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Hunting Admissible Kneading pairs of a Real Rational Map

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ABSTRACT— ***The importance of symbolic systems is that they give us the possibility of simplifying some dynamical systems. Our expectations is that our work can help to understand better the behavior, under iteration, of the Real Rational Map, f(x)=(x²-a)/(x²-b), with 0<b<a<1. Kneading Theory is a powerful tool, and we use it to simplify processes arising from the complicated behavior of this map in a more simple way. In this work we establish some rules that bound regions where we can find, or not, Kneading pairs for f(x).***

Keywords— Symbolic Dynamics, Real Rational Maps, Kneading Theory

1. INTRODUCTION

Historically, the matter that today we call "Symbolic Dynamics" appeared with the french mathematician Jacques Hadamard in one attempt to simplify one problem: the use of sequences of symbols to study the distribution of geodesics in certain surfaces. Later in the years 1930-1940, Arnold Hedlund and Marston Morse developed the method in their studies of geodesics of negative curvature, designating these ideas as Symbolic Dynamics.

Using all the work already developed to logistic maps, or polynomial maps with degree bigger than two, that we can find on the scientific publications related to the study of its iteration, and the work of Cabral [2] that describes a small difference on the behavior of rational maps from these ones, this work intends to be a small contribution to understand better the behavior, under iteration, of the Real Rational Map, $f(x)=\left(x^2-a\right)/\left(x^2-b\right)$, and for start we work only with the parameters $0<b<a<1$, without any loss of generalization. Milnor had published a lot of work regarding the dynamics of real rational maps, but all of them are continuous maps, and with rational degree bigger than two. In general, Milnor worked with real maps that are topological conjugate to the logistic map. Our maps are clearly rational maps discontinuous and with rational degree zero, as the case study by Cabral [2]. So we intend to add, with this paper, more information to the knowledge of how these kinds of maps *f(x)* behave.

Adler in [1], made the question "How and to what extent can a dynamical system be represented by a symbolic one?" This question as a very difficult incompatibility to solve: as, usual, the time evolution models are presented in a phase space of differential varieties but any symbolic system is totally disconnected. So, it is impossible to associate this two realities in one-to-one relation, but as said by Marcus, [3] "A symbolic system will be, always, one good approximation of the model that we want to study.". The main question will be always to find the best possible approximation, that is, to define the set of points on the phase space that can be considered equivalent and so represented by an unique symbol. This identification is possible trough the Markov partitions.

So, we will use Markov partitions and Kneading Theory to accomplish our goal of codifying the iteration under *f(x)* of one object.

The most interesting symbolic systems are the ones, from a practical point of view, that allow us without many difficulties to characterize their dynamical aspects, and our map *f(x)* is one of this cases.

To avoid confusion in symbols, during this work, we use the underlined symbol $\underline{N}$ to represent the natural numbers set, and the underlined symbol $\underline{R}$ to represent the real numbers set. We also will differentiate the use of the symbols and "$<$", "$\leq$" using them when the order is the usual on the real numbers set, and will use them slightly modified as "$\prec$", "$\preccurlyeq$", respectively, when ordering elements on the symbolic set *S*.

2. INITIAL BACKGROUND

The fundamental ingredient of a symbolic system is, naturally, the set of symbols *A*, that we call alphabet. This set *A* has a finite number of elements. The simplest symbolic system that we can create is the full shift system, that is, the space of all infinite sequences of symbols in *S*, with $S^{\underline{N}}=\{x=(x_i):x_i\in S,\ i=1,2,3,\ldots\}$.

This space works as the set of orbits of the dynamical system, and the alphabet S is its phase space. With the introduction of the shift application σ: $S^{\underline{N}} \to S^{\underline{N}}$, where each element i of the sequence $y=\sigma(x)$ is given by $y_i=x_{i+1}$, we can recuperate usual concepts of dynamical systems as the periodic orbits of period p identified as $\sigma^{k+p}(x)=\sigma^k(x)$, for all $k, p \in \underline{N}$. The symbolic system for which is not allowed any symbol repetition is a sub shift.

From all the possible sequences generated by combining the elements of S, using standard combinatory, as we can see in Milnor and Thurston [5], only a few describe the dynamics of a map f. We will call these the group of "admissible sequences". If we denote Σ_f as the space of all admissible sequences for f, then we have $\sigma(\Sigma_f)=\Sigma_f$. In the construction of symbolic spaces Σ_f, the family of words that are finite receives the name of sub shift of finite type. So, given a symbolic system $\Sigma=\Sigma_f$, let $B_m(\Sigma)$ be the set of all words of length m present in the sequences of Σ. The complexity of a symbolic system Σ will be related with the growth rate of the number of elements of $B_m(\Sigma)$ as the length m grows to infinite, see Milnor and Thurston [5]. This number gives an interpretation of the growing of diversity of sequences, that is, the dynamics of the system.

The following definition 2.1, definition 2.2 and theorem 2.1 are due to Milnor and Thurston [5], but they can be found also in the works of Adler [1], Martins, Severino and Sousa Ramos [4], among others.

Definition 2.1: Let Σ_f be a symbolic space. We call entropy of Σ_f to the limit

$$h(\Sigma_f) = \lim_{m\to\infty} \frac{1}{m} \log_2 \left|B_m(\Sigma_f)\right|,$$

where $\left|B_m(\Sigma_f)\right|$ represents the number of elements of $B_m(\Sigma_f)$.

Since $\Sigma_f \subset S^{\underline{N}}$, then $|B_m(\Sigma_f)|\le|B_m(S^{\underline{N}})|=|S|^m$, and we have $h(\Sigma_f)\le log_2|S|$, the upper limit to entropy.

Definition 2.2: Let $A=(a_{ij})$, the *n-dimension* square matrix with elements in the set {0,1}. Using the alphabet S, we call Topological Markov Chain *(TMC)* to the symbolic system Σ_A with elements $x=(x_i)$ such that $a_{x_i,x_{i+1}}=1$.

Theorem 2.3: (Adler [1]) Let Σ_A be a *TMC*, with an irreducible matrix A. Then its topological entropy is given by $h(\Sigma_A)=log(\lambda_p(A))=\rho(A)$ where $\lambda_p(A)$ is the *Perron Eigen Value* of the matrix A, and $\rho(A)$ the spectral radius of A.

3. CODIFYING THE PHASE SPACE

Codifying the phase space of the dynamical system is the fundamental point to initiate the symbolic representation of its dynamics. It is necessary to identify the critical points, the discontinuity points and the discontinuity points of the first derivative. In this article we study the dynamics of the map $f(x) = \left(x^2-a\right)/\left(x^2-b\right)$.

So, let's define the sets $M=\{x \in \underline{R}: x<-\sqrt{b}\}$, $D1=\{-\sqrt{b}\}$, $L=\{x \in \underline{R}: -\sqrt{b}<x<0\}$, $C=\{0\}$, $R=\{x \in \underline{R}: 0<x<\sqrt{b}\}$, $D2=\{\sqrt{b}\}$, $N=\{x \in \underline{R}: \sqrt{b}<x<1\}$, $U=\{1\}$, and $F=\{x \in \underline{R}: x>1\}$.

Definition 3.1: Let $x \in \underline{R}$, and an application f, we define the itinerary $it(x)$ of x, under f, to the sequence of symbols of the kneading alphabet S, $it(x)=ad(x)ad(f(x))ad(f^2(x))...$ We have that $it(x)$ are the elements of Σ. ■

Example 3.1.1 For example, if $a=1$ and $b=0,5$, we have for the critical point $x=0$, the iterate $it(0)=ad(0)ad(f(0))ad(f^2(x))\ldots=CFN\ldots$, having $\sigma(it(0))=FN\ldots$.

Definition 3.2: We define parity of a sequence $(S_1 S_2... S_n)$ as $\rho(S_1 S_2... S_n)$ with $\rho(S_1 S_2... S_n)= -1$ if the sum of the symbols M and L in $(S_1 S_2... S_n)$ is odd, and $\rho(S_1 S_2... S_n)= +1$ if it is even.■

Example 3.2.1 We have $\rho(FNM)=-1$ and $\rho(FNML)=+1$.

Definition 3.3: The relation of order in the set Σ is defined by $M \lhd D1 \lhd L \lhd C \lhd R \lhd D2 \lhd N \lhd U \lhd F \lhd \infty$, ordering the elements of S, following their respective positions on the real axis.

Let $P, Q \in \Sigma$, $P_n \neq Q_n$. We say that $P \prec Q$ *if* $\rho(P_1P_2...P_{n-1})=+1 \wedge P_n \prec Q_n$ or $\rho(P_1P_2...P_{n-1})= -1 \wedge Q_n \prec P_n$. ■

Example 3.3.1: If we want to order the sequences *FNMC* and *FNML*, we start with the first position and we see that they a common symbol, as in the second and third position, but since $L \lhd C$, we have $FNML \lhd FNMC$.

The symbolic order is related with the natural order of the real numbers by the lemma 3.4.

Lemma 3.4 (Sousa Ramos [7]) If f is a modal application on the interval, and if $x, y \in I \subset \underline{R}$, $x \neq y$, then $x<y \Rightarrow it(x) \preccurlyeq it(y)$ and $it(x) \prec it(y) \Rightarrow x<y$. ■

Obviously not all elements of Σ are iterations of $x \in \underline{R}$, under some application f. To the ones that correspond to the iterations of some point we will call it admissible sequences.

The relevance that symbolic dynamics achieved in the last two decades is due to the works of Milnor and Thurston [5] and Sousa Ramos [7]. Milnor and Thurston gave us the possibility of the classification of the dynamics through the iterations of some points, using kneading sequences. To study the dynamics of $f(x)$ we need to study the dynamics of $x=-\sqrt{b}$, $x=0$ and $x=\sqrt{b}$. But, since $f^2(-\sqrt{b})=f^2(\sqrt{b})=1$, we will have $f^3(\pm\sqrt{b})=f(1)$ so it is enough to study the point $x=1$ and the critical point $x=0$ to understand the behavior of $f(x)$ and we will do it at the same time defining a Kneading Invariant that uses both $it(1)$ and $it(f(0))$, joined in a pair of elements.

Definition 3.5: Let f be an application. We define the kneading invariant of f, $K(f)$ as the pair $K(f)=(K_0, K_1)$ with $K_0=it(f(0))$ and $K_1=it(1)$. ■

Proposition 3.6: If $x \in \underline{R}$ and f a modal application in each interval of its domain, we have $it(f(x)) \preccurlyeq K_1$ or $K_0 \preccurlyeq it(f(x))$, with $K(f)=(K_0,K_1)$. ■

Proof: Let $f(x)=\left(x^2-a\right)/\left(x^2-b\right)$, with $0<b<a<1$. If $x \in]-\sqrt{b},\sqrt{b}[$, the function f is a modal application on the interval, and we can see that $f(x) \geq f(0)$ and $f(x) \geq 1$. We have by lemma 3.4 that $f(0) \leq f(x) \Rightarrow it(f(0)) \preccurlyeq it(f(x))$, but $it(f(0))=K_0$, so $K_0 \preccurlyeq it(f(x))$. If $x \notin]-\sqrt{b},\sqrt{b}[$, we have $f(x)<1$, and again using lemma 3.4 we have $f(x)<1 \Rightarrow it(f(x)) \prec it(1)$. But, $it(1)=K_1$, so $it(f(x)) \prec K_1$. ■

One of the main contributions of Sousa Ramos [7] in this area was the construction, through the kneading invariant of the application, of a Markov Matrix. Indeed, for any pair of finite kneading sequences, it is possible to find a Markov partition of the phase space, that is, a finite collection $C=\{I_0, I_1,\ldots,I_n\}$ of disjoint open sets, such the closure of its union matches all phase space, and the image by the application of each one of this sets is the union of some elements of C. We have $I_j \cap I_k = \emptyset,\ j \neq k$, and $\bigcup_{j=0}^{n} I_j = \mathbb{R}$ with $f(I_j)=\bigcup_{j_1}^{j_m} I_k$.

The Markov transition matrix $A_{K(f)}=(a_{ij})$ associated to the kneading invariant is defined as $a_{ij}=1$, if $f(I_i) \supset I_j$ and $a_{ij}=0$ otherwise.

Example 3.7: In our function f, for $a \approx 0{,}543237$ and $b \approx 0{,}317003$ we have $K(f)=(K_0,K_1)=(FND2,UNMC)$, a periodic orbit of period 8. Making $x_0=0$ and $y_0=\infty$, with $x_{n+1}=f(x_n)$ and $y_{n+1}=f(y_n)$, we have the following values ordered in the real axis: $-\infty<y_3<-\sqrt{b}<0<\sqrt{b}<y_2<x_2<1<x_1<\infty$. Now we can create the finite collection $C=\{I_0,I_1,I_2,I_3,I_4,I_5,I_6,I_7,I_8\}$ with $I_0=]-\infty,y_3[$, $I_1=]y_3,-\sqrt{b}[$, $I_2=]-\sqrt{b},0[$, $I_3=]0,\sqrt{b}[$, $I_4=]\sqrt{b},y_2[$, $I_5=]y_2,x_2[$, $I_6=]x_2,1[$, $I_7=]1,x_1[$ and $I_8=]x_1,\infty[$. Applying f we have $f(I_0)=I_3 \cup I_4 \cup I_5 \cup I_6$, $f(I_1)=I_0 \cup I_1 \cup I_2$, $f(I_2)=I_8$, $f(I_3)=I_8$, $f(I_4)=I_0$, $f(I_5)=I_1 \cup I_2 \cup I_3$, $f(I_6)=I_4$, $f(I_7)=I_5$ and $f(I_8)=I_6$. Then the transition matrix of the phase space is

$$A_{(FN\mathbf{D2},UNMC)} = \begin{bmatrix} 0&0&0&1&1&1&1&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&1&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \end{bmatrix}$$

and using the spectral radius we can calculate its growth number $s \approx 1,80709$.

4. THE SYMBOLIC WAY

As is well known, the numerical calculus, especially if we deal with iterations of rational functions, can return a lot of errors hard to control, due to the sensitive iteration of small values under f. So it is important that if we could build the transition matrix using only the symbolic space and its dynamics.

In our function f, we have periodic orbits of $x=0$ and $x=1$, that belong to the same orbit, characterized by Cabral [2], where it is given the influence on the calculus of entropy of the function f, but we have also distinct orbits, or one orbit is periodic and the other pre-periodic, telling us that the study of the function f dynamics will be a challenging work in the near future. So, in this paper we will dedicate our attention only to the cases where $K(f)=(K_0, K_1)$ is a pair of finite sequences.

Let (x_{ik}) be the sequences given by $x_{0k}=\sigma^k(K_0)$, with $k=1,2,\ldots n_0$ and $x_{1k}=\sigma^{1k}(K_1)$, with $k=1,2,\ldots n_1$. The values n_0 and n_1 are the length of K_0 and K_1, respectively. The points (x_{ik}) will belong to real intervals where the function is increasing or decreasing. We calculate the symbols iteration using the following rules:

$$\begin{array}{|c|c|} \text{(a) if } f \text{ is increasing} & \text{(b) if } f \text{ is decreasing} \\ f(]x_{ik},x_{pq}[=]x_{i,k+1};x_{p,q+1}[& f(]x_{ik},x_{pq}[=]x_{p,q+1};x_{i,k+1}[\\ f(]x_{ik},\mathbf{D2}[=]x_{i,k+1};+\infty[& f(]-\infty,x_{ik}[=]x_{i,k+1};U[\\ f(]\mathbf{D2},x_{ik}[=]-\infty;x_{i,k+1}[& f(]x_{ik},\mathbf{D1}[=]-\infty;x_{i,k+1}[\\ f(]x_{ik},+\infty[=]x_{i,k+1};U[& f(]\mathbf{D1},x_{ik}[=]x_{i,k+1};+\infty[\end{array}$$

Then we build the transition matrix A and can calculate the growth number. To understand better this process let us see the next example.

Example 4.1: Let $K(f)=(FNNMC,UNMMD2)$. So we have $K_0=CFNNMC$, $K_1=\infty UNMMD2$ and through shifting

$$\begin{array}{c|c} x_{01}=\sigma^1(K_0)=FNNMC & x_{11}=\sigma^1(K_1)=UNMM\mathbf{D2} \\ x_{02}=\sigma^2(K_0)=NNMC & x_{12}=\sigma^2(K_1)=NMM\mathbf{D2} \\ x_{03}=\sigma^3(K_0)=NMC & x_{13}=\sigma^3(K_1)=MM\mathbf{D2} \\ x_{04}=\sigma^4(K_0)=MC & x_{14}=\sigma^4(K_1)=M\mathbf{D2} \\ x_{05}=\sigma^5(K_0)=C & x_{15}=\sigma^5(K_1)=\mathbf{D2} \end{array}$$

Comparing the first symbol of each x_{ik} and using the monotonicity of f we can order them on the real axis. We can see that x_{14}, x_{04}, and x_{13} are the lowest values since that f is decreasing in M. Next we look at the second symbol, and we see that $M \lhd C \lhd D2$ in the real axis partition, but before f was decreasing so $D2 \lhd C \lhd M$, thus we have $MD2 \lhd MC \lhd MMD2$. Doing the same to x_{03}, x_{12} and x_{02} we can see that they have N as first symbol and f is increasing, there so, the order will be to the second symbol $M \lhd N$ and we have $x_{03}, x_{12} \lhd x_{02}$. Now we need to use the third symbol of x_{03} and x_{12}, regarding the aspect that NM has parity -1, so f is decreasing, and the order will be to the third symbol $C \lhd M$. In conclusion we have $x_{03} \lhd x_{12} \lhd x_{02}$ and repeating the process we have the Markov partition

$$x_{14} \lhd x_{04} \lhd x_{13} \lhd D1 \lhd x_{05} \lhd x_{15} \lhd x_{03} \lhd x_{12} \lhd x_{02} \lhd x_{11} \lhd x_{01}$$

and we can build the partition matrix

$$A_{(FNNMC,UNMM\mathbf{D2})} = \begin{bmatrix} 0&0&0&0&0&0&1&1&1&1&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&1&1&1&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 1&1&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

Until now this calculation is suppressing only the errors in the iteration of the elements, but we need to know K_0 and K_1 by numeric calculations, so its calculation can have also some errors. How to avoid this? Is there a way to calculate K_0 and K_1 without using numerical calculus? The answer is yes! The way is to build a set of rules that allow us to distinguish the admissible sequences from the ones that are not. This way is known as *Kneading Sequence Combinatory*, and this work intend to contribute to the generalization to the real rational functions, since it is already well known to other types of functions as the polynomial quadratic function, see Martins, Severino and Sousa Ramos [4], Marcus [3], Sousa Ramos [7].

5. KNEADING SEQUENCE COMBINATORY

As we can evaluate from the graphic of the function $f(x)=\left(x^2-a\right)/\left(x^2-b\right)$, with $0<b<a<1$, as seen on the example in Figure 1,

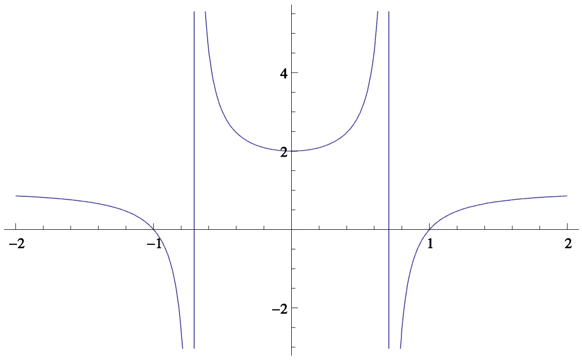

Figure 1: Graphic of $f(x)=(x^2-1)/(x^2-0.5)$

under iteration the symbol U can shift to R or $D2$ or N, that is, $U >> R, D2, N$. Resuming we have:

$M>>M, D1, L, C, R, D2, N$	$D2>>\infty$
$D1>>\infty$	$N>>M, D1, L, C, R, D2, N$
$L>>F$	$U>>R, D2, N$
$C>>F$	$F>>R, D2, N$
$R>>F$	$\infty>>U$

For simplify the results and to avoid long tables, we will consider only the sequences of 4 symbols for K_1 and 5 symbols for K_0, at maximum. And with this restriction we can build universal rules for all sequences K_0 and K_1, in order to both sequences be admissible.

The following proposition 5.1 and theorem 5.2. establish two rules in our construction that we call *Rule 1* and *Rule 2*, respectively.

Proposition 5.1: For $K(f)=(K_0,K_1)$ we have (1) $\sigma^n(K_0)\leqslant K_1$ or $K_0\leqslant\sigma^n(K_0)$, and (2) $\sigma^n(K_1)\leqslant K_1$ or $K_0\leqslant\sigma^n(K_1)$. ■

Proof: This result comes directly from the properties of the function f and its monotonicity. ■

Theorem 5.2: Let (K_0, K_1) be a admissible kneading sequence of the real rational function $f(x)=\left(x^2-a\right)/\left(x^2-b\right)$, with $0<b<a<1$, then the following chain of inequalities is satisfied

$$\ldots \prec \sigma^m(K_0) \prec \sigma^{m-2}(K_1) \prec \sigma^{m-1}(K_0) \prec \ldots \prec \sigma^2(K_1) \prec \sigma^3(K_0) \prec \sigma(K_1) \prec \sigma^2(K_0) \prec K_0 \prec \sigma(K_0).$$

The chain stops when the first symbol of one of the sequences $\sigma^m(K_0)$ or $\sigma^m(K_1)$ is not greater than N, for $x>\sqrt{b}$. ■

Proof: Since $b<a<1$ we have $f^2(0)<1<f(0)$, so $\sigma^2(K_0) \prec K_1 \prec \sigma(K_0)$. For $x>\sqrt{b}$, the function is increasing and working with lemma 3.4 we will have sequentially

$\sigma^2(K_0) \prec K_1 \prec \sigma(K_0);$

$\sigma^3(K_0) \prec \sigma(K_1) \prec \sigma^2(K_0) \prec K_1 \prec \sigma(K_0);$

$\sigma^4(K_0) \prec \sigma^2(K_1) \prec \sigma^3(K_0) \prec \sigma(K_1) \prec \sigma^2(K_0) \prec K_1 \prec \sigma(K_0)$

...

$\sigma^m(K_0) \prec \sigma^{m-2}(K_1) \prec \sigma^{m-1}(K_0) \prec \ldots \prec \sigma^2(K_0) \prec K_1 \prec \sigma(K_0)$. ■

The conditions of *Rule 2* to the initial iterates of $x=f(0)$ and $x=1$ allow us to characterize the symbolic space where, fixed a kneading sequence, will be possible to identify the existing kneading pairs (K_0,K_1). Comparing this to what happens in the symbolic space of bimodal applications in the interval, see Martins, Severino and Sousa Ramos [4], this *Rule 2* is a result with a new characteristic: it is possible to determine an upper and lower limit to the region where we can find the kneading pairs (K_0,K_1). Also the *Rule 2* presents a condition to a pair (K_0,K_1) be a kneading invariant.

Resulting from the theorem 5.2 we have the next corollary:

Corollary 5.2.1: Fixing a kneading sequence K_i, the kneading pairs will occupy a region in the symbolic set defined by $\sigma^m(K_i)$ and $\sigma^{m-1}(K_i)$ with m the lowest integer such that the first symbol of the sequence $\sigma^m(K_i)$ is inferior to N. ■

Proof: We know that N represent values inferior to one. So, by Theorem 5.1 we have

$\sigma^m(K_0) \prec \ldots \prec \sigma^{m-1}(K_0)$ and $\sigma^{m-2}(K_1) \prec \ldots \prec \sigma^{m-3}(K_1)$ then we have the desired result. ■

To exemplify the utility of this rules in creating boundaries to the region where we can find the pair (K_0,K_1), we present two examples of application and an important result in the form of proposition that isolates a big region where we cannot find the desired pair.

Example 5.3: Let $K_1=UNND1$. In this case the chain of symbolic inequalities is

$$\sigma^4(K_1) \prec \sigma^4(K_0) \prec \sigma^2(K_1) \prec \sigma^3(K_0) \prec \sigma(K_1) \prec \sigma^2(K_0) \prec K_1 \prec \sigma(K_0).$$

We conclude that the kneading sequence K_0 must have the form $K_0=FNN\ldots$, but, since $\sigma^3(K_1) \prec \sigma^4(K_0) \prec ND1$ then the kneading sequences K_0, which can create kneading pairs with $K_1=UNND1$ are in the symbolic region $FNND1 \prec K_0 \prec FNNND1$.

Example 5.4 : Let $K_0=FNNMD2$. We have $K_1=UN\ldots$ and from theorem 5.2, and since $\sigma^4(K_0)=MD2 \prec \sigma^2(K_1) \prec \sigma^3(K_0)=NMD2$ the kneading sequences K_1, which can create kneading pairs with K_0 are in the symbolic region $UNMD2 \prec K_1 \prec UNNMD2$.

Proposition 5.5 For $K_1=UNC$, there is no kneading sequence K_0 between $FNND1$ and $FNNMMNLFNND1$. ■

Proof: Let's suppose that exists a finite sequence $S=S_1S_2\ldots S_{m-1}X$, with $X=D1$ or $X=D2$. We show that this sequence violates the necessary conditions for (K_0,UNC) be a kneading invariant.

Let $FNNMMNLFNND1 \prec S_1S_2\ldots S_{m-1}X \prec FNND1$.

The first four symbols are already determined and so $S=FNNMS_5S_6\ldots S_{m-1}X$. Since the parity of FNN is +1 we can write $MMNNL\ldots \prec MS_5S_6\ldots$ and so, $S_5=M$, $S_6=N$ and $S_7=N$. This way we can write $NLF\ldots \prec NS_8S_9$. The alternatives for S_8 are L, R, $D2$ or N. But $K_1=UNC$, so the itinerary of a point in $\sqrt{b}<x<1$ will necessarily be lower than this one, $NS_8 \prec \sigma(UNC)=NC$. The only possibility to S_8 is L, and $S_9=N$.

Until now $S=FNNMMNNLFNS_{11}S_{12}\ldots$. By other hand the itinerary of a point $x>1$ must be bigger that the itinerary of the critical point, so we have $FNNM\ldots \prec FNS_{11}S_{12}\ldots$ and this assure us that $S_{11}=N$ and the symbol S_{12} can be $M,D1,L,R,D2$ or N. But by Theorem 5.1 we have $\sigma^2(FNNM) \prec \sigma^2(FNS_{12}\ldots) \prec \sigma(UNC) \prec \sigma(FNNM\ldots) \prec \sigma(FNS_{12}\ldots)$, that is $NM\ldots \prec S_{12}S_{13}\ldots \prec NC$.

We conclude that $S_{12}=N$ and $S_{13}=M$, $D1$ or L. But, since the parity of the initial subsequence of S is odd than we have $S_{13} \lhd D1$, and follows that $S_{13}=M$. This is not possible and the sequence should be $S=(FNNMMNNL)^{\infty}$, and this contradicts the fact that S should be finite, with a symbol $D1$ or $D2$. Then between $FNND1$ and $FNNMMNLFNND1$ there is no compatible kneading sequence with $K_1=UNC$. ■

We can observe that $(FNNMMNNL)^{\infty}$ corresponds to the itinerary of the periodic orbit which kneading invariant is $FNND1\infty UNC$.

6. CONCLUSIONS

We built in this work two important rules that allow us to create some boundaries on possible regions where we can find the kneading pairs, or to exclude some as they do not have any of this pairs. We opened the door to the study of the behavior of the real rational maps of degree zero, avoiding the use of numerical calculus, which could drive us into errors, using the Kneading Sequence Combinatory. This work will continue in order to find more and better *criteria* to bind in a more efficient way the regions of existence of the Kneading Pairs.

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