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New Algorithm of Obtaining Order and Error Constants of Third Order Linear Multi-Step Method (LMM)

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ABSTRACT— ***The method of obtaining order and error constant presented in [2] for first and second order linear multi-step methods were extended to derived a similar method for obtaining the order and error constant of the third order linear multi-step methods. Specifically the method is meant for the LMM schemes on grid and off grid points to determine their Order and Error constants easily.***

Keywords— LMM, Order, Error constants, Third Order odes, Grid and off grid points

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1. INTRODUCTION

In the past, efforts have been made by eminent scholars to derive method of obtaining Order and Error constants especially the first and second linear multi-step methods (see [2]).Though there exist Taylor's series expansion approximation used for obtaining order and error constants which is too cumbersome compared to this proposed approach.

2. METHODOLOGY

Review of method of Order and Error constant of first and Second order LMM

The order and error constant of linear multi-step methods for first and second order differential equations according to [2] is that given a linear multi-step

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (2.1)$$

We associate the linear difference operator

$$L\,[y(x);h] = \sum_{j=0}^{k} [\ \alpha_j y(x+jh) - h\beta_j y'(x+jh)\ ] \qquad (2.2)$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b]. Expanding the test function $y(x+jh)$ and its derivative $y'(x+jh)$ as Taylor series about x and collecting terms in (2.2) gives

$$L\,[y(x);h] = C_0 y(x) + C_1 h y'(x) + \cdots \quad + C_q h^q y^q(x) \qquad (2.3)$$

where C_q are constants.

A simple calculation yields the following formulae for the constants C_q in term of the coefficients α_j, β_j .

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \ \dots\ \alpha_k$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$C_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 + \ldots + k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \ldots + k^{q-1}\beta_k), \quad q = 2,3,\ldots \quad (2.4)$$

Following Definition 1.0, we say that the method has order P if

$$C_0 = C_1 = C_2 = \cdots C_p = 0, \quad C_{p+1} \neq 0$$

Also with a linear multi-step method

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \quad (2.5)$$

We associate the linear differential operator

$$L[y(x); h] = \sum_{j=0}^{k} [\alpha_j y(x + jh) - h^2 \beta_j y''(x + jh)] \quad (2.6)$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding the test function $y(x + jh)$ and it's second derivative $y''(x + jh)$ as Taylor series about x and collecting terms in (2.6) gives

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + \cdots + C_q h^q y^q(x) + \cdots \quad \textbf{(2.7)}$$

where C_q are constants.

A simple calculation yields the following formulae for the constants C_q in term of the coefficients α_j, β_j.

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots \alpha_k$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + k\alpha_k$$

$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 + \ldots + k^2\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$C_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 + \ldots + k^q\alpha_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{q-2}\beta_2 + \ldots + k^{q-2}\beta_k),$$

$$q = 3,4,\ldots \quad (2.8)$$

Following Definition 1.0, we say that the method has order P if

$C_0 = C_1 = C_2 = \cdots C_p = C_{p+1} = 0$, $C_{p+2} \neq 0$ C_{p+2} is then the error constant and $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$ the principal local truncated error at the point x_n.

Definition 1.0

A linear multistep method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j}$$

$k \geq 3$ is said to be of order P if $C_0 = C_1 = C_2 = \cdots C_p = C_{p+1} = C_{p+2} = 0$

but $C_{p+3} \neq 0$ and C_{p+3} is called error constant.

3. PROPOSED PROCEDURE FOR OBTAINING ORDER AND ERROR CONSTANTS FOR THIRD ORDER SCHEMES IN [1]

Since most of life and physical problems can be modeled into differential equations, we need some numerical Algorithms to obtain its approximate solution. Also we need to analyze the order and error constants of the Algorithms to be used.

Following [2] , we define the local truncated error associated with equation (2.1) to be linear difference operator L as

$$L[y(x),h] = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} - h^3 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (3.1)$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand equation (3.1) as a Taylor series about the point x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + \cdots + C_q h^q y^q(x) + \cdots \qquad \textbf{(3.2)}$$

where the constant $q = 4,5, \ldots$ are given as

$$C_0 = \sum_{j=0}^{k} \alpha_j \qquad (3.3)$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j \qquad (3.4)$$

$$C_2 = \sum_{j=1}^{k} j^2\alpha_j \qquad (3.5)$$

$$C_3 = \frac{1}{3!}\left(\sum_{j=1}^{k} j^3\alpha_j\right) - \left(\sum_{j=1}^{k} \beta_j\right) \qquad (3.6)$$

$$C_4 = \frac{1}{4!}\left(\sum_{j=1}^{k} j^4\alpha_j\right) - \left(\sum_{j=1}^{k} j\beta_j\right) \qquad (3.7)$$

$$C_5 = \frac{1}{5!}\left(\sum_{j=1}^{k} j^5\alpha_j\right) - \left(\frac{1}{2!}\sum_{j=1}^{k} j^2\beta_j\right) \qquad (3.8)$$

$$C_q = \frac{1}{q!}\left(\sum_{j=1}^{k} j^q\alpha_j\right) - \left(\frac{1}{(q-3)!}\sum_{j=1}^{k} j^{q-3}\beta_j\right) \qquad (3.9)$$

Collecting the corresponding terms together gives the form

$$C_0 = \propto_0 + \propto_1 + \propto_2 + \propto_3 + \ldots \propto_k$$

$$C_1 = \propto_1 + 2\propto_2 + 3\propto_3 + \ldots + k\propto_k$$

$$C_2 = \frac{1}{2!}(\propto_1 + 2^2\propto_2 + 3^2\propto_3 + \ldots + k^2\propto_k)$$

$$C_3 = \frac{1}{3!}(\propto_1 + 2^3\propto_2 + 3^3\propto_3 + \ldots + k^3\propto_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$C_4 = \frac{1}{4!}(\propto_1 + 2^4\propto_2 + 3^4\propto_3 + \ldots + k^4\propto_k) - (\beta_1 + 2\beta_2 + 3\beta_3 + \cdots + k\beta_k)$$

$$C_5 = \frac{1}{5!}(\propto_1 + 2^5\propto_2 + 3^5\propto_3 + \ldots + k^5\propto_k) - \frac{1}{2!}(\beta_1 + 2^2\beta_2 + 3^2\beta_3 + \cdots + k^2\beta_k)$$

$$C_q = \frac{1}{q!}(\propto_1 + 2^q\propto_2 + 3^q\propto_3 + \ldots + k^q\propto_k) - \frac{1}{(q-3)!}(\beta_1 + 2^{q-3}\beta_2 + 3^{q-3}\beta_3 \quad \ldots + k^{q-3}\beta_k \ , \quad q = 4,5, \ldots \qquad \textbf{(3.11)}$$

Following Definition 2.0, we say that the method has order P if

$C_0 = C_1 = C_2 = \cdots C_p = C_{p+1} = C_{p+2} = 0$, $C_{p+3} \neq 0$ C_{p+3} is then the error constant and $C_{p+3}h^{p+3}y^{(p+3)}(x_n)$ the principal local truncated error at the point x_n .

4. VERIFICATION OF THE PROPOSED METHOD

We shall verify the proposed method with some linear multi-step schemes to ascertain its levels of accuracy.

The LMM of the form

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{2}[f_{n+2} + f_{n+1}] \quad \textbf{(4.1)}$$

The scheme (4.1) is of order 4 with error constants $C_{P+3} = \frac{1}{240}$

From the scheme (4.1) ,

$\propto_0 = -1, \propto_1 = 3, \ \propto_2 = -3, \propto_3 = 1 \ , \ \beta_0 = 0, \beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{2}, \beta_3 = 0$

Then we observed that

$C_0 = 0, \ C_2 = 0, \ C_3 = 0, C_4 = 0, C_5 = 0 \ and \ \ C_6 = 0$

But $C_7 \neq 0$ **and** $C_{P+3} = C_7 = \frac{1}{240}$

Hence the order $= 4$, **Error constant** $= \frac{1}{240}$

The LMM of the form

$$y_{n+4} - \frac{8}{3}y_{n+3} + 2y_{n+2} - \frac{1}{3}y_n = \frac{h^3}{15120}[63f_n + 2289f_{n+1} + 10374f_{n+2} + 7434f_{n+3} - 21f_{n+4} + 21f_{n+5}] \quad \textbf{(4.2)}$$

The scheme (4.2) is of order 6 with error constants $C_{P+3} = -\frac{61}{90720}$ **(see [3])**

From the scheme (4.2),

$\propto_0 = -\frac{1}{3}, \propto_1 = 0, \ \propto_2 = 2, \propto_3 = -\frac{8}{3}, \propto_4 = 1, \ \beta_0 = \frac{63}{15120}, \beta_1 = \frac{2289}{15120}, \beta_2 = \frac{10374}{15120}, \beta_3 = \frac{7434}{15120}, \beta_4 = \frac{-21}{15120}, \beta_5 = \frac{21}{15120}$

Then we observed that

$C_0 = 0, \ C_2 = 0, \ C_3 = 0, C_4 = 0, C_5 = 0, C_6 = 0, C_7 = 0 \ and \ C_8 = 0$

$But \ \ C_9 = -\frac{61}{90720}$

$C_{P+3} = C_9 = -\frac{61}{90720}$

Hence the order $= 6$, **Error constant** $= -\frac{61}{90720}$

The LMM of the form

$$\frac{1}{8}y_{n+2} - \frac{3}{4}y_{n+1} + y_{n+\frac{1}{2}} - \frac{3}{8}y_n = \frac{h^3}{57600}\left[f_{n+3} + 430f_{n+\frac{3}{2}} + 1845f_{n+1} + 1314f_{n+\frac{1}{2}} + 10f_n\right] \quad \textbf{(4.3)}$$

The scheme (4.3) is of order 5 with error constants $C_{P+3} = -\frac{13}{2457600}$ **(see [1])**

From the scheme (4.3)

$\propto_0 = -\frac{3}{8}, \propto_1 = -\frac{3}{4}, \ \propto_2 = \frac{1}{8}, \propto_{\frac{1}{2}} = 1, \ \beta_0 = \frac{10}{57600}, \beta_1 = \frac{1845}{57600}, \beta_2 = 0, \beta_3 = 1, \beta_{\frac{3}{2}} = \frac{40}{57600}, \beta_{\frac{1}{2}} = \frac{1314}{57600}$

Then we observed that

$C_0 = 0, \ C_2 = 0, \ C_3 = 0, C_4 = 0, C_5 = 0, C_6 = 0 \ \ and \ C_7 = 0$

$$But\ C_8 = -\frac{13}{2457600}$$

$$C_{P+3} = C_8 = -\frac{13}{2457600}$$

Hence the order $= 5$, **Error constant** $= -\frac{13}{2457600}$

5. DISCUSSION OF RESULTS

The three Linear multi-step methods demonstrated with this proposed method confirmed the accuracy of the method of obtaining the Order and error constants of the third order LMM schemes. This new approach is less cumbersome and simple in its implementation.

6. REFERENCES

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