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Application of homotopy analysis method for solving nonlinear fractional partial differential equations

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Abstract

In this article, the homotopy analysis method is applied to solve nonlinear fractional partial differential equations. Based on the homotopy analysis method, a scheme is developed to obtain the approximate solution of the nonlinear fractional heat conduction, Kaup–Kupershmidt, Fisher and Huxley equations with initial conditions, introduced by replacing some integer-order time derivatives by fractional derivatives. The solutions of the studied models are calculated in the form of convergent series with easily computable components. The results of applying this procedure to the studied cases show the high accuracy and efficiency of the new technique. The fractional derivative is described in the Caputo sense. Some illustrative examples are presented to observe some computational results.

Keywords: Analytical solution; Nonlinear fractional heat conduction, Kaup-Kupershmidt, Fisher, Huxley; Fractional partial differential equations (FPDEs), Homotopy analysis method (HAM)

1 Introduction

The concept of the differentiation operator $\mathrm{D} = \mathrm{d}/\mathrm{dx}$ is familiar with all who have studied the elementary calculus. And for suitable functions f, the nth derivative of f, namely $\mathrm{D^n f(x)} = \mathrm{d^n f(x)/dx^n}$ is well defined– provided that n is a positive integer. In 1695 L'Hôpital inquired Leibniz what meaning could be ascribed to $\mathrm{D^n f(x)}$ if n were a fraction. But it was not until 1884 that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematician. By then the theory had been extended to include operators D^{ν}, where ν could be rational or irrational, positive or negative, real or complex [1]. Leibniz, Euler, Laplace, Lacroix and Fourier made mention of derivatives of arbitrary order, but the first use of fractional operations was made by Niels Henrik Abel in 1823. Abel applied the fractional calculus in the solution of the tautochrone problem [1]. Perhaps the first serious attempt to give a logical definition of a fractional derivative is due to Liouville; he published nine papers on the subject between 1832 and 1837, the last in the field in 1855. In recent years, the fractional calculus has been found that derivatives of non–integer order are very effective for the description of many physical phenomena such as damping laws and diffusion process [1, 2, 3, 4, 5]. Some fundamental works on various aspects of the fractional calculus are given by Caputo [6], Debanth [7], Jafari and Seifi [8], Kemple and Beyer [9], Kilbas and Trujillo [10], Momani and Shawagfeh [11], Oldham and Spanier [12], etc. Several methods have been used to solve fractional partial differential equations, such as Laplace transform method, Fourier transform method [9], Adomian's decomposition method [11], homotopy analysis method [13] and so on. A substantial amount of research work has been directed to the study of the nonlinear fractional heat conduction, Kaup–Kupershmidt, Fisher and Huxley equations given by

$$D_t^{\alpha}u - au_{xx}^3 - u + u^3 = 0, \quad 0 < \alpha \le 1, \tag{1.1}$$

$$D_t^{\alpha}u + u_{xxxxx} + 45u^2u_x - \frac{75}{2}u_xu_{xx} - 15uu_{xxx} = 0, \quad 0 < \alpha \le 1, \tag{1.2}$$

$$D_t^{\alpha}u - u_{xx} - u + u^2 = 0, \quad 0 < \alpha \le 1, \tag{1.3}$$

and

$$D_t^{\alpha}u - u_{xx} + u - 2u^2 + u^3 = 0, \quad 0 < \alpha \le 1, \tag{1.4}$$

respectively. Authors of [13] have applied homotopy analysis method for solving nonlinear fractional partial differential equations. Wazwaz [20] has investigated exact solitary solutions for the nonlinear equation of heat conduction in two dimensions. Babolian et. al [21] have obtained analytic approximate solutions to a class of nonlinear PDEs such as Burgers, Fisher, Huxley equations and two combined forms of these equations using the homotopy analysis method. Analytic study on Burgers, Fisher, Huxley equations and combined forms of these equations with the tanh-coth method is used to determine these sets of travelling wave solutions by Wazwaz [22]. Öziş et. al [23] has applied Exp-function method for solving the Fisher equation. In this work, the homotopy analysis method developed by Liao in [25] will be used to conduct an analytic study on the nonlinear fractional heat conduction, Kaup–Kupershmidt, Fisher, Huxley, Burgers–Fisher and Burgers–Huxley equations. Also, homotopy analysis method has successfully applied to partial differential equations and extended by authors [8, 14, 16, 36, 37] to solve different types of nonlinear partial differential equations. This method gives rapidly convergent successive approximations of the exact solution if such a solution exist, or else the approximations can be used for numerical purposes. The homotopy analysis method, a new analytic technique is proposed to solve nonlinear partial differential equations with fractional order. The HAM is useful to obtain exact and approximate solutions of nonlinear partial differential equations. The current paper is organized as follows: In Section 2, we describe basic definitions. In Section 3, the homotopy analysis method will be introduced briefly and this technique will be applied to solve fractional partial differential equations. Section 4 contains some test problems to show the efficiency and accuracy of the new method. In addition, a conclusion is given in Section 5. Finally some references are given at the end of this paper.

2 Basic definitions

In this section, we give some definitions and properties of the fractional calculus [3].

Definition 1. A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in R$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^n, if and only if $f^{(n)} \in C_\mu$, $n \in N$ [3].

Definition 2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \ge 0$, of a function $f \in C_\lambda,\ \lambda \ge -1$, is defined as [3]

$$J^\alpha f(t) = D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,\ (\alpha > 0), \qquad J^0 f(t) = f(t), \tag{2.1}$$

where $\Gamma(z)$ is the well-known Gamma function. Some of the properties of the operator (J^α), which we will need later, are given in the following:
For $f \in C_\lambda,\ \lambda \ge -1,\ \alpha, \beta \ge 0$ and $\gamma \ge -1$:

$$(1) J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \qquad (2) J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t), \qquad (3) J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{}$$

Definition 3. The fractional derivative (D^α) of f(t) in the Caputo's sense is defined as [3]

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \ (\alpha > 0), \tag{2.2}$$

for $n-1 < \alpha \leq n, \ n \in N, \ t > 0 \ , \ f \in C^n_{-1}$. The following are two basic properties of the Caputo's fractional derivative [6]:

(1) *Let* $f \in C^n_{-1}, \ n \in N$. *Then* $D^\alpha f, \ 0 \leq \alpha \leq n$, *is well defined and* $D^\alpha f \in C_{-1}$.

(2) *Let* $n-1 < \alpha \leq n, \ n \in N$ *and* $f \in C^n_\lambda, \lambda \geq -1$. *Then*

$$(J^\alpha D^\alpha) f(t) = f(t) - \sum_{k=0}^{n-1} f^k(0^+) \frac{t^k}{k!}. \tag{2.3}$$

In this paper only real and positive α will be considered. Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operation [36, 37]

$$D^\alpha(\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t), \tag{2.4}$$

in which λ, μ are constants, and satisfy the so-called Leibnitz rule

$$D^\alpha \big(f(t) g(t)\big) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} g^{(k)}(t) D^{\alpha-k} f(t), \tag{2.5}$$

if $f(\tau)$ is continuous in $[0, t]$ and $g(\tau)$ has $(n+1)$ continuous derivatives in $[0, t]$.

Definition 4. For n to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$, is defined as [3, 36, 37]

$$D^\alpha_t u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \big(t-\tau\big)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, & if \quad n-1 < \alpha < n, \\ \frac{\partial^n u(x,t)}{\partial t^n}, & if \quad \alpha = n \in N. \end{cases} \tag{2.6}$$

For more information on the mathematical properties of fractional derivatives and integrals one may refer to [3, 6].

3 The homotopy analysis method

In this paper, we use the homotopy analysis method to solve the problem described in Section 1. This method proposed by a Chinese mathematician J.S. Liao [25]. We apply Liao's basic ideas to the nonlinear fractional partial differential equations. Let us consider the nonlinear fractional partial differential equation

$$\mathcal{NFD}(u(x,t)) = 0, \tag{3.1}$$

where $\mathcal{NFD}$ is a nonlinear fractional partial differential operator, x and t denote independent variables and $u(x,t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the same way. Based on the constructed zero-order deformation equation by Liao [27], we give the following zero-order deformation equation in the similar way

$$(1-q)\mathcal{L}[v(x,t;q) - u_0(x,t)] = qh\mathcal{NFD}[v(x,t;q)], \tag{3.2}$$

where $q \in [0,1]$ is the embedding parameter, h is a nonzero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear non-integer order operator and it possesses the property $\mathcal{L}(C) = 0$, $u_0(x,t)$ is an initial guess of $u(x,t)$,

$\mathrm{v(x,t;q)}$ is an unknown function on independent variables x, t, q. It is important to know that one has great freedom to choose auxiliary parameter h in HAM. The q = 0 and q = 1, give respectively

$$\mathrm{v(x,t;0)=u_0(x,t),\qquad v(x,t;1)=u(x,t).} \tag{3.3}$$

Thus as q increases from 0 to 1, the solution $\mathrm{v(x,t;q)}$ varies from the initial guess $\mathrm{u_0(x,t)}$ to the solution $\mathrm{u(x,t)}$. Expanding $\mathrm{v(x,t;q)}$ in Taylor series with respect to q, one has

$$\mathrm{v(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m,} \tag{3.4}$$

where

$$\mathrm{u_m(x,t)=\frac{1}{m!}\frac{\partial^m v(x,t;q)}{\partial q^m}\bigg|_{q=0}.} \tag{3.5}$$

If the auxiliary linear non-integer order operator, the initial guess, and the auxiliary parameter h are so properly chosen, the series Eq. (3.4), converges at q = 1. Hence we have

$$\mathrm{u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t),} \tag{3.6}$$

which must be one of the solution of the original nonlinear equation, as proved by [27]. As $\mathrm{h}=-1$, Eq. (3.2) becomes

$$\mathrm{(1-q)\mathcal{L}[v(x,t;q)-u_0(x,t)]+q\mathcal{NFD}v(x,t;q)=0,} \tag{3.7}$$

which is used mostly in the homotopy perturbation method (HPM). Thus, HPM is a special case of HAM. According to Eq. (3.4), the governing equation can be deduced from the zero-order deformation Eq. (3.2). Define the vector

$$\mathrm{\vec{u}_n(x,t)=\{u_0(x,t),u_1(x,t),...,u_n(x,t)\}.} \tag{3.8}$$

Differentiating Eq. (3.2), m times with respect to the embedding parameter q and then setting q = 0 and finally dividing them by m!, we have the so-called mth-order deformation equation

$$\mathrm{\mathcal{L}[u_m(x,t)-\chi_m u_{m-1}(x,t)]=h\mathcal{NFR}(\vec{u}_{m-1}(x,t)),} \tag{3.9}$$

where

$$\mathrm{\mathcal{NFR}(\vec{u}_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{NFD}\big(\vec{v}(x,t;q)\big)}{\partial q^{m-1}}\bigg|_{q=0},} \tag{3.10}$$

and

$$\chi_{\mathrm{m}}=\begin{cases}0, & \mathrm{m}\le 1,\\ 1, & \mathrm{m}>1.\end{cases} \tag{3.11}$$

Finally, for the purpose of computation, we will approximate the HAM solution Eq. (3.6) by the following truncated series:

$$\phi_{\mathrm{m}}=\sum_{\mathrm{k}=0}^{\mathrm{m}-1}\mathrm{u_k(x,t)}. \tag{3.12}$$

The mth-order deformation Eq. (3.9), is linear and thus can be easily solved, especially by means of a symbolic computation software such as Mathematica, Maple, Matlab, Maxima and so on.

4 Illustrative examples

In this section, we present several examples to illustrate the applicability of HAM to solve non-linear partial differential equations introduced in Section 1.

Example 1: **Consider the time-dependent one dimensional heat conduction equation [20]** as follows:

$$\mathrm{D}_t^{\alpha}\mathrm{u}(x,t) - a(u^3)_{xx}(x,t) - u(x,t) + u^3(x,t) = 0,\ u(x,0) = \exp\left(\frac{x}{3\sqrt{a}}\right),\quad 0 < \alpha \leq 1. \tag{4.1}$$

To solve the general homogeneous nonlinear equation with the HAM, we choose the linear non-integer order operator in the form

$$\mathcal{L}[v(x,t;q)] = \mathrm{D}_t^{\alpha} v(x,t;q). \tag{4.2}$$

Furthermore, Eq. (4.1), suggests to define the nonlinear fractional partial differential operator as follows:

$$\mathcal{NFD}[v(x,t;q)] = \mathrm{D}_t^{\alpha} v(x,t;q) - a(v^3)_{xx}(x,t;q) - v(x,t;q) + v^3(x,t;q). \tag{4.3}$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[v(x,t;q) - u_0(x,t)] = qh\mathcal{NFD}v(x,t;q). \tag{4.4}$$

Obviously, when $q = 0$ and $q = 1$ respectively, we get

$$v(x,t;0) = u_0(x,t) = u(x,0), \qquad v(x,t;1) = u(x,t). \tag{4.5}$$

According to Eqs. (3.9)–(3.11), we gain the mth-order deformation equation

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\mathcal{NFR}(\overrightarrow{u}_{m-1}(x,t)), \tag{4.6}$$

where

$$\mathcal{NFR}(\overrightarrow{u}_{m-1}(x,t)) = \mathrm{D}_t^{\alpha} u_{m-1}(x,t) - a\sum_{i=0}^{m-1}\sum_{k=0}^{i}(u_k u_{i-k} u_{m-1-i})_{xx}(x,t) \tag{4.7}$$

$$-u_{m-1}(x,t) + \sum_{i=0}^{m-1}\sum_{k=0}^{i}(u_k u_{i-k} u_{m-1-i})(x,t).$$

Now the solution of Eq. (4.6), for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\mathcal{L}^{-1}\mathcal{NFR}[\overrightarrow{u}_{m-1}(x,t)]. \tag{4.8}$$

From Eqs. (4.1), (4.5) and (4.8), we now successively obtain

$$u_0(x,t) = u(x,0) = \exp\left(\frac{x}{3\sqrt{a}}\right), \tag{4.9}$$

$$u_1(x,t) = h\mathrm{D}_t^{-\alpha}\left[\mathrm{D}_t^{\alpha}u_0 - a(u_0^3)_{xx} - u_0 + u_0^3\right] = \tag{4.10}$$

$$-h\mathrm{D}_t^{-\alpha}\left[\exp\left(\frac{x}{3\sqrt{a}}\right)\right] = -\frac{h}{\Gamma(\alpha+1)}t^{\alpha}\left[\exp\left(\frac{x}{3\sqrt{a}}\right)\right],$$

$$u_2(x,t) = (h+1)u_1(x,t) + h\mathrm{D}_t^{-\alpha}\left[-a(3u_0^2u_1)_{xx} - u_1 + 3u_0^2u_1\right] = \tag{4.11}$$

$$-\frac{h(h+1)}{\Gamma(\alpha+1)}t^{\alpha}\left[\exp\left(\frac{x}{3\sqrt{a}}\right)\right] + \frac{h^2}{\Gamma(2\alpha+1)}t^{2\alpha}\left[\exp\left(\frac{x}{3\sqrt{a}}\right)\right],$$

$$u_3(x,t) = (h+1)u_2 + h\mathrm{D}_t^{-\alpha}\left[-a(3u_0^2u_2 + 3u_0u_1^2)_{xx} - u_2 + 3u_0^2u_2 + 3u_0u_1^2\right], \tag{4.12}$$

and so on. For $h = -1$, we obtain $u(x,t)$ as follows

$$u(x,t) = e^{\frac{x}{3\sqrt{a}}} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} e^{\frac{x}{3\sqrt{a}}} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} e^{\frac{x}{3\sqrt{a}}} + \ldots = e^{\frac{x}{3\sqrt{a}}} \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{4.13}$$

In the above terms, we substitute $h = -1$ then the dominant terms remain and the higher order terms vanish. For $\alpha = 1$, we have

$$u_0(x,t) = \exp\left(\frac{x}{3\sqrt{a}}\right), \quad u_1(x,t) = \exp\left(\frac{x}{3\sqrt{a}}\right) t, \quad u_2(x,t) = \exp\left(\frac{x}{3\sqrt{a}}\right) \frac{t^2}{2!}, \tag{4.14}$$

and so on. Thus, we get the exact solution as

$$u(x,t) = \exp\left(\frac{x}{3\sqrt{a}}\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right) = \exp\left(\frac{x + 3\sqrt{a}t}{3\sqrt{a}}\right). \tag{4.15}$$

Example 2: Consider the fractional Kaup-Kupershmidt equation [24] as the following form

$$D_t^{\alpha} u(x,t) + u_{xxxxx}(x,t) + 45u^2(x,t)u_x(x,t) - \frac{75}{2} u_x(x,t)u_{xx}(x,t) - \tag{4.16}$$

$$15u(x,t)u_{xxx}(x,t) = 0, \qquad u(x,0) = \frac{2}{3} + \tan^2(x), \qquad 0 < \alpha \leq 1.$$

Using the above definition, we gain mth-ordedr nonlinear fractional operator as follows;

$$\mathcal{NFR}(\overrightarrow{u}_{m-1}) = D_t^{\alpha} u_{m-1} + (u_{m-1})_{5x} + 45 \sum_{i=0}^{m-1} \sum_{k=0}^{i} u_k u_{i-k} (u_{m-1-i})_x \tag{4.17}$$

$$-\frac{75}{2} \sum_{i=0}^{m-1} (u_i)_x (u_{m-1-i})_{xx} - 15 \sum_{i=0}^{m-1} u_i (u_{m-1-i})_{xxx}.$$

Consequently, the first few terms of the FHAM series solutions are as follows,

$$u_0(x,t) = u(x,0) = \frac{2}{3} + \tan^2(x), \tag{4.18}$$

$$u_1(x,t) = hD_t^{-\alpha}\left[D_t^{\alpha} u_0 + (u_0)_{5x} + 45u_0^2 u_{0x} - \frac{75}{2} u_{0x} u_{0xx} - 15u_0 u_{0xxx}\right] = \tag{4.19}$$

$$2hD_t^{-\alpha}\left[\tan(x)\sec^2(x)\right] = \frac{2ht^{\alpha}}{\Gamma(\alpha+1)} \tan(x)\sec^2(x),$$

$$u_2(x,t) = (h+1)u_1(x,t) + hD_t^{-\alpha}\Big[(u_1)_{5x} + 45u_0^2 u_{1x} + 90u_0 u_{0x} u_1 - \frac{75}{2} u_{0x} u_{1xx} \tag{4.20}$$

$$-\frac{75}{2} u_{1x} u_{0xx} - 15u_0 u_{1xxx} - 15u_1 u_{0xxx}\Big]$$

$$= \frac{2h(h+1)t^{\alpha}}{\Gamma(\alpha+1)} \tan(x)\sec^2(x) - \frac{2h^2 t^{2\alpha}}{\Gamma(2\alpha+1)} \left(2\sec^2(x) - 3\sec^4(x)\right),$$

$$u_3(x,t) = (h+1)u_2 + hD_t^{-\alpha}\big[(u_2)_{5x} + 45u_0^2 u_{2x} + 90u_0 u_{1x} u_1 + 90u_{0x} u_0 u_2 + 45u_1^2 u_{0x} \tag{4.21}$$

$$-\frac{75}{2} u_{0x} u_{2xx} - \frac{75}{2} u_{20x} u_{0xx} - \frac{75}{2} u_{1x} u_{1xx} - 15u_0 u_{2xxx} - 15u_1 u_{1xxx} - 15u_2 u_{0xxx}\Big],$$

and so on. For $h = -1$, we obtain $u(x,t)$ is as follows,

$$u(x,t) = \frac{2}{3} + \tan^2(x) - \frac{2t^{\alpha}}{\Gamma(\alpha+1)}\tan(x)\sec^2(x) - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}\left(2\sec^2(x) - 3\sec^4(x)\right) \tag{4.22}$$

$$+\frac{4t^{3\alpha}}{\Gamma(3\alpha+1)}[\tan(x)\sec^2(x)(-3780\sec^4(x) + 3780\sec^6(x) - 726\sec^2(x) + 2)]$$

$$+\frac{4t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}\left[\tan(x)\sec^4(x)(360 - 1890\sec^2(x) + 1890\sec^4(x))\right] + \dots.$$

For $h = -1$ and $\alpha = 1$, we get

$$u_0(x,t) = \frac{2}{3} + \tan^2(x), \tag{4.23}$$

$$u_1(x,t) = -2\tan(x)\sec^2(x)t,$$

$$u_2(x,t) = \left(-2\sec^2(x) + 3\sec^4(x)\right)t^2,$$

and so on. Thus, the exact solution is as follows,

$$u(x,t) = \frac{2}{3} + \tan^2(x) - 2\tan(x)\sec^2(x)t - \left(2\sec^2(x) - 3\sec^4(x)\right)t^2 + \tag{4.24}$$

$$\frac{4t^3}{3}\left(\tan(x)\sec^2(x)[3780\sec^6(x) - 3780\sec^4(x) - 3\sec^2(x) + 1]\right) + \dots.$$

Therefore, using Taylor series we obtain the following closed form solution

$$u(x,t) = \frac{2}{3} + \tan^2(x-t). \tag{4.25}$$

Example 3: Consider the fractional Fisher's equation [21, 22] as follows,

$$D_t^{\alpha}u(x,t) - u_{xx}(x,t) - u(x,t) + u^2(x,t) = 0, \quad 0 < \alpha \le 1, \tag{4.26}$$

$$u(x,0) = \frac{1}{4}\left[1 - \tanh\left(\frac{x}{2\sqrt{6}}\right)\right]^2.$$

By similar procedure as previous examples, we gain mth-order nonlinear fractional operator as follows,

$$\mathcal{NFR}(\vec{u}_{m-1}) = D_t^{\alpha}u_{m-1} - (u_{m-1})_{xx} - u_{m-1} + \sum_{i=0}^{m-1} u_i u_{m-1-i}. \tag{4.27}$$

Consequently, the first few terms of the FHAM series solutions satisfy

$$u_0(x,t) = u(x,0) = \frac{1}{4}\left[1 - \tanh\left(\frac{x}{2\sqrt{6}}\right)\right]^2, \tag{4.28}$$

$$u_1(x,t) = hD_t^{-\alpha}\left[D_t^{\alpha}u_0 - (u_0)_{xx} - u_0 + u_0^2\right] = \tag{4.29}$$

$$\frac{ht^{\alpha}}{\Gamma(\alpha+1)}\left\{\frac{5}{24}\left[\operatorname{sech}^2\left(\frac{x}{2\sqrt{6}}\right)\left(\tanh\left(\frac{x}{2\sqrt{6}}\right) - 1\right)\right]\right\},$$

$$u_2(x,t) = (h+1)u_1(x,t) + hD_t^{-\alpha}\left[-(u_1)_{xx} - u_1 + 2u_0u_1\right] = \tag{4.30}$$

$$\frac{h(h+1)t^{\alpha}}{\Gamma(\alpha+1)}\left\{\frac{5}{24}\left[\operatorname{sech}^2\left(\frac{x}{2\sqrt{6}}\right)\left(\tanh\left(\frac{x}{2\sqrt{6}}\right) - 1\right)\right]\right\}$$

$$+\frac{25}{288}\frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\left[\cos\left(\frac{x}{2\sqrt{6}}\right)-\sinh\left(\frac{x}{2\sqrt{6}}\right)\right]\left[\cosh\left(\frac{x}{2\sqrt{6}}\right)+3\sinh\left(\frac{x}{2\sqrt{6}}\right)\right]}{\cosh^4\left(\frac{x}{2\sqrt{6}}\right)},$$

$$u_3(x,t)=(h+1)u_2+hD_t^{-\alpha}\left[-(u_2)_{xx}-u_2+2u_0u_2+u_1^2\right], \tag{4.31}$$

and so on. For $h=-1$, $u(x,t)$ is as follows,

$$u(x,t)=\frac{1}{4}\left[1-\tanh\left(\frac{x}{2\sqrt{6}}\right)\right]^2- \tag{4.32}$$

$$\frac{t^\alpha}{\Gamma(\alpha+1)}\left\{\frac{5}{24}\left[\text{sech}^2\left(\frac{x}{2\sqrt{6}}\right)\left(\tanh\left(\frac{x}{2\sqrt{6}}\right)-1\right)\right]\right\}+$$

$$\frac{25}{288}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\left[\cos\left(\frac{x}{2\sqrt{6}}\right)-\sinh\left(\frac{x}{2\sqrt{6}}\right)\right]\left[\cosh\left(\frac{x}{2\sqrt{6}}\right)+3\sinh\left(\frac{x}{2\sqrt{6}}\right)\right]}{\cosh^4\left(\frac{x}{2\sqrt{6}}\right)}$$

$$+\frac{25}{1728}\left(\cosh\left(\frac{x}{2\sqrt{6}}\right)-\sinh\left(\frac{x}{2\sqrt{6}}\right)\right)\left\{\text{sech}^3\left(\frac{x}{2\sqrt{6}}\right)[25+\right.$$

$$15\tanh\left(\frac{x}{2\sqrt{6}}\right)-24\text{sech}^2\left(\frac{x}{2\sqrt{6}}\right)-6\tanh\left(\frac{x}{2\sqrt{6}}\right)\text{sech}^2\left(\frac{x}{2\sqrt{6}}\right)\Big]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\left.-3\text{sech}^5\left(\frac{x}{2\sqrt{6}}\right)\left(1-\tanh\left(\frac{x}{2\sqrt{6}}\right)\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}\right\}.$$

For $h=-1$ and $\alpha=1$, we get

$$u_0(x,t)=\frac{1}{4}\left[1-\tanh\left(\frac{x}{2\sqrt{6}}\right)\right]^2, \tag{4.33}$$

$$u_1(x,t)=-\left\{\frac{5}{24}\left[\text{sech}^2\left(\frac{x}{2\sqrt{6}}\right)\left(\tanh\left(\frac{x}{2\sqrt{6}}\right)-1\right)\right]\right\}t,$$

$$u_2(x,t)=\frac{25}{576}\frac{\left[\cos\left(\frac{x}{2\sqrt{6}}\right)-\sinh\left(\frac{x}{2\sqrt{6}}\right)\right]\left[\cosh\left(\frac{x}{2\sqrt{6}}\right)+3\sinh\left(\frac{x}{2\sqrt{6}}\right)\right]}{\cosh^4\left(\frac{x}{2\sqrt{6}}\right)}t^2,$$

and so on. Thus, using Taylor series we obtain the following closed form solution

$$u(x,t)=\frac{1}{4}\left\{1-\tanh\left[\frac{1}{2\sqrt{6}}\left(x-\frac{5}{\sqrt{6}}t\right)\right]\right\}^2. \tag{4.34}$$

Example 4: **Consider the fractional Huxley equation [21, 22] as follows,**

$$D_t^\alpha u(x,t)-u_{xx}(x,t)+u(x,t)-2u^2(x,t)+u^3(x,t)=0,\quad 0<\alpha\le 1, \tag{4.35}$$

$$u(x,0)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right).$$

By similar procedure as previous examples, we gain mth-order nonlinear fractional operator as follows,

$$\mathcal{NFR}(\vec{u}_{m-1})=D_t^\alpha u_{m-1}-(u_{m-1})_{xx}+u_{m-1}-2\sum_{i=0}^{m-1}u_iu_{m-1-i} \tag{4.36}$$

$$+\sum_{i=0}^{m-1}\sum_{k=0}^{i} u_k u_{i-k} u_{m-1-i}.$$

Consequently, the first few terms of the FHAM series solutions satisfy

$$u_0(x,t) = u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right), \tag{4.37}$$

$$u_1(x,t) = hD_t^{-\alpha}\left[D_t^{\alpha}u_0 - (u_0)_{xx} + u_0 - 2u_0^2 + u_0^3\right] = \frac{ht^{\alpha}}{8\Gamma(\alpha+1)}\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right), \tag{4.38}$$

$$u_2(x,t) = (h+1)u_1(x,t) + hD_t^{-\alpha}\left[-(u_1)_{xx} + u_1 - 4u_0u_1 + 3u_0^2u_1\right] = \tag{4.39}$$

$$\frac{h(h+1)t^{\alpha}}{8\Gamma(\alpha+1)}\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right) - \frac{h^2t^{2\alpha}}{16\Gamma(2\alpha+1)}\tanh\left(\frac{x}{2\sqrt{2}}\right)\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right),$$

$$u_3(x,t) = (h+1)u_2 + hD_t^{-\alpha}\left[-(u_2)_{xx} + u_2 - 4u_0u_2 - 2u_1^2 + 3u_0^2u_2 + 3u_0u_1^2\right], \tag{4.40}$$

and so on. For $h = -1$, $u(x,t)$ is as follows,

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right) - \frac{t^{\alpha}}{8\Gamma(\alpha+1)}\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right) - \tag{4.41}$$

$$\frac{t^{2\alpha}}{16\Gamma(2\alpha+1)}\tanh\left(\frac{x}{2\sqrt{2}}\right)\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right) + \frac{1}{128}\left\{2\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right)\right.$$

$$\times\left[-2 + 2\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right) + 3\tanh\left(\frac{x}{2\sqrt{2}}\right)\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right)\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\left.+\text{sech}^4\left(\frac{x}{2\sqrt{2}}\right)\left(1 - 3\tanh\left(\frac{x}{2\sqrt{2}}\right)\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}\right\}.$$

For $h = -1$ and $\alpha = 1$, we get

$$u_0(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right), \tag{4.42}$$

$$u_1(x,t) = -\frac{1}{8}\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right)t,$$

$$u_2(x,t) = -\frac{1}{32}\tanh\left(\frac{x}{2\sqrt{2}}\right)\text{sech}^2\left(\frac{x}{2\sqrt{2}}\right)t^2,$$

and so on. Therefore, using Taylor series we obtain the following closed form solution

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{1}{2\sqrt{2}}\left(x - \frac{t}{\sqrt{2}}\right)\right]. \tag{4.43}$$

5 The HAM Convergence

A series is often of no use if it is convergent in a rather restricted region. In general, one can prove that the series (3.6), given by the homotopy analysis method converges to the solution, it must be the solution of the considered nonlinear problem. The discussion about the convergence of HAM may refer to [27].

THEOREM 1. If the series (3.6) converges, where $u_m(x,t)$ is governed by the high order deformation equations (3.9) and (4.1) under the definitions (3.11) and (4.7), then it is the exact solution of Equation

(4.1).

Proof: If the series $u_m(x,t)$ converges and

$$S(x,t) = \sum_{m=0}^{+\infty} u_m(x,t). \tag{5.1}$$

where

$$\lim_{m\to+\infty} u_m(x,t) = 0. \tag{5.2}$$

By definition (3.11) of χ_m, we have

$$\sum_{m=1}^{n} [u_m(x,t) - \chi_m u_{m-1}(x,t)] = u_1 + (u_2 - u_1) + ...(u_n - u_{n-1}) = u_n, \tag{5.3}$$

which gives us, according to (5.2),

$$\sum_{m=1}^{+\infty} [u_m(x,t) - \chi_m u_{m-1}(x,t)] = \lim_{n\to+\infty} u_n = 0. \tag{5.4}$$

Therefore, using the above expression and the definition of $\mathcal{L}$, we have

$$\sum_{m=1}^{+\infty} \mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \mathcal{L}\sum_{m=1}^{+\infty} [u_m(x,t) - \chi_m u_{m-1}(x,t)] = 0. \tag{5.5}$$

From the above expression and Equation (3.9), we obtain

$$\sum_{m=1}^{+\infty} \mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\sum_{m=1}^{+\infty} \mathcal{FR}(\overrightarrow{u}_{m-1}(x,t)) = 0, \tag{5.6}$$

which gives, since $h \neq 0$, that

$$\sum_{m=1}^{+\infty} \mathcal{NFR}(\overrightarrow{u}_{m-1}(x,t)) = 0. \tag{5.7}$$

From (4.7), it holds

$$\sum_{m=1}^{+\infty} \mathcal{NFR}(\overrightarrow{u}_{m-1}(x,t)) = \sum_{m=1}^{+\infty} \Bigg[D_t^{\alpha} u_{m-1}(x,t) - a\sum_{i=0}^{m-1}\sum_{k=0}^{i} (u_k u_{i-k} u_{m-1-i})_{xx}(x,t) \tag{5.8}$$

$$-u_{m-1}(x,t) + \sum_{i=0}^{m-1}\sum_{k=0}^{i} (u_k u_{i-k} u_{m-1-i})(x,t)\Bigg] = \sum_{m=0}^{+\infty} D_t^{\alpha} u_m(x,t) - \sum_{m=0}^{+\infty} u_m(x,t) -$$

$$a\sum_{m=1}^{+\infty}\sum_{i=0}^{m-1}\sum_{k=0}^{i} (u_k u_{i-k} u_{m-1-i})_{xx}(x,t) + \sum_{m=1}^{+\infty}\sum_{i=0}^{m-1}\sum_{k=0}^{i} u_k u_{i-k} u_{m-1-i}(x,t) = \sum_{m=0}^{+\infty} D_t^{\alpha} u_m(x,t)$$

$$-\sum_{m=0}^{+\infty} u_m(x,t) - a\frac{d^2}{dx^2}\sum_{m=1}^{+\infty}\sum_{i=0}^{m-1} u_{m-1-i}\sum_{k=0}^{i} u_k u_{i-k}(x,t) + \sum_{m=1}^{+\infty}\sum_{i=0}^{m-1} u_{m-1-i}\sum_{k=0}^{i} u_k u_{i-k}(x,t)$$

$$= \sum_{m=0}^{+\infty} D_t^{\alpha} u_m(x,t) - \sum_{m=0}^{+\infty} u_m(x,t) - a\frac{d^2}{dx^2}\sum_{i=0}^{+\infty}\sum_{m=i+1}^{+\infty} u_{m-1-i}\sum_{k=0}^{i} u_k u_{i-k}(x,t)$$

$$+\sum_{i=0}^{+\infty}\sum_{m=i+1}^{+\infty} u_{m-1-i}\sum_{k=0}^{i} u_k u_{i-k}(x,t) = \sum_{m=0}^{+\infty} D_t^{\alpha} u_m(x,t) - \sum_{m=0}^{+\infty} u_m(x,t)$$

$$-a\frac{d^2}{dx^2}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} u_j(x,t)\sum_{k=0}^{i} u_k u_{i-k}(x,t) + \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} u_j(x,t)\sum_{k=0}^{i} u_k u_{i-k}(x,t)$$

$$= \sum_{m=0}^{+\infty} D_t^{\alpha} u_m(x,t) - \sum_{m=0}^{+\infty} u_m(x,t) - a\frac{d^2}{dx^2}\sum_{j=0}^{+\infty} u_j(x,t)\sum_{k=0}^{+\infty}\sum_{k=i}^{+\infty} u_k u_{i-k}(x,t)$$

$$+\sum_{j=0}^{+\infty} u_j(x,t)\sum_{k=0}^{+\infty}\sum_{k=i}^{+\infty} u_k$$

$$-a\frac{d^2}{dx^2}\sum_{j=0}^{+\infty} u_j(x,t)\sum_{k=0}^{+\infty} u_k(x,t)\sum_{l=0}^{+\infty} u_l(x,t) + \sum_{j=0}^{+\infty} u_j(x,t)\sum_{k=0}^{+\infty} u_k(x,t)\sum_{l=0}^{+\infty} u_l(x,t)$$

$$= D_t^{\alpha} S(x,t) - S(x,t) - a\frac{d^2}{dx^2} S^3(x,t) + S^3(x,t).$$

From Equations (5.7) and (5.8), we get

$$D_t^{\alpha} S(x,t) - S(x,t) - a\frac{d^2}{dx^2} S^3(x,t) + S^3(x,t) = 0, \quad t > 0, \quad 0 < \alpha \le 1. \tag{5.9}$$

From Equations (4.1) and $u_m(x,0) = 0$, it holds

$$S(x,0) = \sum_{m=0}^{+\infty} u_m(x,0) = u_0(x,0) + \sum_{m=1}^{+\infty} u_m(x,0) = u(x,0) = \exp\left(\frac{x}{3\sqrt{a}}\right). \tag{5.10}$$

Therefore, according to the above two expressions, $S(x,t)$ must be the exact solution of Equation (4.1). This ends the proof. For Examples 2-5, according to theorem 1, similar conclusion holds. The parameter h determines the convergence region and rate of the approximation for HAM which is shown in Tables 1–3. If we take $h = -1$, we obtain the exact results which are presented in Tables 1-5. For $h = -1$, we obtain the best results of the case $\alpha = 1$ which has an exact solution.

Table 1: The absolute error, $|u - \phi_8|$, for the Kaup-Kupershmidt equation when $h = -1$ and $\alpha = 1$.

error = $|u(x,t) - \phi_8(x,t)|$

t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	2.335×10^{-9}	5.792×10^{-7}	1.451×10^{-5}	1.429×10^{-4}	8.482×10^{-4}
0.2	3.87×10^{-9}	9.329×10^{-7}	2.266×10^{-5}	2.159×10^{-4}	1.235×10^{-3}
0.3	7.716×10^{-9}	1.834×10^{-6}	4.394×10^{-5}	4.123×10^{-4}	2.319×10^{-3}
0.4	1.709×10^{-8}	4.029×10^{-6}	9.572×10^{-5}	8.913×10^{-4}	4.976×10^{-3}
0.5	4.12×10^{-8}	9.642×10^{-6}	2.276×10^{-4}	2.106×10^{-3}	1.169×10^{-2}

6 Conclusion

In this paper, we applied the homotopy analysis method for solving the nonlinear fractional heat conduction, Kaup-Kupershmidt, Fisher and Huxley equations. The validity of the method has been successfully applied to study several types of partial differential equations. In addition, this method allows us to the perform complicated and tedious algebraic calculations through the computer. The obtained results of applying

Table 2: The absolute error, $|u - \phi_{11}|$, for the Fisher's equation when $h = -1$ and $\alpha = 1$.

error = $\lvert u(x,t) - \phi_{11}(x,t)\rvert$					
t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	3.936×10^{-16}	4.011×10^{-13}	2.297×10^{-11}	4.046×10^{-10}	3.729×10^{-9}
0.2	3.958×10^{-16}	4.053×10^{-13}	2.333×10^{-11}	4.128×10^{-10}	3.825×10^{-9}
0.3	3.877×10^{-16}	3.989×10^{-13}	2.307×10^{-11}	4.103×10^{-10}	3.819×10^{-9}
0.4	3.698×10^{-16}	3.824×10^{-13}	2.223×10^{-11}	3.973×10^{-10}	3.717×10^{-9}
0.5	3.429×10^{-16}	3.565×10^{-13}	2.084×10^{-11}	3.744×10^{-10}	3.521×10^{-9}

Table 3: The absolute error, $|u - \phi_8|$, for the Huxley equation when $h = -1$ and $\alpha = 1$.

error = $\lvert u(x,t) - \phi_8(x,t)\rvert$					
t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	4.853×10^{-16}	1.137×10^{-13}	2.645×10^{-12}	2.371×10^{-11}	1.252×10^{-10}
0.2	9.855×10^{-16}	2.425×10^{-13}	5.963×10^{-12}	5.701×10^{-11}	3.244×10^{-10}
0.3	1.432×10^{-15}	3.581×10^{-13}	8.955×10^{-12}	8.718×10^{-11}	5.059×10^{-10}
0.4	1.802×10^{-15}	4.544×10^{-13}	1.147×10^{-11}	1.267×10^{-10}	6.601×10^{-10}
0.5	2.078×10^{-15}	5.271×10^{-13}	1.338×10^{-11}	1.323×10^{-10}	7.797×10^{-10}

this procedure show the high accuracy and rapid convergent of the homotopy analysis method. Homotopy analysis method provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter h. This is an obvious advantage of the homotopy analysis method. In this way, we obtain solutions in power series. Also, we obtained the exact solutions in the special case $\alpha = 1$, $h = -1$, for some equations. However, it is well-known that a power series often has a small convergence radius. It should be emphasized that, in the frame of the homotopy analysis method, we have great freedom to choose the initial guess and the auxiliary linear operator $\mathcal{L} = D^{\alpha}$. This work shows that the homotopy analysis method is a very efficient and powerful tool for solving the nonlinear fractional partial differential equations.

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