $U(y,i) = U(y)$ for every $i \in \mathcal{M}$, $y \in [0,\infty)$, then the value function is given by $V(x,i) = v(x,i)$.

Theorem 3.2. *Suppose that $U(0,i) = -\infty\ \forall i \in \mathcal{M}$. Let $v(\cdot,i) \in C^2(0,\infty)$, $i \in \mathcal{M}$, be an increasing and concave function in $(0,\infty)$ such that $v(0,i) = \frac{U(0,i)}{\delta} = -\infty\ \forall i \in \mathcal{M}$. If the function $v(\cdot,i)$ satisfies the Hamilton-Jacobi-Bellman equation*

$$\sup_u \{\mathcal{L}_i^u v(x,i) + U(c,i) + \lambda_i \mathbb{E}[v(x - L + I(L), i) - v(x,i)]\} = -\sum_{j\in\mathcal{M}} q_{ij} v(x,j)\ , \tag{6}$$

for every $x > 0$, and the control $\hat{u} = (\hat{\pi}, \hat{c}, \hat{I})$ defined by

$$\hat{u}_t := \arg\sup_u \big(\mathcal{L}_{S_t}^u v(\hat{X}_t, S_t) + U(c,S_t) + \lambda_{S_t}\mathbb{E}[v(\hat{X}_t - L_t + I(L_t), S_t) - v(\hat{X}_t, S_t)]\big) 1_{0\le t<\theta}\ ,$$

is admissible, then $\hat{u}$ is an optimal control to problem (2.1) and the value function is given by $V(x,i) = v(x,i)$.

4 EXPLICIT SOLUTIONS OF VALUE FUNCTION

The objective of this section consists in obtaining explicit solutions to optimal consumption, investment and insurance problem for particular choices of the utility function. Consider an utility function of *HARA* (Hyperbolic Absolute Risk Aversion) type, which is characterized by the following functional form:

$$U(x) = \frac{1-\alpha}{\alpha}\left(\frac{ax}{1-\alpha} + b\right)^\alpha, \quad x \in \mathbb{R}^+$$

where a, b and the risk aversion parameter α, are given constants. Note that this class of utility functions includes the logarithmic utility $U(x) = \log(ax)$, takig the limit $\alpha \to 0$ with $b = 0$, as well as the smaller class of CRRA utility functions. Suppose that the insurable loss L is proportional to the investor's wealth, namely $L(t, S_t, X_t) = \eta_{S_t} l_t X_t$, where for every $i \in \mathcal{M}$, $\eta_i > 0$ measures the intensity of the insurable loss in regime i, and for every $t \ge 0$, $l_t \in (0,1)$ is an $\mathcal{F}_t$-measurable process indicating the loss proportion at time t. In the sequel, we will assume that the loss proportion do not depend on time t, and in particular that l is constant or uniformly distributed on $(0,1)$. In this context, we note that the HJB equation (5) is equivalent to

$$\sup_{\pi\in\mathbb{R}}\big[f(\pi,x,i)\big] + \sup_{c\ge 0}\big[g(c,x,i)\big] + \lambda_i \sup_{I\ge 0}\big[h(I,x,i)\big] = (\delta+\lambda_i)v(x,i) - r_i x v'(x,i) - \sum_{j\in\mathcal{M}} q_{ij}\left(v(x,j) - \frac{U(0,j)}{\delta}\right), \tag{7}$$

while the HJB equation (6) is equivalent to

$$\sup_{\pi\in\mathbb{R}}\big[f(\pi,x,i)\big] + \sup_{c\ge 0}\big[g(c,x,i)\big] + \lambda_i \sup_{I\ge 0}\big[h(I,x,i)\big] = (\delta+\lambda_i)v(x,i) - r_i x v'(x,i) - \sum_{j\in\mathcal{M}} q_{ij} v(x,j), \tag{8}$$

where $v' := \partial v/\partial x$, $v'' := \partial^2 v/\partial x^2$ and

$$f(\pi,x,i) := (\mu_i - r_i)\pi x v'(x,i) + \frac{1}{2}\sigma_i^2 \pi^2 x^2 v''(x,i)\ ;\ g(c,x,i) := U(c,i) - cv'(x,i); \tag{9}$$

$$h(I,x,i) := \mathbb{E}\big[v(x - \eta_i l x + I(\eta_i l x), i)\big] - (1+\theta_i)\mathbb{E}\big[I(\eta_i l x)\big]v'(x,i). \tag{10}$$

If $v(\cdot,i)$ is strictly increasing and concave for every $i \in \mathcal{M}$, i.e. $v' > 0$ and $v'' < 0$ for all $x > 0$, $i \in \mathcal{M}$, then a candidate for the *optimal investment strategy* is

$$\hat{\pi}(x,i) := \arg\sup_{\pi\in\mathbb{R}}\{f(\pi,x,i)\} = -\frac{(\mu_i - r_i)v'(x,i)}{\sigma_i^2 x v''(x,i)}, \tag{11}$$

and a candidate for the *optimal consumption strategy* is given by

$$\hat{c}(x,i) := \arg\sup_{c\ge 0}\{g(\pi,x,i)\} = (U')^{-1}(v'(x,i), i). \tag{12}$$

Note that U' is strictly decreasing, so the inverse of U' exists; in particular, for HARA utility functions such inverse always exists on $(0,\infty)$. For the *optimal insurance policy*, we have the following theorem:

Theorem 4.1. *The optimal insurance, denoted with $\hat{I}$, is either no insurance or deductible insurance (a.s.)*

- *The optimal insurance is no insurance, i.e.* $\hat{I}(x,i,l) = 0\ \forall i \in \mathcal{M}$, *if* $(1+\theta_i)v'(x,i) \geq v'\big((1-\eta_i\, ess\sup(l))x, i\big)$.
- *The optimal insurance is deductible insurance, i.e.* $\hat{I}(x,i,l) = (\eta_i l x - d_i)^+\ \forall i \in \mathcal{M}$, *if* $\exists d_i := d_i(x) \in (0,x)$ *such that* $(1+\theta_i)v'(x,i) = v'(x-d_i,i)$.

Proof. See [19], Lemma 4.1 and Theorem 4.1. □

The result in theorem 4.1 is well known in static models: indeed, Schlesinger and Gollier proved in [16] that if one considers per-claim insurance and if the premium is proportional to the expected payout $\mathbb{E}[I_t(L_t)]$, then optimal insurance is deductible insurance. Moore and Young were the first to show that the same result holds in a dynamic setting, see [13], Proposition 3.1. Note that if $\theta_i = 0$, then the deductible is null ($d_i = 0$), i.e. full insurance is optimal if the premium rate is actuarially fair. Furthermore, the optimal insurance $\hat{I}$ satisfies some usual properties, namely it is an increasing function of the loss, a decreasing function of the price, and it vanishes when insurable loss is zero. In order to find an explicit function $v(\cdot,i)$, $i \in \mathcal{M}$, that satisfies either equation (5) or equation (6), we consider four utility functions of HARA type: the first three are very common in financial economics and do not depend on the market regimes, whereas the last one changes its parameters according to the regime of the economy.

1. $U(x,i) = \ln(x)$, $x > 0$,
2. $U(x,i) = -x^\alpha,\ x > 0,\ \alpha < 0$,
3. $U(x,i) = x^\alpha,\ x > 0,\ 0 < \alpha < 1$,
4. $U(x,i) = \beta_i\sqrt{x},\ x > 0,\ \beta_i > 0,\ i \in \mathcal{M}$.

It is possible to verify that these functions satisfy all the requirements for being utility functions: in particular, for a fixed $i \in \mathcal{M}$ each of them is $C^2(0,\infty)$, strictly increasing and strictly concave, and satisfy the linear growth condition (2) for suitable constants $K > 0$. To be more specific, $K = 1$ for the first three utility functions and $K = \max_i\{\beta_i\}$ for the last one.

4.1 $U(x,i) = \ln(x)$, $x > 0$

In this case, a solution to the HJB equation (6) is given by $\hat{v}(x,i) = \frac{1}{\delta}\ln(\delta x) + \hat{A}_i$, $\ i \in \mathcal{M}$,where the constants $\hat{A}_i$, $i \in \mathcal{M}$, will be determined below. In order to find the equations for optimal investment, consumption and insurance policies, compute first $\hat{v}'(x,i) = \frac{1}{\delta x}$, $\hat{v}''(x,i) = -\frac{1}{\delta x^2}$ and $(U')^{-1}(x,i) = \frac{1}{x}$; with these resuts, by (11) and (12), we obtain that $\hat{\pi}(x,i) = \frac{\mu_i - r_i}{\sigma_i^2}$ and $\hat{c}(x,i) = \delta x$, then $\frac{1+\theta_i}{\delta x} = \frac{1}{\delta(x-d_i)}$, which gives $d_i = \frac{\theta_i}{1+\theta_i}x$ and Since there exists $d_i := d_i(x) \in (0,x)$ satisfying $(1+\theta_i)v'(x,i) = v'(x-d_i,i)$, then, by Th.4.1, we have $\hat{I}(x,i,l) = \left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+ x$. Substituting the optimal controls in HJB equation (6), we obtain that the constants $\hat{A}_i$, $i \in \mathcal{M}$, has to satisfy the following equation:

$$\frac{1}{\delta}(r_i + \gamma_i + \lambda_i\hat{\Lambda}_i - \delta) = \delta\hat{A}_i - \sum_{j\in\mathcal{M}} q_{ij}\hat{A}_j, \tag{13}$$

where $\gamma_i := (\mu_i - r_i)^2/(2\sigma_i^2)$ and

$$\hat{\Lambda}_i := \mathbb{E}\left[\ln\left(1 - \eta_i l + \left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+\right)\right] - (1+\theta_i)\mathbb{E}\left[\left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+\right]. \tag{14}$$

Note that the $\hat{\Lambda}_1, \hat{\Lambda}_2, \ldots, \hat{\Lambda}_M$, for $M \in \{2,3,4,\ldots\}$, are the only quantities not directly given by the market; in particular, in section 5 we will compute the value of Λ_i, $\forall i \in \mathcal{M}$, from equation (14), assuming a constant loss proportion l and, without loss of generality, that $\frac{\theta_1}{\eta_1(1+\theta_1)} < \frac{\theta_2}{\eta_2(1+\theta_2)} < \ldots < \frac{\theta_M}{\eta_M(1+\theta_M)}$.

Proposition 4.2. *The function* $\hat{v}(\cdot,i)$, $i \in \mathcal{M}$, *given by* $\hat{v}(x,i) = \begin{cases} \frac{1}{\delta}\ln(\delta x) + \hat{A}_i, & x > 0 \\ -\infty, & x = 0, \end{cases}$ *where* $\hat{A}_i$, $i \in \mathcal{M}$, *satisfies the linear system* (13), *is the value function of Problem 2.1 with utility function* $U(x,i) = \ln(x)$, $x > 0$. *Moreover, the optimal policy is given by*

$$\hat{u}_t = (\hat{\pi}_t, \hat{c}_t, \hat{I}_t) = \left(\frac{\mu_{S_t} - r_{S_t}}{\sigma_{S_t}^2}, \delta\hat{X}_t, \left(\eta_{S_t} l_t - \frac{\theta_{S_t}}{1+\theta_{S_t}}\right)^+ \hat{X}_t\right), \tag{15}$$

where $\{\hat{X}_t, t \geq 0\}$ *denotes the wealth process for an investor who chooses the optimal strategy* $\hat{u}_t$ *at every time* $t \geq 0$. *Proof: See [19], pp. 17-18.*

4.2 $U(x,i) = -x^\alpha$, $x > 0$, $\alpha < 0$, $\forall i \in \mathcal{M}$

In this framework, a solution to the HJB equation (6) is given by $\tilde{v}(x,i) = -\tilde{A}_i^{1-\alpha}x^\alpha$, where the constants $\tilde{A}_i > 0$, $i \in \mathcal{M}$ will be determined below. Since $\tilde{v}'(x,i) = -\alpha\tilde{A}_i^{1-\alpha}x^{\alpha-1}$, $\tilde{v}''(x,i) = \alpha(1-\alpha)\tilde{A}_i^{1-\alpha}x^{\alpha-2}$ and $(U')^{-1}(x,i) = (-\frac{\alpha}{x})^{\frac{1}{1-\alpha}}$, we obtain, $\forall i \in \mathcal{M}$, $\hat{\pi}(x,i) = \frac{\mu_i - r_i}{(1-\alpha)\sigma_i^2}$ and $\hat{c}(x,i) = \frac{x}{\tilde{A}_i}$. Concerning the optimal insurance policy, solving the equation $(1+\theta_i)\tilde{v}'(x,i) = \tilde{v}'(x-d_i,i)$ we find $d_i = \nu_i x$, where $\nu_i := 1-(1+\theta_i)^{-\frac{1}{1-\alpha}}$, which implies $\hat{I}(x,i,l) = (\eta_i l - \nu_i)^+ x$. By substituting the candidate control $\hat{u} = (\hat{\pi}, \hat{c}, \hat{I})$ into the HJB equation (6), we obtain the following non-linear system for the constants $\tilde{A}_i$, $i \in \mathcal{M}$:

$$\left(\delta - \alpha r_i - \frac{\alpha}{1-\alpha}\gamma_i + \lambda_i(1-\tilde{\Lambda}_i)\right)\tilde{A}_i^{1-\alpha} - (1-\alpha)\tilde{A}_i^{-\alpha} = \sum_{j\in\mathcal{M}} q_{ij}\tilde{A}_j^{1-\alpha}, \tag{16}$$

where $\tilde{\Lambda}_i := \mathbb{E}\big[(1-\eta_i l + (\eta_i l - \nu_i)^+)^\alpha\big] - \alpha(1+\theta_i)\mathbb{E}\big[(\eta_i l - \nu_i)^+\big]$ can be easily computed assuming that l is constant or uniformly distributed on (0,1), as in previous case. Note that the constants $\tilde{A}_i$, $i \in \mathcal{M}$, must be positive by definition of the optimal consumption process $\hat{c}(x,i)$, and that $\tilde{A}_i \equiv 0$ $\forall i \in \mathcal{M}$ is a trivial solution of system (16), because $1-\alpha$ and $-\alpha$ are both positive constants. Therefore, in order to guarantee the above non-linear system has a unique positive solution, an additional condition is necessary:

Lemma 4.3. The non-linear system (16) has a unique positive solution $\tilde{A}_i$, $i \in \mathcal{M}$, if the parameter δ satisfies

$$\delta > \max_{i\in\mathcal{M}}\left\{\alpha r_i + \frac{\alpha}{1-\alpha}\gamma_i - \lambda_i(1-\tilde{\Lambda}_i)\right\}. \tag{17}$$

Proof. See [18], Lemma 4.1. □

Proposition 4.4. *The function $\tilde{v}(\cdot,i)$, $i \in \mathcal{M}$, given by* $\tilde{v}(x,i) = \begin{cases} -\tilde{A}_i^{1-\alpha}x^\alpha, & x > 0 \\ -\infty, & x = 0 \end{cases}$*, where $\tilde{A}_i$, $i \in \mathcal{M}$ is the unique solution to the linear system* (16) *with condition* (17)*, is the value function of Problem 2.1 with utility function $U(x,i) = -x^\alpha$, $x > 0$, $\alpha < 0$. Moreover, the optimal policy is given by*

$$\hat{u}_t = (\hat{\pi}_t, \hat{c}_t, \hat{I}_t) = \left(\frac{\mu_{S_t} - r_{S_t}}{(1-\alpha)\sigma_{S_t}^2}, \frac{\hat{X}_t}{\tilde{A}_{S_t}}, (\eta_{S_t} l_t - \nu_{S_t})^+\hat{X}_t\right), \quad t \geq 0. \tag{18}$$

4.3 $U(x,i) = x^\alpha$, $x > 0$, $0 < \alpha < 1$, $\forall i \in \mathcal{M}$

We have to consider the HJB equation (5), since $U(0,i) = 0$ $\forall i \in \mathcal{M}$, and a solution is given by the function $\bar{v}(x,i) = \bar{A}_i^{1-\alpha}x^\alpha$, where the constants $\bar{A}_i > 0$, $i \in \mathcal{M}$, satisfies again the non linear system (16), with the only difference that now $0 < \alpha < 1$. This can be easily proven by showing that for every $i \in \mathcal{M}$ the candidates for optimal investment, consumption and insurance policies are the same as before.

Lemma 4.5. System (16) with $0 < \alpha < 1$ has a unique positive solution $\bar{A}_i$, $i \in \mathcal{M}$, if the parameter δ satisfies

$$\delta > \max_{i\in\mathcal{M}}\left\{\alpha r_i + \frac{\alpha}{1-\alpha}\gamma_i\right\}. \tag{19}$$

Proof. See [18], Lemma 4.2. □

Proposition 4.6. *The function $\bar{v}(x,i) := \bar{A}_i^{1-\alpha}x^\alpha$, $x \geq 0$, $i \in \mathcal{M}$, where $\bar{A}_i$, $i \in \mathcal{M}$ is the unique solution to the linear system* (16) *with condition* (19)*, is the value function of Problem 2.1 with utility function $U(x,i) = x^\alpha$, $x > 0$, $0 < \alpha < 1$. Moreover, the optimal policy is given by*

$$\hat{u}_t = (\hat{\pi}_t, \hat{c}_t, \hat{I}_t) = \left(\frac{\mu_{S_t} - r_{S_t}}{(1-\alpha)\sigma_{S_t}^2}, \frac{\hat{X}_t}{\bar{A}_{S_t}}, (\eta_{S_t} l_t - \nu_{S_t})^+\hat{X}_t\right), \quad t \geq 0. \tag{20}$$

4.4 $U(x,i) = \beta_i\sqrt{x}$, $x > 0$, $\beta_i > 0$, $\forall i \in \mathcal{M}$

Consider a power utility function with $\alpha = \frac{1}{2}$ and with the particularity that it depends explicitly on the state of the economy. In this scenario, the value function that solves the HJB equation (5) is $\check{v}(x,i) = (\check{A}_i x)^{\frac{1}{2}}$, where the constants $\check{A}_i > 0$, $i \in \mathcal{M}$ satisfy a non-linear system that will be determined below.

From $\check{v}'(x,i) = \frac{1}{2}\check{A}_i^{\frac{1}{2}} x^{-\frac{1}{2}}$, $\check{v}''(x,i) = -\frac{1}{4}\check{A}_i^{\frac{1}{2}} x^{-\frac{3}{2}}$ and $(U')^{-1}(x,i) = \left(\frac{\beta_i}{2x}\right)$ we obtain

$$\hat{\pi}(x,i) = \frac{2(\mu_i - r_i)}{\sigma_i^2}, \quad \hat{c}(x,i) = \frac{\beta_i^2 x}{\check{A}_i} \quad \text{and} \quad \hat{I}(x,i,l) = (\eta_i l - \check{\nu}_i)^+ x,$$

where $\check{\nu}_i = 1 - \frac{1}{(1+\theta_i)^2}$ $\forall i \in \mathcal{M}$. Substituting these candidates for optimal policies in the HJB equation (5) yields the following non-linear system for constants $\check{A}_i$

$$\left(\delta - \frac{1}{2}r_i - \gamma_i + \lambda_i(1 - \check{\Lambda}_i)\right)\check{A}_i^{\frac{1}{2}} - \frac{1}{2}\frac{\beta_i^2}{\check{A}_i^{\frac{1}{2}}} = \sum_{j\in\mathcal{M}} q_{ij}\check{A}_j^{\frac{1}{2}}, \tag{21}$$

where $\check{\Lambda}_i := \mathbb{E}\big[(1 - \eta_i l + (\eta_i l - \check{\nu}_i)^+)^{\frac{1}{2}}\big] - \frac{1}{2}(1+\theta_i)\mathbb{E}\big[(\eta_i l - \check{\nu}_i)^+\big]$. One can prove that system (21) has a unique positive solution if condition (19) is satisfied for $\alpha = \frac{1}{2}$.

Proposition 4.7. *For every $i \in \mathcal{M}$, the function $\check{v}(x,i) := (\check{A}_i x)^{\frac{1}{2}}$, $x \geq 0$, where $\check{A}_i$ is the unique solution to the linear system* (21) *with condition* (19) *and $\alpha = \frac{1}{2}$, is the value function of Problem 2.1 with the regime-dependent utility function $U(x,i) = \beta_i x^{\frac{1}{2}}$, $x > 0$, $\beta_i > 0$. Moreover, the optimal policy is given by*

$$\hat{u}_t = (\hat{\pi}_t, \hat{c}_t, \hat{I}_t) = \left(\frac{2(\mu_{S_t} - r_{S_t})}{\sigma_{S_t}^2}, \frac{\beta_{S_t}\hat{X}_t}{\check{A}_{S_t}}, \big(\eta_{S_t} l_t - \check{\nu}_{S_t}\big)^+ \hat{X}_t\right), \quad t \geq 0. \tag{22}$$

5 ECONOMIC ANALYSIS

In the first part of this section we analyze the impact of risk aversion and market parameters on optimal policy from an economical point of view, and we extend the results presented in [19] by assuming that there are three, instead of two, regimes in the economy:

- *regime 1*, or low volatility state, represents a market with good economic conditions, for instance a market in economic boom or in which security prices are rising (*bull market*);
- *regime 2*, or medium volatility state, represents an average economy, in which prices are stable;
- *regime 3*, or high volatility state, corresponds to a market with bad economic conditions, for instance a market in recession or in which security prices are very variable and are expected to fall (*bear market*).

Next, we study the impact of adding an exogenous wage in the investor's wealth process by setting a new optimization problem and comparing the relative value function with the previous one.

5.1 Impact of market parameters and risk aversion

According to the economic theory, the market parameters should satisfy some conditions, among which the assumption that $r_i < \mu_i$ for every $i = 1, 2, 3$ and that the stock returns are higher in better economic conditions, i.e. $\mu_1 > \mu_2 > \mu_3$ (for a complete analysis see, e.g., [5]). On the other hand, both stock volatility and default risk are higher during a period of economic crisis, that is $\sigma_1 < \sigma_2 < \sigma_3$ and $\eta_1 < \eta_2 < \eta_3$. The data of treasury bill rate suggest the risk-free interest rate is higher in good economy, hence $r_1 > r_2 > r_3$, and Haley found in [7] that the underwriting margin is negatively correlated with the interest rate, which implies that during recessions the insurance company will increase the loading factor, i.e. $\theta_1 < \theta_2 < \theta_3$. In order to analyze how the regime switching together with the level of risk aversion affect the decision of the investor, consider only the first three utility functions, for which the optimal investment strategy can be uniformly expressed as

$$\hat{\pi}_t = \frac{1}{1-\alpha}\frac{\mu_{S_t} - r_{S_t}}{\sigma_{S_t}^2}, \tag{23}$$

where $\alpha = 0$ for the utility function $U(x,i) = \ln(x)$, $x > 0$. Equation (23) shows that the optimal proportion invested in the the stock is constant in any given regime, and depends only on the state of economy and on investor's risk aversion parameter α. This result is consistent with [12], in which Merton proved that for constant market coefficients and just one regime, the optimal portfolio investment does not depend on time or wealth, i.e. is constant. As we can see, for every $i = 1, 2, 3$, $\hat{\pi}$ is directly proportional to the expected excess return on the stock $(\mu_i - r_i)$, and inversely proportional to the variance of the stock return (σ_i^2); even tough it has been proven that the expected excess returns are lower in a period of good economic conditions, the effect of volatility in the ratio

$(\mu_i - r_i)/\sigma_i^2$ is strong enough to ensure that the expected excess return over variance is higher when the market conditions are strong (see [5] for an exhaustive analysis on empirical data). Therefore, the investor allocates a higher fraction of his wealth in a period of good economic conditions, that is $\hat{\pi}(x,1) > \hat{\pi}(x,2) > \hat{\pi}(x,3)$. From equation (23) we have that the optimal investment policy is inversely proportional to the relative risk aversion $1-\alpha$, tehrefore that an investor with low risk tolerance ($\alpha < 0$) will invest a smaller proportion of his wealth on the stock. Figure 1 represents the optimal investment as a function of the risk aversion parameter, given market coefficients $r_1 = 0.08$, $r_2 = 0.05$, $r_3 = 0.03$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $\mu_3 = 0.15$, $\sigma_1 = 0.25$, $\sigma_2 = 0.4$ and $\sigma_3 = 0.6$.

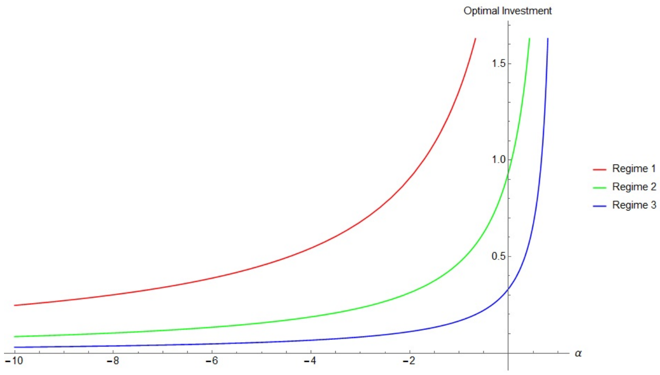

Figure 1: Optimal investment policy for $\alpha \in [-10, 1)$

Let us examine in details the optimal consumption policies for the first three utility functions: since they are all proportional to the wealth process $\hat{X}_t$, $t \geq 0$, we consider the *optimal consumption to wealth ratio*

$$k_t := \frac{\hat{c}_t}{\hat{X}_t} = \begin{cases} \delta, & \text{if } U(x,i) = \ln(x),\ \alpha = 0; \\ \frac{1}{\tilde{A}_{S_t}}, & \text{if } U(x,i) = -x^\alpha,\ \alpha < 0; \\ \frac{1}{\bar{A}_{S_t}}, & \text{if } U(x,i) = x^\alpha,\ 0 < \alpha < 1, \end{cases}$$

where $t \geq 0$ and $\tilde{A}_i$, $\bar{A}_i$ are the positive solutions to system (16) with $\alpha < 0$ and $0 < \alpha < 1$, respectively. Since k_t is positive in all three cases, the economic agent will consume proportionally more when he becomes wealthier, regardless of the market regime: this agrees with the economic theory and with previous studies on the positive effect of wealth on consumption (see, e.g., [14]). In order to study the dependency of the optimal consumption to wealth ratio on the risk aversion parameter α, we are going to analyze separately the three cases $\alpha = 0$, $\alpha < 0$ and $\alpha \in (0,1)$. For moderate risk-avers investors ($\alpha = 0$), k_t is a constant equal to the discount rate δ and does not vary with the regime of the economy; therefore, $\hat{c}(x,1)/x = \hat{c}(x,2)/x = \hat{c}(x,3)/x$ for every $x > 0$, and investors who use a logarithmic utility function will consume the same proportion of their wealth in every market regime. In the case of high risk-averse investors ($\alpha < 0$), we solve numerically system (16) for three regimes and market parameters $\delta = 0.15$, $r_1 = 0.08$, $r_2 = 0.05$, $r_3 = 0.03$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $\mu_3 = 0.15$, $\sigma_1 = 0.25$, $\sigma_2 = 0.4$, $\sigma_3 = 0.6$, $\lambda_1 = 0.1$, $\lambda_2 = 0.15$, $\lambda_3 = 0.2$, $q_{12} = 4.04$, $q_{13} = 4.04$, $q_{21} = 4.4$, $q_{23} = 4.4$, $q_{31} = 4.64$, $q_{32} = 4.64$, $\eta_1 = 0.8$, $\eta_2 = 0.9$, $eta_3 = 1$, $\theta_1 = 0.15$, $\theta_2 = 0.2$ and $\theta_3 = 0.25$. Suppose that the loss proportion l_t is constant and equal to l for every time $t \geq 0$, then from definition of $\tilde{\Lambda}_i$ in (16) one obtains, for $i = 1,2,3$,

$$\tilde{\Lambda}_i = \begin{cases} (1-\eta_i l)^\alpha & \text{if } (\eta_i l - \nu_i)^+ = 0; \\ (1-\nu_i)^\alpha - \alpha(1+\theta_i)(\eta_i l - \nu_i) & \text{if } (\eta_i l - \nu_i)^+ > 0, \end{cases}$$

where $\nu_i := 1 - (1+\theta_i)^{-\frac{1}{1-\alpha}}$ satisfy $\frac{\nu_1}{\eta_1} \leq \frac{\nu_2}{\eta_2} \leq \frac{\nu_3}{\eta_3}$ for every $\alpha \in (-\infty, 0)$. Note that these parameters satisfy also the technical condition (17), which ensures the existence and uniqueness of the positive solution of system (16).

In table 1 are shown the results for $l = 0.5$ and different values of α. We can see that $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$ in every case, which implies $\hat{c}(x,1)/x > \hat{c}(x,2)/x > \hat{c}(x,3)/x$: in other words, a very risk-averse investor will consume proportionally more when the economic conditions are good, i.e. in regime 1. However, for a fixed regime the consumption to wealth rate is still lower than the consumption to wealth rate of a less risk-averse investor, that is, the optimal consumption to wealth ratio is an increasing function of α. In general, we can say that an economic agent with a very low risk tolerance does not consume much, since for almost all values of α, he allocates less then 10% of his initial wealth for consumption. Hence, a change of wealth does not have a strong effect on the investor's consumption rate. In addition, for a fixed value of the risk aversion parameter α, as the loss proportion increases, the economic agent will behave more conservatively by reducing the proportion of wealth spent in consumption. This result is shown in figure 2, where the optimal consumption ratio for $\alpha \in [-2.5, 0)$ and $l = 0.2$, $l = 0.5$ and $l = 0.7$ is represented. In figure 3 a shorter interval for the risk aversion parameter is considered, which allows us to see in a clearer way that the red dashed line (corresponding to regime 1) is upper for every value of α and for every l. Furthermore all the ratios converge to 0.15 for $\alpha \to 0$, which is exactly the optimal consumption to wealth ratio when $\alpha = 0$ and $U(x, i) = \ln(x)$, as reported in fig.3. Note that all these graphs have been obtained by interpolation of the values of α given in table 1.

α	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\hat{c}(x,1)/x$	$\hat{c}(x,2)/x$	$\hat{c}(x,3)/x$
-0.5	8.5721	8.6115	8.6307	0.11665	0.11612	0.11586
-0.8	10.0341	10.0893	10.1179	0.09966	0.09911	0.09883
-1	11.1043	11.1691	11.2037	0.09005	0.08953	0.08925
-1.5	14.1036	14.1916	14.2418	0.07090	0.07046	0.07021
-2	17.6072	17.7192	17.7866	0.05679	0.05643	0.05622
-2.5	21.7159	21.8545	21.9413	0.04604	0.04575	0.04557
-3	26.5847	26.7538	26.8633	0.03761	0.03737	0.03722
-3.5	32.4394	32.6443	32.7808	0.03082	0.03063	0.03050
-4	39.6121	39.8602	40.0296	0.02524	0.02508	0.02498

Table 1: Optimal consumption to wealth ratio for an investor with high risk aversion ($\alpha < 0$) and $l = 0.5$

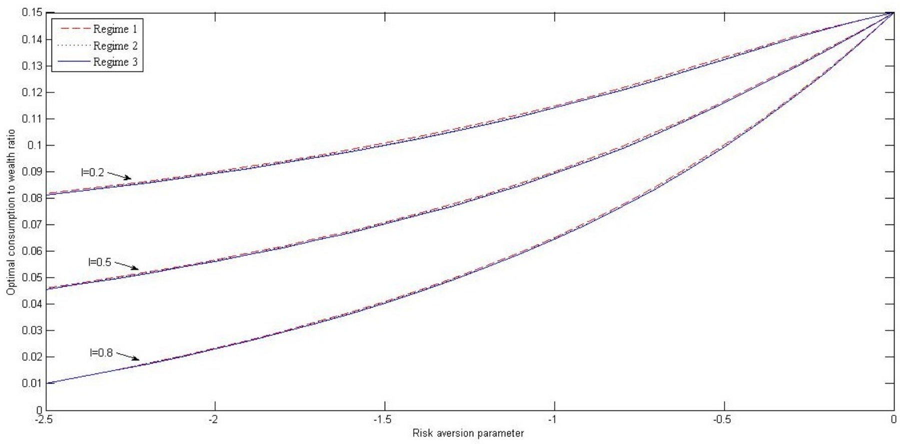

Figure 2: Optimal consumption to wealth ratio when $\alpha < 0$

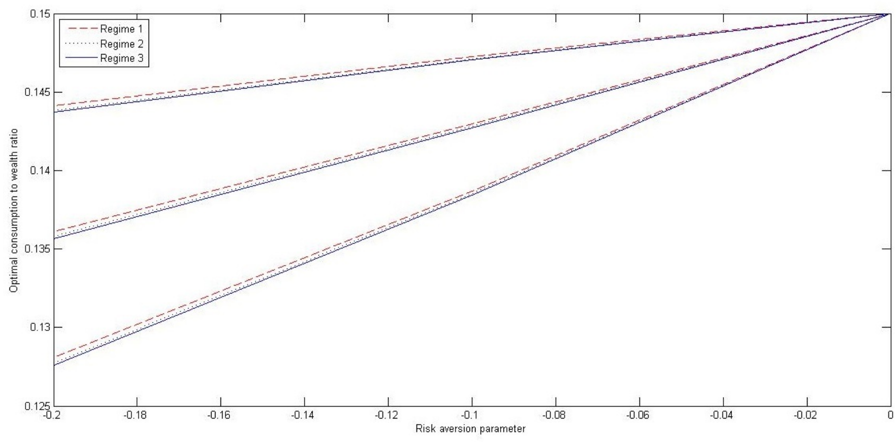

Figure 3: Consumption to wealth ratio for $-0.2 \leq \alpha < 0$

For low risk-averse investors ($0 < \alpha < 1$), the optimal consumption to wealth ratio is given by $\hat{c}(x,i) = 1/\bar{A}_i$, where the constant $\bar{A}_i$, $i = 1, 2, 3$ are computed from system (16). This time we change some market parameters, setting in particular $\delta = 0.2$, $r_1 = 0.15$, $r_2 = 0.11$, $r_3 = 0.08$, $\sigma_1 = 0.4$, $\sigma_2 = 0.6$ and $\sigma_3 = 0.8$, for which condition (19) is satisfied. As we can see in table 2, which exhibits the results of solving numerically system (16) for different values of $\alpha \in (0,1)$, and loss proportion $l = 0.5$, also in this case the optimal consumption to wealth ratio is an increasing function of α. However, the difference is that now $\bar{A}_1 > \bar{A}_2 > \bar{A}_3$, which implies $\hat{c}(x,1)/x < \hat{c}(x,2)/x < \hat{c}(x,3)/x$: in other words, an investor with high risk tolerance will spend a greater proportion of wealth on consumption under bad economic conditions than under strong market conditions. Then, by interpolating the values in this table, we obtain the plot in figure 4, that shows a more evident difference between regimes and a faster growth of the optimal consumption to wealth ratio with respect to the case of a very risk averse investor.

α	$\bar{A}_1$	$\bar{A}_2$	$\bar{A}_3$	$\hat{c}(x,1)/x$	$\hat{c}(x,2)/x$	$\hat{c}(x,3)/x$
0.1	4.6156	4.6115	4.6083	0.216656	0.21685	0.216998
0.2	4.2235	4.2148	4.2083	0.236769	0.237257	0.237623
0.4	3.4195	3.3995	3.3852	0.29244	0.294164	0.2954
0.5	3.0121	2.9843	2.9653	0.331991	0.335087	0.337232
0.6	2.6102	2.5714	2.5463	0.383118	0.388898	0.392727
0.8	2.0042	1.8980	1.8424	0.49894	0.526859	0.542785

Table 2: Optimal consumption to wealth ratio for an investor with low risk aversion ($0 < \alpha < 1$) and $l = 0.5$

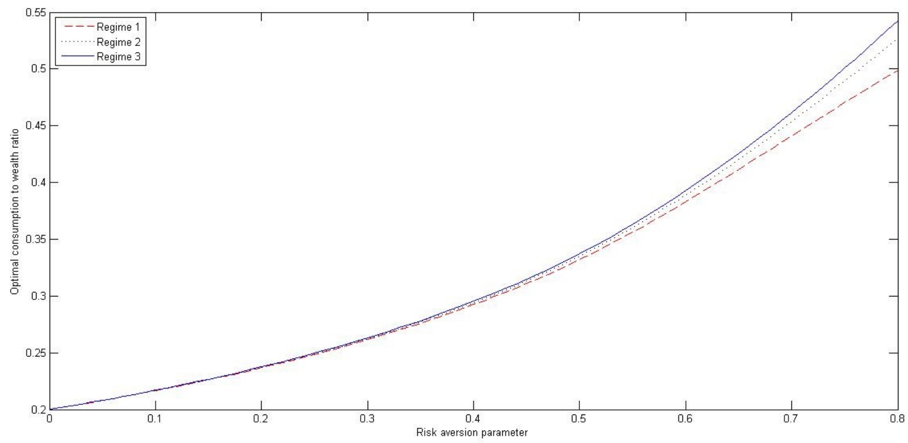

Figure 4: Optimal consumption to wealth ratio when $0 < \alpha \leq 0.8$

The optimal insurance policy is proportional to the investor's wealth $\hat{X}_t$ for every $t \geq 0$, and is given by

$$\hat{I}_t = \left[\eta_{S_t} l_t - 1 + (1+\theta_{S_t})^{-\frac{1}{1-\alpha}}\right]^+ \hat{X}_t, \tag{24}$$

where we assume $\alpha = 0$ for the log-utility function. Note that it is optimal to buy insurance if and only if

$$\eta_{S_t} l_t > 1 - (1+\theta_{S_t})^{-\frac{1}{1-\alpha}}, \tag{25}$$

or equivalently if and only if the insurable loss ($\eta_{S_t} l$) is large, the cost of insurance (θ_{S_t}), and the investor is very risk averse (α small). When condition (25) is satisfied, then $\hat{I}_t = \left[\eta_{S_t} l_t - 1 + (1+\theta_{S_t})^{-\frac{1}{1-\alpha}}\right]\hat{X}_t$, from which we note that optimal insurance is directly proportional to η_{S_t} and l, as expected. Furthermore, since

$$\frac{\partial \hat{I}_t}{\partial \theta_{S_t}} = -\left[\frac{1}{1-\alpha}(1+\theta_{S_t})^{-\frac{2-\alpha}{1-\alpha}}\right]\hat{X}_t < 0 \ ; \ \frac{\partial^2 \hat{I}_t}{\partial \theta_{S_t}^2} = \left[\frac{2-\alpha}{(1-\alpha)^2}(1+\theta_{S_t})^{\frac{2\alpha-3}{1-\alpha}}\right]\hat{X}_t > 0,$$

the optimal insurance is a decreasing and convex function of θ. That is, as the premium load increases, the investor will reduce the purchase of insurance, and the amount of reduction in insurance decreases as θ rises. Moreover, the optimal insurance is a decreasing function of the risk aversion parameter α, which implies that investors with high risk tolerance will spend a small amount of money on insurance. Figure 5 shows in particular the optimal insurance to wealth ratio $K_t := \hat{I}_t/\hat{X}_t$ for the first set of market coefficients and for $\alpha \in [-5, 1)$; as we can see, for $\alpha > 0.6$ the optimal strategy is no insurance in every regime.

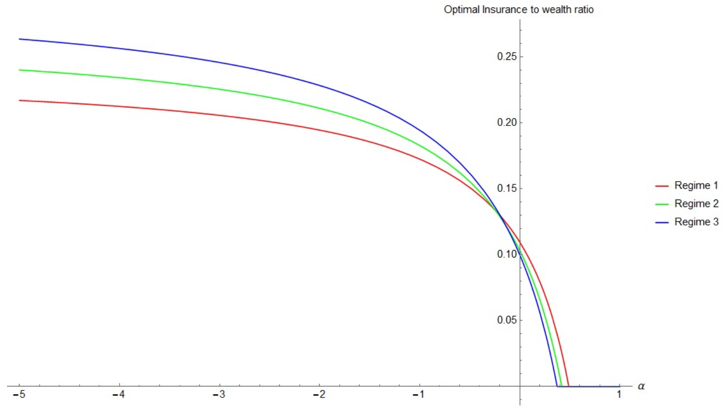

Figure 5: Optimal insurance to wealth ratio when $\alpha \in [-5, 1)$

5.2 Impact of adding a wage

The purpose of this subsection is to examine the impact of adding a non-zero, exogenous wage $\omega = \{\omega_t, t \geq 0\}$ to the wealth process $X = \{X_t, t \geq 0\}$. To achieve this objective, we calculate the value function $v^\omega(x, i)$ for three particular HARA utility functions when the investor is subject to a positive income, and we compare it with the value function of problem 2.1. In the case of an exogenous wage, the dynamics of the wealth process becomes

$$dX_t = (r_{S_t} X_t + (\mu_{S_t} - r_{S_t})\pi_t X_t - c_t + \omega_t - P_t)dt + \sigma_{S_t}\pi_t X_t dW_t - (L_t - I_t(L_t))dN(t), \tag{26}$$

with the standard initial conditions $X_0 = x > 0$ and $S_0 = i \in \mathcal{M}$.

For this reason, the operator $\mathcal{L}_i^u$ defined at the beginning of section 3 is now given by

$$\mathcal{L}_i^u(\psi) := (r_i x + (\mu_i - r_i)\pi x - c + \omega - \lambda_i(1 + \theta_i)\mathbb{E}[I(L)])\frac{\partial \psi}{\partial x} + \frac{1}{2}\sigma_i^2\pi^2 x^2 \frac{\partial^2 \psi}{\partial x^2} - \delta\psi, \tag{27}$$

and the Hamilton-Jacobi-Bellman equation (8) becomes

$$\sup_{\pi \in \mathbb{R}}\big[f(\pi, x, i)\big] + \sup_{c \geq 0}\big[g(c, x, i)\big] + \lambda_i \sup_{I \geq 0}\big[h(I, x, i)\big] = (\delta + \lambda_i)v(x, i) - (r_i x + \omega)v'(x, i) - \sum_{j \in \mathcal{M}} q_{ij} v(x, j), \tag{28}$$

where f, g and h are defined as before.

Remark 5.1. Note that we are considering only the case in which $U(0, i) = -\infty$ or $U(0, i) \equiv 0 \ \forall i = 1, 2, 3$, since all the utility functions considered in previous section are of this form; however, we would have the same results for an utility function with $U(0, i)$ finite and different from zero.

Let us first consider the logarithmic utility function $U(x, i) = \ln(x)$, for $x > 0$. If we substitute in the HJB equation (28) the value function $\hat{v}(x, i) = \frac{1}{\delta}\ln(\delta x) + \hat{A}_i^\omega$, $\quad i = 1, 2, 3$, and the corresponding optimal policies (which are equal to those defined in subsection 4.1), we obtain that, for every $i = 1, 2, 3$, $\hat{A}_i^\omega$ must now satisfy the following equation: $\frac{1}{\delta}(r_i + \frac{\omega}{x} + \gamma_i + \lambda_i\hat{\Lambda}_i - \delta) = \delta\hat{A}_i^\omega - \sum_{j \in \{1,2,3\}} q_{ij}\hat{A}_j^\omega$, where $\gamma_i := \frac{(\mu_i - r_i)^2}{2\sigma_i^2}$ and $\hat{\Lambda}_i$, $i = 1, 2, 3$, is defined in equation (14). Now the values of $\hat{A}_i^\omega$ depend on $x > 0$, and become constant only for a fixed initial wealth. In particular, if we consider the first set of market coefficients with loss proportion $l = 0.3$ and exogenous wage $\omega = 1$, we have $\hat{A}_1^\omega = -1.22951 + \frac{44.4}{x}$, $\hat{A}_2^\omega = -1.34173 + \frac{44.4}{x}$, $\hat{A}_3^\omega = -1.38509 + \frac{44.4}{x}$. This implies that $\hat{A}_1^\omega > \hat{A}_2^\omega > \hat{A}_3^\omega \ \forall x > 0$, and consequently that $\hat{v}^\omega(x, 1) > \hat{v}^\omega(x, 2) > \hat{v}^\omega(x, 3)$. Moreover, since for the same parameter set but with $\omega \equiv 0$, we obtain $\hat{A}_1 = -1.22951$, $\hat{A}_2 = -1.34173$, $\hat{A}_3 = -1.38509$, then we note that for every $x > 0$ and $i = 1, 2, 3$ the value function for the problem with a positive income is greater than the value function of the problem with no wage. Figure 6 shows the two functions for $i = 2$: as we can see, for $x \to \infty$, i.e. as the ratio ω/x goes to zero, the value function $\hat{v}^\omega$, in red color, tends to $\hat{v}$.

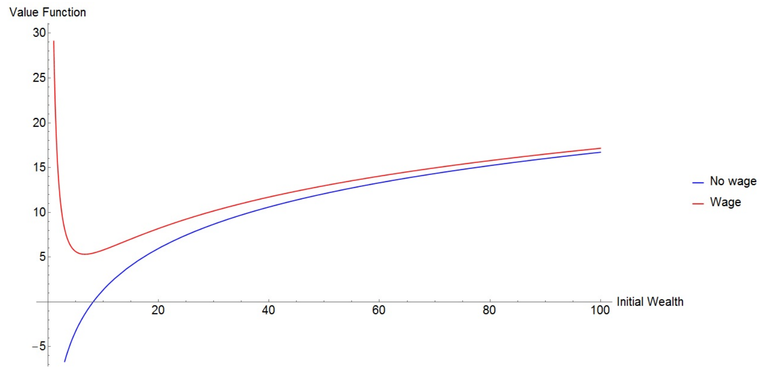

Figure 6: $\hat{v}^{\omega}(x, 2)$ (wage $\omega = 1$) vs $\hat{v}(x, 2)$ (no wage)

If we do not fix a wage, the value of $\hat{A}_i^{\omega}$, $i = 1, 2, 3$ will depend on ω and x, and the value function is now a surface, as shown in the three-dimensional plot of figure 7. Note that the value function is very high for small values of the initial wealth and high values of the wage, and in particular when ω is bigger than x: reasonably, a wage that is three times the investor's initial wealth will affect considerably the value function, since it corresponds to a substantial addition of money and a change in agent's life.

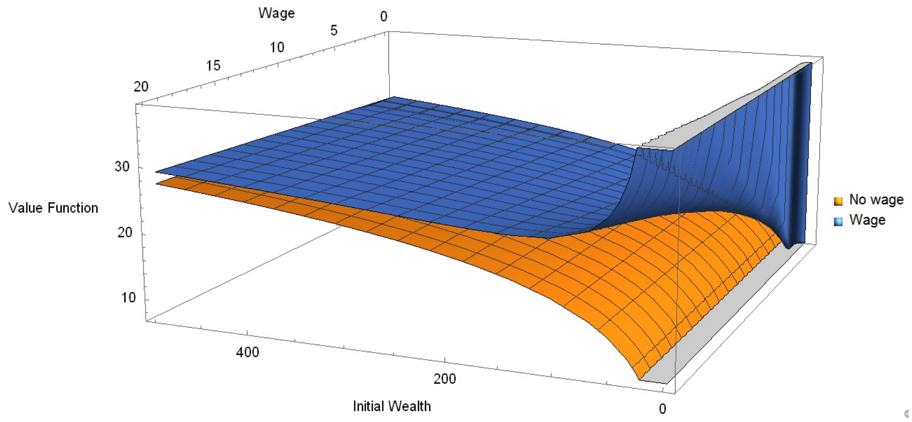

Figure 7: $\hat{v}^{\omega}(x, \omega, 2)$ vs $\hat{v}(x, \omega, 2)$, 3-dimensional plot

Remark 5.2. An investor with a logarithmic utility function does not change his optimal strategies if he is subject to an external input of money. Indeed, optimal investment, consumption and insurance policies do not depend on the values of $\hat{A}_i$, but only on market parameters. This is due to the risk-neutrality of the economic agent: we are going to see next that for an investor with high risk aversion ($U(x, i) = -x^{\alpha}$, $\alpha < 0$) or low risk aversion ($U(x, i) = x^{\alpha}$, $0 < \alpha < 1$) the optimal consumption policy changes with the addition of a wage in the wealth process.

Consider now high risk-averse investors with utility function $U(x, i) = -x^{\alpha}$ and $\alpha < 0$. Following the same procedure, we substitute in the HJB equation (28) the value function $\tilde{v}^{\omega}(x, i) = -(\tilde{A}_i^{\omega})^{1-\alpha} x^{\alpha}$, and we get the following non-linear system for $\tilde{A}_i^{\omega}$, $i = 1, 2, 3$:

$$\left(\delta - \alpha r_i - \alpha \frac{w}{x} - \frac{\alpha}{1-\alpha}\gamma_i + \lambda_i(1 - \tilde{\Lambda}_i)\right)(\tilde{A}_i^{\omega})^{1-\alpha} - (1-\alpha)(\tilde{A}_i^{\omega})^{-\alpha} = \sum_{j \in \{1,2,3\}} q_{ij} (\tilde{A}_j^{\omega})^{1-\alpha}, \tag{29}$$

where for every $i = 1, 2, 3$, $\tilde{\Lambda}_i$ are defined in section 4.2. In order to ensure that system (29) has a unique positive solution for every fixed $x > 0$, we have to slightly modify condition (17) for the parameter δ, which becomes

$$\delta > \max_{i \in \{1,2,3\}} \left\{ \alpha r_i + \alpha \frac{w}{x} + \frac{\alpha}{1-\alpha}\gamma_i - \lambda_i(1 - \tilde{\Lambda}_i) \right\}. \tag{30}$$

In this case we cannot find the values of $\tilde{A}_i^{\omega}$ as function of the initial wealth x, since system (29) does not have a closed form solution. Therefore, we have fixed different values of x and solved numerically the system with the usual market parameters, $\alpha = -1$ and $\omega = 1$, obtaining the results shown in table 3. On the other hand, constants $\tilde{A}_i$ for the problem without income and with the same risk aversion parameter are given by $\tilde{A}_1 = 9.38109$, $\tilde{A}_2 = 9.4304$, $\tilde{A}_3 = 9.45366$, which are bigger than the corresponding values for an investor with positive wage, for all values of x considered in the table. It can be shown that this is true $\forall x > 0$, which implies that the values function for the problem with wage is always greater than the other, for every $i = 1, 2, 3$. Figure 8, obtained by interpolating the data in table 3, compares the value functions $\tilde{v}^\omega(x, 1)$ and $\tilde{v}(x, 1)$, showing again that for $x \to \infty$ the difference between them tends to be zero.

x	$\tilde{A}_1^\omega$	$\tilde{A}_2^\omega$	$\tilde{A}_3^\omega$	$\tilde{v}^\omega(x,1)$	$\tilde{v}^\omega(x,2)$	$\tilde{v}^\omega(x,3)$
0.1	0.195257	0.195989	0.196347	-0.381252	-0.384117	-0.38552
0.6	1.06025	1.06548	1.06796	-1.87356	-1.89207	-1.9009
1.5	2.26615	2.27775	2.28324	-3.42363	-3.45878	-3.47547
4	4.3083	4.33072	4.34131	-4.64036	-4.68878	-4.71174
10	6.3774	6.41078	6.42654	-4.06712	-4.10982	-4.13005
25	7.8939	7.93532	7.95487	-2.49254	-2.51877	-2.5312
75	8.82677	8.87314	8.89502	-1.03883	-1.04977	-1.05495
100	8.95912	9.00619	9.0284	-0.802658	-0.811114	-0.81512

Table 3: Solutions to system (29) for different values of x, investor with $\alpha = -1$ and exogenous wage $\omega = 1$

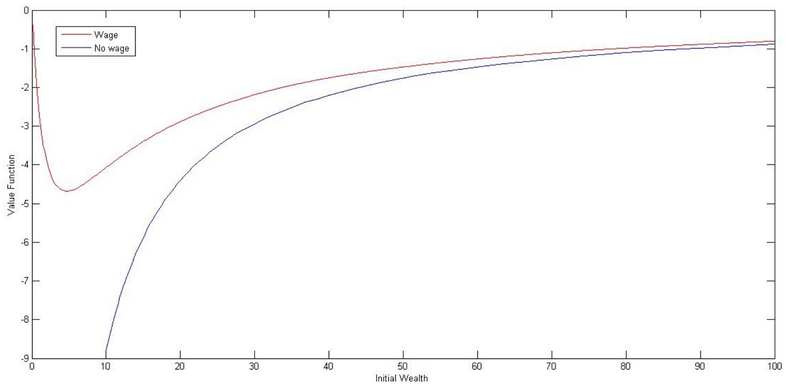

Figure 8: $\tilde{v}^\omega(x, 1)$ vs $\tilde{v}(x, 1)$ for $\alpha = -1$

Since the optimal consumption policy is given by $\hat{c}(x, i) = x/\tilde{A}_i^\omega$, we get that an economic agent will consume proportionally more in periods of good economic conditions and, more important, that the optimal consumption to wealth ratio is higher for an investor with a positive income. For the case of a low risk-averse investor, we consider the utility function described in section 4.4, that is $U(x, i) = \beta_i\sqrt{x}$ with $x > 0$ and $\beta_1 = 1.5$, $\beta_2 = 1$ and $\beta_3 = 0.5$: in particular, this utility function depend on the state of the economy and can be related to an investor with a counter-cyclical risk-aversion, i.e. an investor that is less risk averse during periods of boom or good economic condition than during periods of crisis. With this choice a solution to the HJB equation (28) is given by the value function $\check{v}^\omega(x, i) = (\check{A}_i^\omega x)^{\frac{1}{2}}$, where now, for $i = 1, 2, 3$, $\check{A}_i^\omega$ satisfies the following nonlinear system

$$\left(\delta - \frac{1}{2}r_i - \alpha\frac{w}{x} - \gamma_i + \lambda_i(1 - \check{\Lambda}_i)\right)(\check{A}_i^\omega)^{\frac{1}{2}} - \frac{1}{2}\frac{\beta_i^2}{(\check{A}_i^\omega)^{\frac{1}{2}}} = \sum_{j\in\{1,2,3\}} q_{ij}(\check{A}_j^\omega)^{\frac{1}{2}}. \quad (31)$$

The above system, with $\check{\Lambda}_i$ given in section 4.4, has certainly a positive solution if we change condition (19) for $\alpha = 1/2$ into the following one: $\delta > \max_{i\in\{1,2,3\}}\left\{\frac{1}{2}r_i + \alpha\frac{w}{x} + \gamma_i\right\}$.As in the previous case, we choose $\omega = 1$ and solve numerically system (31) for different values of x, obtaining the results in table 4. Figure 9 compares the value function of the problem with exogenous wage with the one of the original stochastic optimization problem, which is given by $\check{v}(x, i) = (\check{A}_i x)^{\frac{1}{2}}$ and $\check{A}_1 = 4.13656$, $\check{A}_2 = 4.00495$, $\check{A}_3 = 3.9331$.

x	$\check{A}_1^\omega$	$\check{A}_2^\omega$	$\check{A}_3^\omega$	$\check{v}^\omega(x,1)$	$\check{v}^\omega(x,2)$	$\check{v}^\omega(x,3)$
3.4	217.911	216.026	214.887	27.2195	27.1015	27.0299
3.6	55.7464	55.1895	54.8594	14.1664	14.0955	14.0533
4	24.6402	24.3384	24.1636	9.92778	9.86679	9.83129
7	7.8510	7.6877	7.59695	7.41331	7.3358	7.29237
15	5.30338	5.16161	5.08377	8.91912	8.7991	8.7325
50	4.42822	4.29404	4.22069	14.8799	14.6527	14.527
100	4.27735	4.1445	4.07193	20.6818	20.3581	20.179

Table 4: Solutions to system (31) for different values of x and $\omega = 1$

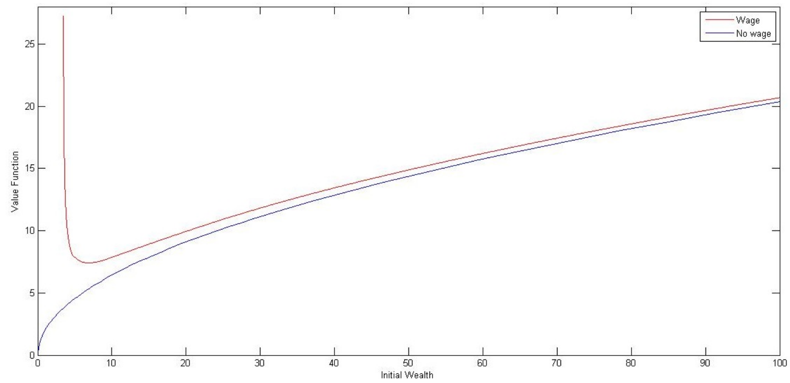

Figure 9: $\check{v}^\omega(x,1)$ vs $\check{v}(x,1)$

Even if only the value functions related to regime 1 are plotted, as in previous cases we have $\check{v}^\omega(x,i) > \check{v}(x,i)$ for every $i = 1,2,3$. Moreover, we can observe from the table that also in this case the value function of an investor subjected to a positive profit is always bigger during periods of economic growth, i.e. $\check{v}^\omega(x,1) > \check{v}^\omega(x,2) > \check{v}^\omega(x,3)$. On the other hand, let us stress the fact that, since the optimal consumption to wealth ratio is given by $\hat{c}(x,i) = 1/\check{A}_i^\omega$ and for every $x > 0$ fixed we have $\check{A}_i^\omega > \check{A}_i$, $i = 1,2,3$, a low risk-averse investor who earns a wage will consume a smaller proportion of his wealth than a similar investor without salary. A possible explanation for this counterintuitive situation can be found in the choice of saving money that can be invested, for example, in the risky asset: his propensity to the risk induces the economic agent not to spend the additional input in consumption, but to increase his wealth in order to have a bigger amount invested in the stock at the next step.

Conclusions

Following the approach developed by Zou and Cadenillas in [19], the present paper exhibits a novel numerical analysis concerning the impact of regimes, market coefficients and investor's risk aversion on optimal insurance policies. Such results extends those in [19] since we allow for a third, intermediate regime characterizing the state of the considered economy. In the same framework, we also provide new results concerning the impact of adding a non-zero, exogenous wage in investor's wealth equation and we outline a comparison with respect to the standard problem.

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