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Tensor Formulation of Dirac Equation through Divisors

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ABSTRACT--- ***In previous works, spinor Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell´s like equations through two complex isotropic vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}'=\vec{E}' - i\vec{H}'$. It has been proved, that the fields $\vec{E}$ and $\vec{H}$ have the same properties as those of the electric and magnetic fields $\vec{E}$ and $\vec{H}$, which are components of a single electromagnetic field. In particular, the solution of these non-linear equations for free particle as well fulfils Maxwell´s equations for vacuum (with zero at the right side). This gives us the right to interpret electron field as a system of two electromagnetic fields $(\vec{E}, \vec{H})$ and $(\vec{E}', \vec{H}')$, propagating with the phase velocity equal to the velocity of light and the velocity of the particle is the group velocity. In the present work, we shall write these non-linear Maxwell´s like equations, representing Dirac equation in covariant form through divisors K_μ and m_μ .This form directly reveals their relativistic invariance and can open the possibilities to future generalizations.***

Keywords---- Dirac equation, spinor, tensor, divisors.

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1. INTRODUCTION

It is known that spinors were introduced in physics mainly for the following reasons:

1. To obtain observable spectrum of physical quantities (energy, momentum, angular momentum, spin, coordinate and velocity) for half-spin particles.
2. To obtain a very simple linear first order wave equation for description of half-spin particles.
3. To ensure positive determination of the probability density conserved in time.
4. To elaborate relativistic covariant theory.

However, unlike these advantages of spinor formalism, the use of spinors in the description of fermions fields leads to many problems. For example, the definition of a spinor is related to the choice of a special coordinate system, and therefore, spinors can be defined only in a limited class of space-time manifolds. Spinor "ψ" and spinor "-ψ" describe the same physical state; i.e., there is a problem of single-valuedness "spinor"↔"physical state". In addition, the use of spinors leads to many difficulties in calculation of matrix elements.

However, all these disadvantages of spinor formalism can be overcome by mapping tridimensional (but two components) spinor ξ to isotropic complex vector $\vec{F} = \vec{E} + i\vec{H}$, satisfying non-linear condition $\vec{F}^2 = 0$, equivalent to two conditions for real quantities $\vec{E}^2 - \vec{H}^2 = 0$ and $\vec{E}.\vec{H} = 0$. These last conditions are obtained by equating to zero separately real and imaginary parts of equality $\vec{F}^2 = 0$. This method is known in literature as Cartan map. Isotropic vectors can be defined in any space-time manifold. In addition, unlike spinors, vectors $\vec{E}$ and $\vec{H}$ can be used as observable physical quantities. For example, in electrodynamics they play the role of strengths of electric and magnetic fields.

Via Cartan map, Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell´s like equations for two fields $(\vec{E}, \vec{H})$ and $(\vec{E}', \vec{H}')$. These fields have the same properties as $\vec{E}, \vec{H}$, components of tensor $F_{\mu\nu}$of electromagnetic field. It has been proved, that the solution of these non-linear equations for free particle as well fulfils Maxwell´s equations for vacuum (with zero at the right side).

In this work, in development of the above ideas, we shall write these non-linear equations, representing Dirac equation in covariant form through divisors.

2. RESEARCH METHOD

In this work, we shall investigate Dirac equation in tensor formalism. Using Cartan map, Dirac equation for half spin particle has been written in tensor form, in the form of non-linear Maxwell's like equations for two "electromagnetic fields" $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$. Using the method of tensor analysis, we shall introduce two second rank tensors $F_{\mu\nu}$ and $F'_{\mu\nu}$ and we shall write these non-linear tensor equations in covariant form through divisors.

Dirac Equation through Divisors

In previous works, Dirac equation for half-spin particle with rest mass m

$$(\gamma_\mu\partial_\mu\text{-}m)\ \psi=0, \qquad (1)$$

has been written in tensor form, through isotropic complex vectors $\vec{F}=\vec{E}+i\vec{H}$ and

$\vec{F}'=\vec{E}'-i\vec{H}'$ as follows

$$\begin{cases} D_0\vec{F}+v_i(\vec{D}\,F_i)-i\vec{D}\times\vec{F}=-\frac{m}{\sqrt{2}}\frac{\vec{F}\times\vec{F}'}{(\vec{F}\vec{F}')^{1/2}} \\ D_0\vec{F}'-v'_i(\vec{D}F_i')+i\vec{D}\times\vec{F}'=-\frac{m}{\sqrt{2}}\frac{\vec{F}\times\vec{F}'}{(\vec{F}\vec{F}')^{1/2}} \end{cases}. \qquad (2)$$

Where

$$D_0=\frac{i}{2}\frac{\partial}{\partial t},$$

$$\vec{D}=-\frac{i}{2}\vec{\nabla}, \qquad (3)$$

$$\vec{v}=\frac{\vec{E}\times\vec{H}}{\vec{E}^2}.$$

Here we use the system of units in which c=ħ=1.

Separating real and imaginary parts in equations (2), we obtain a system of non-linear Maxwell´s like equations for two electromagnetic fields $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$

$$\begin{cases} \mathrm{rot}\,\vec{E}+\frac{\partial\vec{H}}{\partial t}=v_i(\vec{\nabla}H_i)+\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\,\vec{H}')^{1/2}} \\ \mathrm{rot}\,\vec{H}-\frac{\partial\vec{E}}{\partial t}=-v_i(\vec{\nabla}E_i)+\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\,\vec{H}')^{1/2}} \\ \mathrm{rot}\,\vec{E}'+\frac{\partial\vec{H}'}{\partial t}=-v'_i(\vec{\nabla}H'_i)-\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\,\vec{H}')^{1/2}} \\ \mathrm{rot}\,\vec{H}'-\frac{\partial\vec{E}'}{\partial t}=v'_i(\vec{\nabla}E'_i)+\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\,\vec{H}')^{1/2}} \end{cases}. \qquad (4)$$

In this work we shall write equations (4) through divisors

$$K_\mu=\left(|\vec{E}|,\frac{\vec{E}\times\vec{H}}{|\vec{E}|}\right), \qquad m_\mu=\left(0,\frac{\vec{F}}{|\vec{E}|}\right),$$

$$K'_\mu = \left(|\vec{E}'|, \frac{\vec{E}'\times\vec{H}'}{|\vec{E}'|}\right), \qquad m'_\mu = \left(0, \frac{\vec{F}'}{|\vec{E}'|}\right). \tag{5}$$

Let us introduce bivector fields $F_{\mu\nu}$ and $F'_{\mu\nu}$ as follows

$$F_{\mu\nu} = K_\mu \wedge m_\nu, \qquad F'_{\mu\nu} = K'_\mu \wedge m'_\nu. \tag{6}$$

Here the sign "∧" means antisymmetrisation over indices μ and ν.

Then equations (4) written through divisors take the form

$$\begin{cases} D_\nu F^{\mu\nu} = K^\nu \partial^\mu m_\nu - \frac{m}{\sqrt{2}}(K_\nu + K'_\nu)\tilde{G}^{\mu\nu} \\ D_\nu F'^{\mu\nu} = -K'^\nu \partial^\mu m'_\nu + \frac{m}{\sqrt{2}}(K_\nu + K'_\nu)\tilde{G}^{\mu\nu} \end{cases} \tag{7}$$

Here

$$\tilde{G}^{\mu\nu} = \frac{1}{2}\,\mathcal{E}^{\mu\nu\sigma\tau}G_{\sigma\tau}. \tag{8}$$

Where

$$G_{\mu\nu} = m_\mu \wedge m'_\nu. \tag{9}$$

3. DISCUSSION AND CONCLUSION

In this work, we studied non-linear Maxwell's like equations, representing spinor Dirac equation for half-spin particle. These equations have been obtained by using different methods. In particular, they have been obtained by using Cartan map. In our work, we wrote these non-linear equations in covariant form through divisors in the form given by equations (7). These equations show that spinor Dirac equation in tensor form differs from Maxwell's equations for electromagnetic field only by the presence of nonlinear right side, which can be regarded as an additional current. Once again, this result shows the interconnection between Dirac theory for electron and Maxwell's theory for electromagnetic field.

4. REFERENCES

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