Analytical Modal Solutions on Optical Inhomogeneous Fibers: the Frobenius-Taylor Method (FTM)

Isabel Ventim Neves[1], Manuel Guerreiro Neves[2]

[1]DEE, FCT-UNL
Quinta da Torre,
2829-516 Caparica, Portugal
Email: ivn {at} uninova.pt

[2]Instituto Superior Técnico
Av. Rovisco Pais,
1049-001 Lisboa, Portugal
Email: manuel.neves {at} ist.utl.pt

ABSTRACT– ***Analytical solutions for wave propagation in a general inhomogeneous optical fiber profile are built using serial expansions about regular points of differential equations for radial variation of field components, called as Frobenius' and Taylor's, whose coefficients are evaluated recursively. Using both types of series expansion, a general and fast convergent algorithm for computing modal characteristics is presented: the Frobenius-Taylor Method (FTM), together with a set of alternative functions to the second-kind Bessel ones (used in the cladding), closely related but numerically more stables. The complete algorithm is presented and can be applied to both high and low values of the index difference, showing a fast convergence for frequencies just over, very near and far from cutoff.***

Keywords – Inhomogeneous Optical Fibers, Modal Solutions, Dispersion, FTM

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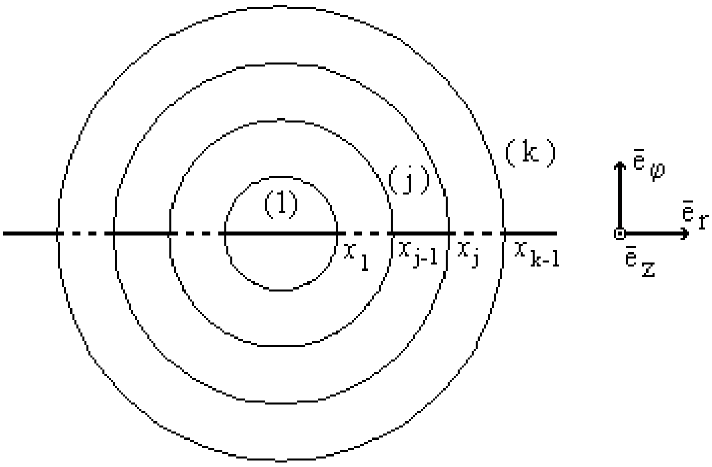
1. INTRODUCTION

The design of an optical fiber profile will greatly benefit from the use of analytical solutions for field components and a correspondent algorithm to compute modal characteristics. In fact, such an algorithm is free of spurious solutions and allows fast evaluation of waveguide dispersion characteristics. Analytical solutions can be constructed using Frobenius-type series expansions, first proposed by Kirchhoff [1] under a slow index variation approach. This method is referenced in classical books and is mainly applied to the α-power profiles [2] but, for more general profiles, the series convergence was reported to be very poor. However, using both Frobenius and Taylor's series expansions, a general fast convergent algorithm can be constructed, named as Frobenius-Taylor Method (FTM). Frobenius solutions in the core and Taylor solutions in an intermediate layer of the fiber structure are first presented in [3] and a set of Frobenius' solutions is defined in the cladding. Those solutions are related to the modified Bessel functions of second kind but can be used very near and far from cutoff, remaining stable and fast convergent.

The complete algorithm is included here and it is important to point out that neither a slow variation of the refractive index or small values for the index difference are imposed, which allows the algorithm's application to general profiles. In an example where a fast profile variation is considered, cutoff values and modal solutions for both small and high values of index differences are presented.

2. GENERAL

The fiber is composed by k coaxial isotropic layers $(k \geq 2)$, where a general layer j is defined by the relative permittivity radial function $\varepsilon_j(r)$ and the outer medium k extends to infinity and is homogeneous: the cladding, $\varepsilon_k(r) = \varepsilon_k$. The fiber is inserted in a cylindrical coordinate system, where a normalised radial variable x is used instead of r, where x= r/a and a is a reference length. All layers do have $\mu=\mu_0$, are considered without losses, and the time excitation is $\exp(j\omega t)$.



(1) inner medium, $\varepsilon = \varepsilon_1(x)$

(j) intermediate medium $\varepsilon = \varepsilon_j(x)$

(k) outer medium $\varepsilon = \varepsilon_k$

Fig. 1: The fiber structure in a cylindrical coordinate system

Modal solutions will be built using series development in for solving the radial x-variation of the radial and azimuthal magnetic field in each medium of the Fig.1. Those equations are [3]:

(2.1.a)

$$\left\{\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+s(x)-\frac{m^2+1}{x^2}\right\}H_r(x)=-\frac{2m}{x^2}H_\varphi(x)$$

$$\left\{\frac{d^2}{dx^2}+\left(\frac{1}{x}-g(x)\right)\frac{d}{dx}+s(x)-\frac{g(x)}{x}-\frac{m^2+1}{x^2}\right\}H_\varphi(x)=-\left(\frac{2m}{x^2}+m\frac{g(x)}{x}\right)H_r(x)$$

(2.1.b)

$$s(x)=(k_0a)^2\left[\varepsilon(x)-\beta^2\right] \qquad g(x)=\frac{1}{\varepsilon(x)}\frac{d}{dx}[\varepsilon(x)]$$

In (2.1) $\beta = k_z / k_0$, k_0 is the free space wave number and m is the azimuthal order – related to the well-known azymuthal field-variation with sin(mφ) or cos(mφ). For the differential equations (2.1), the points $x = 0$ and $x = \infty$ are singular, the first a regular one and the second irregular, and any other point is regular [4]. The solutions will be written as series expansions about a regular point x_0, of the Frobenius' type if this point is the singular one, and of the Taylor's type otherwise [4]. In those expansions the series' coefficients are built using the Taylor´s coefficients of the functions s(x) and g(x) in (2.1), about the point x_0. Starting with the Taylor´s coefficients of a general function ε(x),

(2.2.a)

$$\varepsilon(x)=\sum_{k=0}^{\infty}\varepsilon_k(x-x_0)^k\ ,\quad \varepsilon_k=\frac{1}{k!}\left[\frac{d^k}{dx^k}\varepsilon(x)\right]_{x=x_0}$$

the s_k and g_k coefficients are calculated as follows:

(2.2.b)

$$\text{for } k=0\ ,\qquad s_0=(k_0a)^2(\varepsilon_0-\beta^2)\ \text{, with } \varepsilon_0=\varepsilon(x=0)$$

$$\text{for } k\geq 1\ ,\qquad \begin{cases} s_k=(k_0a)^2\varepsilon_k \\ g_{k-1}=\dfrac{1}{\varepsilon_0}\left(k\varepsilon_k-\displaystyle\sum_{i=1}^{k-1}g_{i-1}\varepsilon_{k-i}\right) \end{cases}$$

Using from now on matricial notations, the functions s(x) and g(x) in (2.1.b) are defined by

(2.2.c)

$$\begin{bmatrix} s(x) \\ g(x) \end{bmatrix}=\sum_{k=0}^{\infty}\begin{bmatrix} s_k \\ g_k \end{bmatrix}(x-x_0)^k$$

and the field components in (2.1) are built using series expansions about x_0, with the form

$$\begin{bmatrix} H_r(x) \\ H_\varphi(x) \end{bmatrix} = (x - x_0)^\gamma \sum_{n=0}^{\infty} \begin{bmatrix} a_n \\ b_n \end{bmatrix} (x - x_0)^n \tag{2.3}$$

where γ is a constant, [4].

3. FROBENIUS' SOLUTIONS IN THE INNER MEDIUM

Choosing $x_0 = 0$, the pair of functions $H_r(x)$ and $H_\varphi(x)$ in (2.3) becomes:

$$\begin{bmatrix} H_r(x) \\ H_\varphi(x) \end{bmatrix} = x^\gamma \sum_{n=0}^{\infty} \begin{bmatrix} a_n \\ b_b \end{bmatrix} x^n \tag{3.1}$$

where one of the zero-order coefficient is the integration constant. Applying (3.1) to the differential equations (2.1), together with the series expansions of the functions s(x) and g(x) about $x_0 = 0$ in (2.3), and making zero the coefficient of x^n in the resultant series [4], the following system is obtained:

$$\begin{gathered} \left[(\gamma+n)^2 - (m^2+1)\right]\begin{bmatrix} a_n \\ b_n \end{bmatrix} + 2m \begin{bmatrix} b_n \\ a_n \end{bmatrix} + \begin{bmatrix} S(a) \\ S(b) - G(b) \end{bmatrix} = 0 \\ \begin{bmatrix} S(a) \\ S(b) \end{bmatrix} = \sum_{i=0}^{n-2} s_{n-2-i} \begin{bmatrix} a_i \\ b_i \end{bmatrix} \\ G(b) = \sum_{i=0}^{n-1} g_{n-1-i} \left[(\gamma + i + 1)\, b_i - m\, a_i\right] \end{gathered} \tag{3.2}$$

Solving the system (3.2) in order to calculate the coefficients a_n and b_n, it is obtained a recursive formula:

$$\begin{aligned} \Delta(\gamma,n)\, a_n &= -\left[(\gamma+n)^2 - (m^2+1)\right] S(a) + 2m\left[S(b) - G(b)\right] \\ \Delta(\gamma,n)\, b_n &= -\left[(\gamma+n)^2 - (m^2+1)\right]\left[S(b) - G(b)\right] + 2m\, S(a) \\ \Delta(\gamma,n) &= \left[(\gamma+n)^2 - (m+1)^2\right]\left[(\gamma+n)^2 - (m-1)^2\right] \end{aligned} \tag{3.3}$$

where $\Delta(\gamma,n)$ is the determinant in (3.2). Making $\Delta(\gamma, n=0) = 0$ four different values of γ are obtained if $m \neq 0$ and only two if $m = 0$, namely:

$$m > 0 \begin{cases} \gamma_1 = m+1 \\ \gamma_2 = m-1 \\ \gamma_3 = -(m+1) \\ \gamma_4 = -(m-1) \end{cases} \qquad m = 0 \begin{cases} \gamma_1 = 1 \\ \gamma_3 = -1 \end{cases} \tag{3.4}$$

As the inner medium contains the point x = 0 (see Fig.1), only the solutions associated to no negatives values of γ can be considered, in order to avoid infinitive fields in this point. So, the differential equations (2.1) in the inner medium (1) have Frobenius' solutions [3] with the form:

$$m = 0: \begin{cases} H_r(x) = A_1^{(1)}\ x\ \sum_{n=0}^{\infty} a_n^{(1)}\ x^n \\ H_\varphi(x) = B_1^{(1)}\ x\ \sum_{n=0}^{\infty} b_n^{(1)}\ x^n \end{cases} \tag{3.5.a}$$

$$m \geq 1: \quad \begin{bmatrix} H_r(x) \\ H_\varphi(x) \end{bmatrix} = A_1^{(1)} x^{m+1} \sum_{n=0}^{\infty} \begin{bmatrix} a_n^{(1)} \\ b_n^{(1)} \end{bmatrix} x^n + A_1^{(2)} x^{m-1} \sum_{n=0}^{\infty} \begin{bmatrix} a_n^{(2)} \\ b_n^{(2)} \end{bmatrix} x^n \qquad (3.5.b)$$

where A_1, B_1 are the integration constants in (3.5). In order to define and calculate all the intervening coefficients, the following algorithm is used:

(3.6)

Solution (1):

$a_0^{(1)} = -b_0^{(1)} = 1$

$n \geq 1$, $\left(a_n^{(1)}, b_n^{(1)}\right)$ calculated using (3.3) with $\gamma = m+1$

Solution (2):

$a_0^{(2)} = b_0^{(2)} = 1$

$a_1^{(2)} = b_1^{(2)} = 0$

$a_2^{(2)} = b_2^{(2)} = -s_0 / 4m$

$n \geq 3$, $\left(a_n^{(2)}, b_n^{(2)}\right)$ calculated using (3.3) with $\gamma = m - 1$

4. TAYLOR'S SOLUTIONS IN AN INTERMEDIATE MEDIUM (J)

If $x_0 > 0$ it is possible to probe that $\gamma = 0$ in the pair of functions in (2.3), [4]. So, the Taylor's solutions are given by:

$$\begin{bmatrix} H_r(x) \\ H_\varphi(x) \end{bmatrix} \equiv \begin{bmatrix} H_r(x - x_0) \\ H_\varphi(x - x_0) \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} a_n \\ b_n \end{bmatrix} (x - x_0)^n \qquad (4.1)$$

Making $x = (x - x_0) + x_0$ in the differential equations (2.1), with $s(x) = s(x - x_0)$ and $g(x) = g(x - x_0)$, applying to those equations the pair of functions in (4.1) and making zero the coefficient of $(x - x_0)^n$ in the series expansions, it results [3]:

(4.2)

$$x_0^2 \, n \, (n-1) \begin{bmatrix} a_n \\ b_n \end{bmatrix} + \begin{bmatrix} S(a) \\ S(b) - G(b) \end{bmatrix} = 0$$

$$\begin{bmatrix} S(a) \\ S(b) \end{bmatrix} = x_0 (n-1)(2n-3) \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} + \left[(n-2)^2 - (m^2+1)\right] \begin{bmatrix} a_{n-2} \\ b_{n-2} \end{bmatrix} + 2m \begin{bmatrix} b_{n-2} \\ a_{n-2} \end{bmatrix} +$$

$$+ \sum_{i=0}^{n-4} s_{n-4-i} \begin{bmatrix} a_i \\ b_i \end{bmatrix} + 2\, x_0 \sum_{i=0}^{n-3} s_{n-3-i} \begin{bmatrix} a_i \\ b_i \end{bmatrix} + x_0^2 \sum_{i=0}^{n-2} s_{n-2-i} \begin{bmatrix} a_i \\ b_i \end{bmatrix}$$

$$G(b) = \sum_{i=0}^{n-3} g_{n-3-i} \left[(i+1) b_i - m\, a_i\right] + x_0 \sum_{i=0}^{n-2} g_{n-2-i} \left[(2i+1) b_i - m\, a_i\right] + x_0^2 \sum_{i=0}^{n-1} g_{n-1-i} \, i \, b_i$$

The Taylor's solutions in an intermediate medium will have the form:

(4.3.a)

$$m = 0: \begin{cases} H_r(x) = A_j^{(1)} h_r^{(1)}(x - x_0) + A_j^{(3)} h_r^{(3)}(x - x_0) \\ H_\varphi(x) = B_j^{(1)} h_\varphi^{(1)}(x - x_0) + B_j^{(3)} h_\varphi^{(3)}(x - x_0) \end{cases}$$

$$m \geq 1: \begin{bmatrix} H_r(x) \\ H_\varphi(x) \end{bmatrix} = \sum_{i=1}^{4} A_j^{(i)} \begin{bmatrix} h_r^{(i)}(x - x_0) \\ h_\varphi^{(i)}(x - x_0) \end{bmatrix}$$

where the A_j, B_j are the integrations constants and the functions $h_r(x - x_0)$, $h_\varphi(x - x_0)$ are defined as follows

(4.3.b)

$$\begin{bmatrix} h_r^{(i)}(x-x_0) \\ h_\varphi^{(i)}(x-x_0) \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} a_n^{(i)} \\ b_n^{(i)} \end{bmatrix} (x-x_0)^n \ , \quad \text{with } i = 1, 2, 3, 4$$

Now, all the intervening coefficients in (4.3) are defined and calculated using the following algorithm [3]:

(4.4)

Solution (1): $a_0^{(1)} = -b_0^{(1)} = 1,\ a_1^{(1)} = b_1^{(1)} = 0,\ n \geq 2 : (a_n^{(1)}, b_n^{(1)})$ calculated using (4.2)

Solution (2): $a_0^{(2)} = b_0^{(2)} = 1,\ a_1^{(2)} = b_1^{(2)} = 0,\ n \geq 2 : (a_n^{(2)}, b_n^{(2)})$ calculated using (4.2)

Solution (3): $a_0^{(3)} = b_0^{(3)} = 0,\ a_1^{(3)} = -b_1^{(3)} = 1,\ n \geq 2 : (a_n^{(3)}, b_n^{(3)})$ calculated using (4.2)

Solution (4): $a_0^{(4)} = b_0^{(4)} = 0,\ a_1^{(4)} = b_1^{(4)} = 0,\ n \geq 2 : (a_n^{(4)}, b_n^{(4)})$ calculated using (4.2)

5. FROBENIUS' SOLUTIONS IN THE OUTER MEDIUM (K)

In the outer medium, homogeneous (the cladding), it is well-known that finite-valued solutions at $x = \infty$ must be used, named as "proper solutions", related with the Frobenius' series expansions in (3.1) with no positive values of γ in (3.4). Those solutions for $H_r(x)$ and $H_\varphi(x)$ will be defined as follows:

(5.1.a)

$$m = 0: \quad \begin{cases} H_r(x) = A_k^{(3)}\ h_{pr}^{(3)}(x) \\ H_\varphi(x) = B_k^{(3)}\ h_{p\varphi}^{(3)}(x) \end{cases}$$

(5.1.b)

$$m \geq 1: \quad \begin{bmatrix} H_r(x) \\ H_\varphi(x) \end{bmatrix} = A_k^{(3)} \begin{bmatrix} h_{pr}^{(3)}(x) \\ h_{p\varphi}^{(3)}(x) \end{bmatrix} + A_k^{(4)} \begin{bmatrix} h_{pr}^{(4)}(x) \\ h_{p\varphi}^{(4)}(x) \end{bmatrix}$$

In (5.1) A_k and B_k are the integration constants and the functions $h_{pr}(x)$, $h_{p\varphi}(x)$ are the proper solutions (note the lower index p, just from "proper"). Those functions has the same form as in (3.1) on the cutoff condition ($\beta^2 = \varepsilon_k$), as all the n-order coefficients are zero-valued except for n = 0 (in fact $s_k = g_k = 0$ in (3.3), and so $a_n = b_n = 0$ if $n>0$). In the propagation condition ($\beta^2>\varepsilon_k$), the proper solutions are built as in [4], and series expansions with coefficients (a'_n, b'_n) are involved. The results are:

Cutoff condition $(\beta^2 = \varepsilon_k)$:

(5.2.a)

$$\begin{bmatrix} h_{pr}^{(\ell)}(x) \\ h_{p\varphi}^{(\ell)}(x) \end{bmatrix} = \begin{bmatrix} x^{-\alpha_\ell} \\ (-1)^\ell\ x^{-\alpha_\ell} \end{bmatrix} \quad : \quad \ell = 3, 4 \quad \begin{cases} \ell = 3,\ \alpha_\ell = \alpha_3 = -(m+1) \\ \ell = 4,\ \alpha_\ell = \alpha_4 = -|m-1| \end{cases}$$

Propagation condition $(\beta^2 > \varepsilon_k)$:

(5.2.b)

$$m = 0: \quad \begin{cases} H_r(x) = A_k^{(3)}\ h_p^{(3)}(x) \\ H_\varphi(x) = B_k^{(3)}\ h_p^{(3)}(x) \end{cases}$$

$$m \geq 1: \quad \begin{bmatrix} H_r(x) \\ H_\varphi(x) \end{bmatrix} = A_k^{(3)} \begin{bmatrix} h_p^{(3)}(x) \\ -h_p^{(3)}(x) \end{bmatrix} + A_k^{(4)} \begin{bmatrix} h_p^{(4)}(x) \\ h_p^{(4)}(x) \end{bmatrix}$$

(5.2.c)

$$\ell = 3,4 \quad \begin{cases} \ell = 3\,, \ \alpha_\ell = \alpha_3 = -(m+1) \\ \ell = 4\,, \ \alpha_\ell = \alpha_4 = -|m-1| \end{cases}$$

$$h_p^{(\ell)}(x) = F_{\ell-2}\left[(R + \ln x)\ x^{\alpha_\ell} \sum_{n=0}^{\infty} a_{2n}^{(\ell-2)} x^{2n} + x^{\alpha_\ell} \sum_{n=0}^{\infty} a'^{(\ell-2)}_{2n} x^{2n}\right] - x^{-\alpha_\ell} \sum_{n=0}^{\alpha_\ell - 1} a_{2n}^{(\ell)}\ x^{2n}$$

$$R = \gamma_0 + \ln\left[\frac{1}{2}\sqrt{-s_0}\right] - \frac{1}{2}\sum_{i=1}^{\alpha_\ell}\frac{1}{i}$$

$$\gamma_0 = .5772\ldots \quad \text{(Euler constant)}$$

Now, for n > 0 all the intervening coefficients in (5.2.c) are calculated using

(5.3)

$$\begin{cases} 4n(\gamma + n) a_{2n} = -s_0\, a_{2n-2} \\ 4n(\alpha + n) a'_{2n} = -2(\alpha + 2n) a_{2n} - s_0\, a'_{2n-2} \end{cases}$$

and the proper solutions, for each value of ℓ in (5.2), are evaluated as follows:

(5.4)

Solution (3): $\ell = 3: \ \alpha = \alpha_1 = m+1\,, \quad m \geq 0$

$a_0^{(1)} = 1\,, \ a'^{(1)}_0 = 0\,, \ \text{for } n \geq 1: \ (a_{2n,}^{(1)}, a'^{(1)}_{2n})$ calculated using (5.3) with $\gamma = \alpha_1$

$a_0^{(3)} = 1\,, \ \text{for } 1 \leq n \leq m: \ a_{2n}^{(3)}$ calculated by (5.3) with $\gamma = -\alpha_1$, $F_1 = \dfrac{s_0\, a_{2m}^{(1)}}{2\alpha_1}$

Solution (4): $\ell = 4: \ \alpha = \alpha_2 = m - 1 \geq 0$

$m \geq 1: a_0^{(2)} = 1\,, \ a'^{(2)}_0 = 0\,, \ \text{for } n \geq 1: \ (a_{2n,}^{(2)}, a'^{(2)}_{2n})$ calculated by (5.3) with $\gamma = \alpha_2$

$\text{if } m = 1: \ a_0^{(4)} = 0\,, \ F_2 = 1$

$\text{if } m \geq 2: \ a_0^{(4)} = 1\,, \ \text{for } 1 \leq n \leq m-2: a_{2n}^{(4)}$ calculated by (5.3) with $\gamma = -\alpha_2$, $F_2 = \dfrac{s_0\, a_{2m-4}^{(4)}}{2\alpha_2}$

It is possible to verify that the proper functions in (5.2) are proportional to the modified Bessel functions of second kind K_α [5], as follows:

(5.5)

$$K_\alpha(\sqrt{-s_0}\ x) = \begin{cases} -h_p^{(3)}(x)\,, & \alpha = 0 \\ -\dfrac{1}{2}(\alpha - 1)!\left[\dfrac{1}{2}\sqrt{-s_0}\ x\right]^{-\alpha} h_p^{(\ell)}(x)\,, & \alpha \geq 1 \end{cases}$$

6. MODAL EQUATION

The modal equation is obtained making zero the system determinant obtained from the boundary conditions at points x_j in Fig.1, with $1 \leq j \leq k-1$. The boundary conditions will be the continuity of $H_r(x)$, $H_\varphi(x)$, $H_z(x)$, $E_z(x)$. From the Maxwell equations, the x-variation of the longitudinal components $H_z(x)$ and $E_z(x)$ are given by [3]:

(6.1)

$$\begin{bmatrix} j(k_0a)\beta H_z(x) \\ j(k_0a)\dfrac{\varepsilon(x)}{Z_0}E_z(x) \end{bmatrix} = \left(\frac{1}{x}+\frac{d}{dx}\right)\begin{bmatrix} H_r(x) \\ H_\varphi(x) \end{bmatrix} - \frac{m}{x}\begin{bmatrix} H_\varphi(x) \\ H_r(x) \end{bmatrix}$$

where $Z_0 = 120\pi\,[\Omega]$ is the free-space wave resistance.

As the transversal field components $H_r(x)$, $H_\varphi(x)$ are linear combinations of the elementary functions $h_r(x)$, $h_\varphi(x)$ early defined, the longitudinal field components in (6.1) will be also linear combinations of the elementary functions $h_z(x)$, $e_z(x)$, defined as follows:

(6.2)

$$\begin{bmatrix} h_z(x) \\ e_z(x) \end{bmatrix} = \left(\frac{1}{x}+\frac{d}{dx}\right)\begin{bmatrix} h_r(x) \\ h_\varphi(x) \end{bmatrix} - \frac{m}{x}\begin{bmatrix} h_\varphi(x) \\ h_r(x) \end{bmatrix}$$

(this both for Frobenius' and Taylor's solutions, those with argument "x-x_0")

Imposing the continuity of the H_r, H_φ, H_z, E_z field components in each boundary points in the Fig.1, the following system of equations is obtained:

(6.3)

$$[M]\,[A]^T = 0$$

where $[A]^T$ is the integrations constants' column matrix, and $[M]$ is a square matrix. The modal equation is obtained imposing

(6.4)

$$D = \det[M] = 0$$

and a simple criterium for hybrid modes identification (HE or EH if m > 0) can be used [3], just comparing the absolute value of the integration constants in the outer medium (k), as follows:

(6.5)

$$\text{HE modes: } \left|A_k^{(3)}\right| < \left|A_k^{(4)}\right| \qquad \text{EH modes: } \left|A_k^{(3)}\right| > \left|A_k^{(4)}\right|$$

7. AN EXAMPLE

It follows an example of index profile fiber (perhaps not very interesting for practical applications but suitable for exemplify the algorithm application), entering with fiber and modal parameters (defined in the point 7.2).

7.1. Profile definition

Consider an index profile for witch three media are defined, see Fig.2:

inner medium(1), $0 \le x \le x_1$:

$$\varepsilon_1(x) = \varepsilon_m + (\varepsilon_M - \varepsilon_m)\,x/x_1$$

intermediate medium (2), $x_1 \le x \le x_2$:

$$\varepsilon_2(x) = \varepsilon_m + (\varepsilon_M - \varepsilon_m)(x - x_2)/(x_1 - x_2)$$

outer medium (3), $x \ge x_2$: $\varepsilon_3(x) = \varepsilon_k$

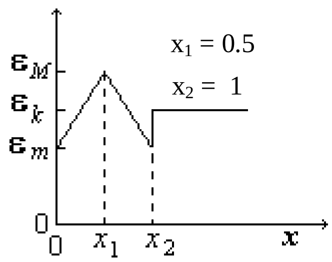


Figure 2: The fiber profile

The Taylor's coefficients ε_i of he function $\varepsilon(x)$, for each one of the considered media in the Fig.2, are given by:

(7.1)

inner medium (1): $\varepsilon_0 = \varepsilon_m$, $\varepsilon_1 = (\varepsilon_M - \varepsilon_m)/x_1$, $\varepsilon_i = 0\ \forall i \geq 2$

intermediate medium (2):

$\varepsilon_0 = \varepsilon_m + (\varepsilon_M - \varepsilon_m)(x_0 - 1)/(x_1 - 1)$, $\varepsilon_1 = (\varepsilon_M - \varepsilon_m)/(x_1 - 1)$, $\varepsilon_i = 0\ \forall i \geq 2$

outer medium (k): $\varepsilon_0 = \varepsilon_k$, $\varepsilon_i = 0\ \forall i \geq 1$

7.2. Profile and modal parameters

Defining the profile parameters Δ and Δ_t as follows:

(7.2.a)

$$\Delta = \frac{\varepsilon_M - \varepsilon_k}{\varepsilon_M} \qquad \Delta_t = \frac{\varepsilon_M - \varepsilon_m}{\varepsilon_M}$$

and considering also the modal parameters V, b:

(7.2.b)

$$V = (k_0 a)\sqrt{\varepsilon_M - \varepsilon_k}\ , \quad b = \frac{\beta^2 - \varepsilon_k}{\varepsilon_M - \varepsilon_k}$$

the s and g Taylor's coefficients, according to (2.2), will be given by:

(7.3)

inner medium (1):

$$s_0 = V^2\left(1 - \frac{\Delta_t}{\Delta} - b\right), \quad s_1 = V^2 \frac{\Delta_t}{\Delta}\frac{1}{x_1}, \quad s_n = 0\ \forall n \geq 2$$

$$g_0 = \frac{\varepsilon_1}{\varepsilon_0}, \quad g_n = -\frac{1}{\varepsilon_0} g_{n-1}\varepsilon_1 = -g_0 g_{n-1}\ \forall n \geq 1$$

intermediate medium (2):

$$s_0 = V^2\left(1 - b + \frac{\Delta_t}{\Delta}\frac{x_0 - x_1}{x_1 - 1}\right), \quad s_1 = V^2 \frac{\Delta_t}{\Delta}\frac{1}{x_1 - 1}, \quad s_n = 0\ \forall n \geq 2$$

$$g_0 = \frac{\varepsilon_1}{\varepsilon_0}, \quad g_n = -\frac{1}{\varepsilon_0} g_{n-1}\varepsilon_1 = -g_0 g_{n-1}\ \forall n \geq 1$$

outer medium (3):

$$s_0 = -V^2 b\ , \quad s_n = 0\ \forall n \geq 1\ , \quad g_n = 0\ \forall n$$

Modal solutions are evaluated using both high and small values for the profile parameters in (7.2.a), as follows:

(7.4)

$$(\Delta_t = 20\%, \Delta = 10\%) \ \text{ and } \ (\Delta_t = 0.2\%, \Delta = 0.1\%)$$

7.3 Numerical criteria

For computation proposes, it is considered that:

- the solution (V, b), in (7.2.b), is found when the determinant value D of M in the modal equation (6.3) obey to $|D| \leq 10^{-6}$. Note that M is a 4×4 matrix for transversal TE and TM modes, and a 8×8 matrix for hybrid HE and EH modes.
- the convergence of a series expansion is obtained when $|T_n / S_n| \leq 10^{-8}$, where T_n is the n^{th} term and $S_n = \sum_{i=0}^{n} T_i$.

7.4 Results

The cutoff value (b = 0) of the normalised frequency V are presented in Table 1, for the first 20 modes, and Fig.3 shows a plot of modal solutions. Now, in Fig.3, it is interesting to note that:

1. For very small index differences, the set of modes ($HE_{L+1,}$ $EH_{L-1,n}$) and (TE_{0n} , TM_{0n} and HE_{2n}) have the same cutoff value (see Table 1) and coincident modal lines (see Fig.3). In fact, they are degenerated and can be considered as the LP_{Ln} modes.
2. The TE_{01} and LP_{11} lines are the same for both groups of Δ and Δ_t values, and this will occur for all the TE_{0n} and LP_{1n} modes. In fact, according to (6.2) and (2.1.a) with *m* = 0, the field components for TE modes only depends on the function *s(x)*, and so on the value of Δ_t / Δ = 2 for both of the considered cases.

Table 1: V cutoff values

	Δ_t = 20% Δ = 10%	Δ_t = 0.2% Δ = 0.1%
HE_{11}	1.296	0.122
TE_{01}	5.154	5.154
HE_{21}	5.256	5.155
TM_{01}	5.372	5.156
EH_{11}	7.348	7.277
HE_{31}	7.371	7.277
EH_{21}	9.355	9.310
HE_{41}	9.402	9.310
EH_{31}	11.351	11.325
HE_{51}	11.416	11.326
EH_{41}	13.348	13.336
HE_{61}	13.427	13.337
HE_{12}	14.089	14.093
HE_{22}	14.764	14.779
TE_{02}	14.779	14.779
TM_{02}	14.806	14.779
EH_{51}	15.347	15.346
HE_{71}	15.437	15.347
HE_{32}	16.155	16.167
EH_{12}	16.184	16.167

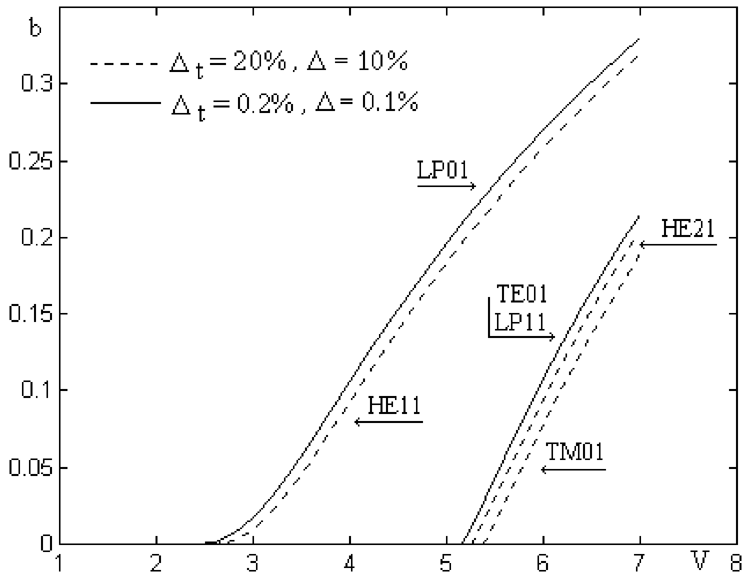


Figure 3: Modal lines for high and very small step index differences.

8. CONCLUSIONS

Using Frobenius and Taylor's series expansions as solutions of radial wave equations for field components, a fast convergent algorithm is presented for calculating modal characteristics. Important aspects are the possible use in a large frequency range, and the inclusion of fast radial variation of the refractive index. A set of modal solution points can be easily obtained for any propagation mode, allowing the evaluation and graphical plots of another parameters of the fiber, just as in [6], [7] and [8]. It is important to point out that FTM is full explained in this paper and can be applied in this domain of inhomogeneous fibers without searching another reference.

9. REFERENCES

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