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On Minimal λ_c-Open Sets

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ABSTRACT— ***In this paper, we introduce and discuss minimal λ_c-open sets in topological spaces. We establish some basic properties of minimal λ_c-open. We obtain an application of a theory of minimal λ_c-open sets and we defined a λ_c-locally finite space.***

Keywords— λ-open sets, λ_c-open sets, minimal λ_c-open, s-regular operation.

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1. INTRODUCTION

The study of semi open sets in topological spaces was initiated by Levine **[7]**. The concept of operation γ was initiated by Kasahara [**3**]. He also introduced γ -closed graph of a function. Using this operation, Ogata [**9**] introduced the concept of γ -open sets and investigated the related topological properties of the associated topology τ_γ and τ. He further investigated general operator approaches of closed graph of mappings. Further Ahmad and Hussain [**1**] continued studying the properties of γ –open (γ-closed) sets. In 2009, Hussain and Ahmad [**2**], introduced the concept of minimal γ-open sets. In 2011[**4**] (resp., in 2013[**6**]) Khalaf and Namiq defined an operation λ called s-operation. They defined λ*-open sets [**8**] which is equivalent to λ-open set [**4**] and λ_s- open set [**6**] by using s-operation. They defined λ_c- open set [**6**] by using s-operation and closed set and also investigated several properties of λ_c-derived, λ_c-interior and λ_c-closure points in topological spaces.

In this paper, we introduce and discuss minimal λ_c-open sets in topological spaces. We establish some basic properties of minimal λ_c-open sets and provide an example to illustrate that minimal λ_c-open sets are independent of minimal open sets.

First, we recall some definitions and results used in this paper.

2. PRELIMINARIES

Throughout, X denotes a topological space. Let A be a subset of X, then the closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset A of a topological space (X,τ) is said to be semi open [7] if $A \subseteq Cl(Int(A))$. The complement of a semi open set is said to be semi closed [7]. The family of all semi open (resp., semi closed) sets in a topological space (X,τ) is denoted by $SO(X,\tau)$or $SO(X)$ (resp. $SC(X,\tau)$ or$SC(X)$). We consider λ as a function defined on $SO(X)$into $P(X)$ and $\lambda: SO(X) \longrightarrow P(X)$ is called an s-operation if $V \subseteq \lambda(V)$ for each non-empty semi open set V. It is assumed that $\lambda(\phi) = \phi$and $\lambda(X) = X$for any s-operation λ. Let X be a topological space and $\lambda: SO(X) \longrightarrow P(X)$ be an s-operation, then a subset A of X is called a λ^*-open set **[8]** which is equivalent to λ –open set [**4**] and λ_s-open set [**6**] if for each $x \in A$ there exists a semi open set U such that $x \in U$and $\lambda(U) \subseteq A$. The complement of a λ^*-open set is said to be λ^*-closed. The family of all λ^*-open (resp., λ^*-closed) subsets of a topological space (X,τ) is denoted by $SO_\lambda(X,\tau)$ or $SO_\lambda(X)$ (resp., $SC_\lambda(X,\tau)$ or $SC_\lambda(X)$).

Definition 2.1. A λ^*-open **[8]** (λ –open **[4]**, λ_s-open **[6]**) subset A of a topological space X is called λ_c-open **[4]** if for each $x \in A$ there exists a closed set F such that $x \in F \subseteq A$. The complement of a λ_c-open set is called λ_c-closed **[4]**. The family of all λ_c-open (resp., λ_c-closed) subsets of a topological space (X,τ) is denoted by $SO_{\lambda c}(X,\tau)$ or $SO_{\lambda c}(X)$(resp. $SC_{\lambda c}(X,\tau)$or $SC_{\lambda c}(X)$) **[4]**.

The following definitions and results are in **[4]**.

Proposition 2.2. For a topological space X, $SO_{\lambda c}(X) \subseteq SO_{\lambda}(X) \subseteq SO(X)$.

The following example shows that the converse of the above proposition may not be true in general.

Example 2.3. Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, X\}$. We define an s-operation $\lambda: SO(X) \longrightarrow P(X)$ as $\lambda(A) = A$ if $b \in A$ and $\lambda(A) = X$ otherwise. Here, we have $\{a, c\}$ is semi open but it is not λ^*-open. And also $\{a, b\}$ is λ^*-open set but it is not λ_c- open .

Definition 2.4. An s-operation λ on X is said to be s-regular which is equivalent to λ -regular [**6**] if for every semi open sets U and V containing $x \in X$, there exists a semi open set Wcontaining x such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V)$.

Definition 2.5. Let A be a subset of X. Then:

(1) The λ_c-closure of A ($\lambda_c Cl(A)$) is the intersection of all λ_c-closed sets containing A.

(2) The λ_c-interior of A ($\lambda_c Int(A)$) is the union of all λ_c-open sets of X contained in A.

Proposition 2.6. For each point $x \in X, x \in \lambda_c Cl(A)$ if and only if $V \cap A \neq \phi$ for every $V \in SO_{\lambda_c}(X)$ such that $x \in V$.

Proposition 2.7. Let $\{A_\alpha\}_{\alpha \in I}$be any collection of λ_c-open sets in a topological space (X, τ), then $\bigcup_{\alpha \in I} A_\alpha$is a λ_c-open set.

Proposition 2.8. Let λ be an s-regular s-operation. If A and B are λ_c-open sets in X, then $A \cap B$ is also a λ_c-open set.

The proof of the following two propositions are in **[5].**

Proposition 2.9. Let $\{A_\alpha\}_{\alpha \in I}$be any collection of λ^*-open sets in a topological space (X, τ), then $\bigcup_{\alpha \in I} A_\alpha$ is a λ^*-open set.

Proposition 2.10. Let λ be s-regular operation. If A and B are λ^*-open sets in X, then $A \cap B$is also λ^*-open .

3. MINIMAL λ_c-OPEN SETS

Definition 3.1. Let X be a space and $A \subseteq X$ be a λ_c-open set. Then A is called a minimal λ_c-open set if ϕ and A are the only λ_c-open subsets of A.

Example 3.2. Let $X = \{a, b, c\}$, and $\tau = P(X)$. We define an s-operation $\lambda: SO(X) \longrightarrow P(X)$ as $\lambda(A) = A$ if $A = \{a, c\}$ and $\lambda(A) = X$ otherwise. The λ_c-open sets are $\phi, \{a, c\}$ and X. We have $\{a, c\}$ is minimal λ_c-open set.

Proposition 3.3. Let A be a nonempty λ_c-open subset of a space X. If $A \subseteq \lambda_c Cl(C)$, then $\lambda_c Cl(A) = \lambda_c Cl(C)$, for any nonempty subset C of A.
Proof. For any nonempty subset C of A, we have $\lambda_c Cl(C) \subseteq \lambda_c Cl(A)$. On the other hand, by hypothesis we have $\lambda_c Cl(A) = \lambda_c Cl(\lambda_c Cl(C))$=$\lambda_c Cl(C)$ implies $\lambda_c Cl(A) \subseteq \lambda_c Cl(C)$.
Therefore, $\lambda_c Cl(A) = \lambda_c Cl(C)$ for any nonempty subset $C$of A.

Proposition 3.4. Let A be a nonempty λ_c-open subset of a space X. If $\lambda_c Cl(A) = \lambda_c Cl(C)$, for any nonempty subset C of A, then A is a minimal λ_c-open set.
Proof. Suppose that A is not a minimal λ_c-open set. Then there exists a nonempty λ_c-open set B such that $B \subseteq A$ and hence there exists an element $x \in A$ such that $x \notin B$.Then we have $\lambda_c Cl(\{x\}) \subseteq X \setminus B$ implies that $\lambda_c Cl(\{x\}) = \lambda_c Cl(A)$. This contradiction proves the proposition

Remark 3.5. In the remainder of this section we suppose that λ is an s–regular operation defined on a topological space X .
Proposition 3.6. The following statements are true:
(1) If A is a minimal λ_c-open set and B a λ_c-open set. Then $A \cap B = \phi$ or $A \subseteq B$.
(2) If B and C are minimal λ_c-open sets. Then $B \cap C = \phi$ or $B = C$.

Proof.(1) Let B be a λ_c-open set such that $A \cap B \neq \phi$. Since A is a minimal λ_c-open set and $A \cap B \subseteq A$, we have $A \cap B = A$. Therefore, $A \subseteq B$.
(2) If $A \cap B \neq \phi$, then by (1), we have $B \subseteq C$ and $C \subseteq B$. Therefore, $B = C$.

Proposition 3.7. Let A be a minimal λ_c-open set. If x is an element of A, then $A \subseteq B$ for any λ_c-open neighborhood B of x.
Proof. Let B be a λ_c-open neighborhood of x such that $A \not\subset B$. Since where λ is s–regular operation, then $A \cap B$ is λ_c-open set such that $A \cap B \subseteq A$ and $A \cap B \neq \phi$. This contradicts our assumption that A is a minimal λ_c-open set.

Proposition 3.8. Let A be a minimal λ_c-open set. Then for any element x of A, $A = \cap\{B: B$ is λ_c-open neighborhood of $x\}$.
Proof. By Proposition 3.4, and the fact that A is λ_c-open neighborhood of x, we have $A \subseteq \cap\{B: B$ is λ_c-open neighborhood of $x\} \subseteq A$. Therefore, the result follows.

Proposition 3.9. If A is a minimal λ_c-open set in X not containing the point x. Then for any λ_c-open neighborhood C of x, either $C \cap A = \phi$ or $A \subseteq C$.
Proof. Since C is a λ_c-open set, we have the result by Proposition 3.3.

Corollary 3.10. If A is a minimal λ_c-open set in X not containing $x \in X$ such that $x \notin A$. If $A_x = \cap\{B: B$ is λ_c-open neighborhood of $x\}$. Then either $A_x \cap A = \phi$ or $A \subseteq A_x$.
Proof. If $A \subseteq B$ for any λ_c-open neighborhood B of x, then $A \subseteq \cap\{B: B$ is λ_c-open neighborhood of $x\}$. Therefore, $A \subseteq A_x$. Otherwise, there exists a λ_c-open neighborhood B of x such that $B \cap A = \phi$. Then we have $A_x \cap A = \phi$.

Corollary 3.11. If A is a nonempty minimal λ_c-open set of X, then for a nonempty subset C of A, we have $A \subseteq \lambda_c Cl(C)$.
Proof. Let C be any nonempty subset of A. Let $y \in A$ and B be any λ_c-open neighborhood of y. By Proposition 3.4, we have $A \subseteq B$ and $C = A \cap C \subseteq B \cap C$. Thus, $B \cap C \neq \phi$ and hence $y \in \lambda_c Cl(C)$. This implies that $A \cap \lambda_c Cl(C)$. Hence the proof.

Combining Corollary 3.11 and Propositions 3.3 and 3.4, we have:
Theorem 3.11. Let A be a nonempty λ_c-open subset of space X. Then the following are equivalent:
(1) A is minimal λ_c-open set, where λ is s-regular.
(2) For any nonempty subset C of A, $A \subseteq \lambda_c Cl(C)$.
(3) For any nonempty subset C of A, $\lambda_c Cl(A) = \lambda_c Cl(C)$.

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4. FINITE λ_c-OPEN SETS

In this section, we study some properties of minimal λ_c-open sets in finite λ_c-open sets and λ_c-locally finite spaces.

Proposition 4.1. Let $B \neq \phi$ be a finite λ_c-open set in a topological space X. Then, there exists at least one (finite) minimal λ_c-open set A such that $A \subseteq B$.
Proof. Suppose that B is a finite λ_c-open set in X. Then, we have the following two possibilities:
(1) B is a minimal λ_c-open set.
(2) B is not a minimal λ_c-open set.
In case (1), if we choose $B = A$, then the proposition is proved. If the case (2) is true, then there exists a nonempty (finite) λ_c-open set B_1 which is properly contained in B. If B_1 is minimal λ_c-open, we take $A = B_1$. If B_1 is not a minimal λ_c-open set, then there exists a nonempty (finite) λ_c-open set B_2 such that $B_2 \subseteq B_1 \subseteq B$. We continue this process and have a sequence of λ_c-open $\ldots \subseteq B_m \subseteq \cdots \subseteq B_2 \subseteq B_1 \subseteq B$. Since B is finite, this process will end in a finite number of steps. That is, for some natural number k, we have a minimal λ_c-open set B_k such that $B_k = A$. This completes the proof.
Definition 4.2. A space X is said to be a λ_c-locally finite space, if for each $x \in X$ there exists a finite λ_c-open set A in X such that $x \in A$.
Corollary 4.3. Let X be a λ_c-locally finite space and B a nonempty λ_c-open set. Then there exists at least one (finite) minimal λ_c-open set A such that $A \subseteq B$, where λ is s-regular.
Proof. Since B is a nonempty set, there exists an element x of B. Since X is a λ_c-locally finite space, we have a finite λ_c-open set B_x such that $x \in B_x$. Since $B \cap B_x$ is a finite λ_c-open set, so by Proposition 4.1, we get a minimal λ_c-open set A such that $A \subseteq B \cap B_x \subseteq B$.

Proposition 4.4. Let X be a space and for any $\alpha \in I$, B_α a λ_c-open set and $\phi \neq A$ a finite λ_c-open set. Then $A \cap (\cap_{\alpha \in I} B_\alpha)$ is a finite λ_c-open set, where λ is s-regular.

Proof. We see that there exists an integer n such that $A \cap (\bigcap_{\alpha\in I} B_\alpha) = A \cap (\bigcap_{i=1}^{n} B_{\alpha i})$ and hence we have the result.

Using Proposition 4.4, we can prove the following:

Theorem 4.5. Let X be a space and for any $\alpha \in I$, B_α is a λ_c-open set and for any $\beta \in J$, B_β is a nonempty finite λ_c-open set. Then, $(\bigcup_{\beta\in J} B_\beta) \cap (\bigcap_{\alpha\in I} B_\alpha)$ is a λ_c open set, where λ is s-regular.

5. MORE PROPERTIES

Let A be a nonempty finite λ_c-open set. It is clear, by Proposition 3.3 and Proposition 4.1, that if λ is s-regular, then there exists a natural number m such that $\{A_1, A_2, \dots, A_m\}$ is the class of all minimal λ_c-open sets in A satisfying the following two conditions:

(1) For any ι, n with $1 \le \iota,\ n \le m$ and $\iota \ne n$, $A_\iota \cap A_n = \phi$.

(2) If C is a minimal λ_c-open set in A, then there exists ι with $1 \le \iota \le m$ such that $C = A_\iota$.

Theorem 5.1. Let X be a space and $\phi \ne A$ a finite λ_c-open set such that A is not a minimal λ_c-open set. Let $\{A_1, A_2, \dots, A_m\}$be a class of all minimal λ_c-open sets in A and $y \in A \setminus (A_1 \cup A_2 \cup \dots \cup A_m)$. Define $A_y = \bigcap\{B : B$ is λ_c-open neighborhood of $x\}$. Then there exists a natural number $k \in \{1,2,3,\dots,m\}$ such that A_k is contained in A_y, where λ is s-regular.

Proof. Suppose on the contrary, that for any natural number $k \in \{1,2,3,\dots,m\}$, $A_k$is not contained in A_y. By Corollary 3.7, for any minimal λ_c-open set A_k in A, $A_k \cap A_y = \phi$. By Proposition 4.4, $\phi \ne A_y$ is a finite λ_c-open set. Therefore, by Proposition 4.1, there exists a minimal λ_c-open set C such that $C \subseteq A_y$. Since $C \subseteq A_y \subseteq A$, we have C is a minimal λ_c-open set in A. By supposition, for any minimal λ_c-open set A_k, we have $A_k \cap C \subseteq A_k \cap A_y = \phi$. Therefore, for any natural number$k \in \{1,2,3,\dots,m\}$, $C \ne A_k$. This contradicts our assumption. Hence the proof.

Proposition 5.2. Let X be a space and $\phi \ne A$ be a finite λ_c-open set which is not a minimal λ_c-open set. Let $\{A_1, A_2, \dots, A_m\}$ be a class of all minimal λ_c-open sets in A and $y \in A \setminus (A_1 \cup A_2 \cup \dots \cup A_m)$. Then there exists a natural number $k \in \{1,2,3,\dots,m\}$ such that for any λ_c-open neighborhood B_y of y, A_k is contained in B_y, where λ is s-regular.

Proof. This follows from Theorem 5.1, as $\bigcap\{B : B$ is λ_c-open of $y\} \subseteq B_y$.

Theorem 5.3. Let X be a space and $\phi \ne A$ be a finite λ_c-open set which is not a minimal λ_c-open set. Let $\{A_1, A_2, \dots, A_m\}$ be the class of all minimal λ_c-open sets in A and $y \in A \setminus (A_1 \cup A_2 \cup \dots \cup A_m)$. Then there exists a natural number $k \in \{1,2,3,\dots,m\}$, such that $y \in \lambda_c Cl(A_k)$, where λ is s-regular.

Proof. Follows from Proposition 5.2, that there exists a natural number $k \in \{1,2,3,\dots,m\}$ such that $A_k \subseteq B$for any λ_c-open neighborhood$B$ of$y$. Therefore, $\phi \ne A_k \cap A_k \subseteq A_k \cap B$ implies $y \in \lambda_c Cl(A_k)$. This completes the proof.

Proposition 5.4. Let $\phi \ne A$ be a finite λ_c-open set in a space X and for each $k \in \{1,2,3,\dots,m\}$, $A_k$is a minimal λ_c-open sets in A. If the class $\{A_1, A_2, \dots, A_m\}$ contains all minimal λ_c-open sets in A, then for any $\phi \ne B_k \subseteq A_k$, $A \subseteq \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$, where λ is s-regular.

Proof. If A is a minimal λ_c-open set, then this is the result of Theorem 3.11 (2). Otherwise, when A is not a minimal λ_c-open set. If x is any element of $A \setminus (A_1 \cup A_2 \cup \dots \cup A_m)$, then by Theorem 5.3, $x \in \lambda_c Cl(A_1) \cup \lambda_c Cl(A_2) \cup \dots \cup \lambda_c Cl(A_m)$. Therefore, by Theorem 3.11 (3), we obtain that $A \subseteq \lambda_c Cl(A_1) \cup \lambda_c Cl(A_2) \cup \dots \cup \lambda_c Cl(A_m) = \lambda_c Cl(B_1) \cup \lambda_c Cl(B_2) \cup \dots \cup \lambda_c Cl(B_m) = \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$.

Proposition 5.5. Let$\phi \ne A$ be a finite λ_c-open set and A_k is a minimal λ_c-open set in A, for each $k \in \{1,2,3,\dots,m\}$. If for any $\phi \ne B_k \subseteq A_k$, $A \subseteq \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$ then$\lambda_c Cl(A) = \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$.

Proof. For any$\phi \ne B_k \subseteq A_k$with$k \in \{1,2,3,\dots,m\}$, we have$\lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m) \subseteq \lambda_c Cl(A)$. Also, we have $\lambda_c Cl(A) \subseteq \lambda_c Cl(B_1) \cup \lambda_c Cl(B_2) \cup \dots \cup \lambda_c Cl(B_m) = \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$.

Therefore,$\lambda_c Cl(A) = \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$ for any nonempty subset $B_k$of$A_k$with $k \in \{1,2,3,\dots,m\}$.

Proposition 5.6. Let $\phi \ne A$ be a finite λ_c-open set and for each $k \in \{1,2,3,\dots,m\}$, $A_k$is a minimal λ_c-open set in A. If for any $\phi \ne B_k \subseteq A_k$, $\lambda_c Cl(A) = \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$, then the class $\{A_1, A_2, \dots, A_m\}$ contains all minimal λ_c-open sets in A.

Proof. Suppose that C is a minimal λ_c-open set in A and $C \ne A_k$ for $k \in \{1,2,3,\dots,m\}$. Then, we have $C \cap \lambda_c Cl(A_k) = \phi$for each$k \in \{1,2,3,\dots,m\}$. It follows that any element of C is not contained in $\lambda_c Cl(A_1 \cup A_2 \cup \dots \cup A_m)$. This is a contradiction to the fact that $C \subseteq A \subseteq \lambda_c Cl(A) = \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$. This completes the proof.

Combining Propositions 5.4, 5.5 and 5.6, we have the following theorem:

Theorem 5.7. Let A be a nonempty finite λ_c-open set and A_k a minimal λ_c-open set in A for each $k \in \{1,2,3,\dots,m\}$. Then the following three conditions are equivalent:

(1) The class $\{A_1, A_2, \dots, A_m\}$ contains all minimal λ_c-open sets in A.

(2) For any $\phi \neq B_k \subseteq A_k$, $A \subseteq \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$.
(3) For any $\phi \neq B_k \subseteq A_k, \lambda_c Cl(A) = \lambda_c Cl(B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m)$, where λ is s-regular.

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