SUBORDINATION AND SUPERORDINATION INVOLVING CERTAIN FRACTIONAL OPERATOR

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Abstract. In this paper we will find the relation between the fractional integral operator $I^{\alpha}f(z)$,and a special case of the integral operator $J_{\eta,\lambda}f(z)$ introduced recently by Salah And Darus [6],then we illustrate some subordination and superordination properties of $J_{\eta,\lambda}f(z)$ in the special case where $\eta = 0$.

Keywords: subordination;superordination; starlike; convex.

1 Introduction

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the area of physics and engineering. There are several kinds of fractional derivatives, which are generalizations for derivatives of integral order. For the purpose of this paper the Caputo's definition of fractional differentiation will be used related to Srivastava and Owa definitions for fractional operators in the complex z-plane.

Definition 1.1(see [6]) Caputo's definition of the fractional-order derivative is defined as

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau$$

where $n-1 < Re(\alpha) \leq n, n \in N$, and the parameter α is allowed to be real or even complex, a is the initial value of function f.

Definition 1.2 (see [7]) The fractional derivative of order α is defined as

$$D_z^{\alpha}f(z) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z \frac{f(t)}{(z-t)^{\alpha}}dt, 0 \leqslant \alpha < 1$$

where f is analytic in simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t)$ to be real when

$(z-t) > 0$.

Definition 1.3(see [7]) The fractional integral of order α is defined by

$$I_z^{\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z f(t)(z-t)^{\alpha-1} dt, 0 \leqslant \alpha < 1$$

where f is analytic in simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$ note that $I_z^{\alpha} f(z) = \frac{z^{\alpha-1}}{\Gamma(\alpha)} f(z)$ if $z > 0$ and 0 if $z \leq 0$

Using Caputo's definition 1.1, Salah and Darus have recently introduced the following integral operator.

Definition 1.4(see [6])

$$J_{\eta,\lambda} f(z) = \frac{\Gamma(2+\eta-\lambda)}{\Gamma(\eta-\lambda)} z^{\lambda-\eta} \int_0^z \frac{\Omega^{\eta} f(\xi)}{(z-\xi)^{\lambda+1-\eta}} d\xi$$

where $\Omega^{\eta} f(z) = \Gamma(2-\eta) z^{\eta} D_z^{\eta} f(z)$ is the generalization operator of Salagean derivative operator and Libera integral operator (see [6]).

if $\eta = 0$ then

$$J_{0,\lambda} f(z) = \frac{\Gamma(2-\lambda)}{\Gamma(-\lambda)} z^{\lambda} \int_0^z f(t)(z-t)^{-\lambda-1} dt, (-1 < \lambda \leqslant 0)$$

setting $-\lambda = \alpha$ we can simply write

$$J_{\alpha} f(z) = \Gamma(2+\alpha) z^{-\alpha} I_z^{\alpha} f(z)$$

,for $0 \leq \alpha < 1$.

The purpose of this paper is to study the fractional differential subordination

$$\left(\frac{z^{\alpha} J_{\alpha} g_1}{\Gamma(2+\alpha)}\right)^{\beta}(z) \prec \left(\frac{z^{\alpha} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\beta}(z) \prec \left(\frac{z^{\alpha} J_{\alpha} g_2}{\Gamma(2+\alpha)}\right)^{\beta}(z)$$

and

$$\left(\frac{z^{\alpha-1} J_{\alpha} g_1}{\Gamma(2+\alpha)}\right)^{\mu}(z) \prec \left(\frac{z^{\alpha-1} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\mu}(z) \prec \left(\frac{z^{\alpha-1} J_{\alpha} g_2}{\Gamma(2+\alpha)}\right)^{\mu}(z), \mu \geqslant 1$$

2 General properties

Theorem 2.1 For $0 \leq \alpha < 1$ and f is continuous function, then

1. $DI_z^{\alpha} f(z) = \frac{z^{\alpha-1}}{\Gamma(\alpha)} f(0) + I_z^{\alpha} Df(z), D = \frac{d}{dz}$
2. $I_z^{\alpha} D_z^{\alpha} f(z) = D_z^{\alpha} I_z^{\alpha} f(z) = f(z)$
3. $DJ_{\alpha} f(z) = \frac{\Gamma(2+\alpha)}{z\Gamma(\alpha)} f(0) - \frac{\alpha}{z} J_{\alpha} f(z) + J_{\alpha} Df(z)$

Proof. For the proof of (1) and (2) see [1] and proof of (3) follows by computing the 1st derivative of $J_\alpha f(z) = \Gamma(2+\alpha) z^{-\alpha} I_z^\alpha f(z)$

Lemma 2.1. Let $f(z)$ be a non-increasing function then for $z_1 \leqslant z_2$ we have

$$J_\alpha f(z_1) \geqslant J_\alpha f(z_2)$$

Proof.

$$J_\alpha f(z_1) = \Gamma(2+\alpha) z_1^{-\alpha} I_z^\alpha f(z_1) = \frac{\Gamma(2+\alpha) f(z_1)}{\Gamma(\alpha) z_1} \geqslant \frac{\Gamma(2+\alpha) f(z_2)}{\Gamma(\alpha) z_2} = J_\alpha f(z_2)$$

3 Subordination and Superordination

Let F, G be analytic functions in the unit disk U.The function $F$is subordinate to G,we write $F \prec G$,if G is univalent , $F(0) = G(0)$ and $F(U) \subset G(U)$.Or given two functions F, G, which are analytic in U, the function F is said to be subordination to G in U if there exists a function w,analytic in U with $w(0) = 0$ and $|w(z)| < 1, \forall z \in U$, such that $F(z) = G(w(z)), \forall z \in U$. Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U.if $p$is analytic in U and satisfies the subordination $\varphi\left(p(z), zp'(z)\right) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination , $p < q$. If $p$and $\varphi\left(p(z), zp'(z)\right)$ are univalent in U and satisfy the differential superordination $h(z) \prec \varphi\left(p(z), zp'(z)\right)$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q < p$. Let H be the class of analytic functions in U and $H[a, n]$ be the subclass of H consisting of functions of the form $f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots$.let A be the subclass of H consisting of functions of the form $f(z) = z + a_2 z^2 + \cdots$.

Definition 3.1(see[2]Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$ where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0, \zeta \in \partial U - E(f)$.

Lemma 3.1(see [3]**)** let $q(z)$ be univalent in the unit disk U and θ, φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z) \varphi(q(z)), h(z) = \theta(q(z)) + Q(z)$ .suppose that

1. $Q(z)$ is starlike univalent in U ,and
2. $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0, z \in U$.If

$\theta(p(z)) + zp'(z) \varphi(p(z)) \prec \theta(q(z)) + zq'(z) \varphi(q(z))$, then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 3.2(see [4] .**)** Let $q(z)$be convex univalent in the unit disk U and $\psi, \gamma \in \mathbb{C}$ with .if $p(z)$ is analytic in U and $\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z)$,then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

lemma 3.3(see [5]**.)** Let $q(z)$ be convex univalent in the unit disk U and ϑ, ϕ be analytic in a domain D containing $q(U)$ .Suppose that

1. $zq^{'}(z)\phi(q(z))$ is starlike in U, and
2. $\Re\left\{\frac{\vartheta^{'}(q(z))}{\phi(q(z))}\right\} > 0, z \in U.$

If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$and $\vartheta(p(z)) + zp^{'}(z)\phi(z)$ is univalent in U and $\vartheta(q(z)) + zq^{'}(z)\phi(q(z)) \prec \vartheta(p(z)) + zp^{'}(z)\phi(p(z))$ then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

lemma 3.4(see [2]**.)** Let $q(z)$ be convex univalent in the unit disk U and $\gamma \in \mathbb{C}$. Further ,assume that $\Re\{\overline{\gamma}\} \succ 0$ .if $p(z) \in H[q(0), 1] \cap Q$,with $p(z) + \gamma zp^{'}(z)$ is univalent in U then $q(z) + \gamma zq^{'}(z) \prec p(z) + \gamma zp^{'}(z)$ implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Theorem 3.1. Let f, g be analytic functions in U such that $f(0) = g(0) = 0, g^{'}(0) = 1$, $\left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha g\right)^\beta(z)$ be convex univalent in U satisfies

$\mathbb{R}\left\{1 + \frac{zg"(z)}{g^{'}(z)} + (\beta - 1)\frac{zg^{'}(z)}{g(z)} + \frac{z}{g^{'}(z)} + \frac{1}{\gamma}\right\} > 0$
$\beta \geqslant 1, g(z) \neq 0, g^{'}(z) \neq 0, z \in U.$

If

$\left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha f\right)^\beta \in A$

and the subordination

$\left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha f\right)^\beta(z)\left[1 + \beta\gamma\frac{zf^{'}(z)}{f(z)}\right] \prec \left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha g\right)^\beta(z)\left[1 + \beta\gamma\frac{zg^{'}(z)}{g(z)}\right], \gamma \in \mathbb{C}$

holds, then

$\left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha f\right)^\beta(z) \prec \left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha g\right)^\beta(z)$
and $\left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha g\right)^\beta(z)$ is the best dominant.

Proof. We are to apply lemma 3.2 setting

$$p(z) = \left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha f\right)^\beta(z) = (I_z^\alpha f)^\beta$$

and

$$q(z) = \left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha g\right)^\beta(z) = (I_z^\alpha g)^\beta.$$

First we show that

$$\mathbb{R}\left\{1 + \frac{zq"(z)}{q'(z)} + \frac{1}{\gamma}\right\} > 0.$$

By using Theorem 2.1 we obtain

$$\mathbb{R}\left\{1 + \frac{zq"(z)}{q'(z)} + \frac{1}{\gamma}\right\} =$$

$$\mathbb{R}\left\{1+\frac{z\beta\left(I_z^\alpha g\right)^{\beta-1}(z)\left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}+I_z^\alpha g''(z)\right]}{\beta\left(I_z^\alpha g\right)^{\beta-1}(z)\,I_z^\alpha g'(z)}+\frac{z\beta(\beta-1)\left(I_z^\alpha g'\right)^2(z)\left(I_z^\alpha g\right)^{\beta-2}(z)}{\beta\left(I_z^\alpha g\right)^{\beta-1}(z)\,I_z^\alpha g'(z)}+\frac{1}{\gamma}\right\}$$

$$=\mathbb{R}\left\{1+\frac{zg''(z)}{g'(z)}+\frac{z(\beta-1)g'(z)}{g(z)}+\frac{z}{g'(z)}+\frac{1}{\gamma}\right\}>0.$$

Hence $q(z)$ is convex univalent function in U , Now we will show that

$$p(z)+\gamma zp'(z)\prec q(z)+\gamma zq'(z)$$

$$\mathbb{R}\{\overline{\gamma}\}>0,\psi=1.$$

By the assumption of the theorem we have

$$p(z)+\gamma zp'(z)=$$

$$\left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha f\right)^\beta(z)\left[1+\beta\gamma\frac{zf'(z)}{f(z)}\right]\prec\left(\frac{z^\alpha}{\Gamma(2+\alpha)}J_\alpha g\right)^\beta(z)\left[1+\beta\gamma\frac{zg'(z)}{g(z)}\right].$$

Thus in view of lemma 3.2, $p(z)\prec q(z)$and q is the best dominant

Theorem 3.2. Let f,g be analytic function in U such that $f(0)=g(0)=0,g'(0)=1$ denote

$G(z)=\frac{z^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{g'(z)}{z}-\frac{g(z)}{z^2}\right]$

such that

$\mathbb{R}\left\{2+\frac{zG'(z)}{G(z)}+\frac{(\mu-1)z^{3-\alpha}\Gamma(\alpha)G(z)}{g(z)}\right\}>0.$

Assume that

$z\left[\left(\frac{z^{\alpha-1}J_\alpha g}{\Gamma(2+\alpha)}\right)^\mu\right]'$ is starlike univalent function in U .If $\left(\frac{z^{\alpha-1}J_\alpha f}{\Gamma(2+\alpha)}\right)^\mu(z)\in A$

and the subordination

$\left(\frac{z^{\alpha-1}J_\alpha f}{\Gamma(2+\alpha)}\right)^\mu(z)\left\{1+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\right\}\prec\left(\frac{z^{\alpha-1}J_\alpha g}{\Gamma(2+\alpha)}\right)^\mu(z)\left\{1+\mu\left(\frac{zg'(z)}{g(z)}-1\right)\right\}$

holds, then

$\left(\frac{z^{\alpha-1}J_\alpha f}{\Gamma(2+\alpha)}\right)^\mu(z)\prec\left(\frac{z^{\alpha-1}J_\alpha g}{\Gamma(2+\alpha)}\right)^\mu(z),\mu\geqslant 1$ and $\left(\frac{z^{\alpha-1}J_\alpha g}{\Gamma(2+\alpha)}\right)^\mu(z)$ is the best dominant

Proof. our aim is to apply lemma 3.1 setting

$$p(z)=\left(\frac{z^{\alpha-1}J_\alpha f}{\Gamma(2+\alpha)}\right)^\mu(z)=\left(\frac{I_z^\alpha f}{z}\right)^\mu(z)$$

and

$$q(z)=\left(\frac{z^{\alpha-1}J_{\alpha}g}{\Gamma(2+\alpha)}\right)^{\mu}(z)=\left(\frac{I_z^{\alpha}g}{z}\right)^{\mu}(z).$$

First we show that

$$\mathbb{R}\left\{2+\frac{zq''(z)}{q'(z)}\right\}>0$$

$$\mathbb{R}\left\{2+\frac{zq''(z)}{q'(z)}\right\}=\mathbb{R}\left\{2+\frac{zG'(z)}{G(z)}+\frac{(\mu-1)\,z^{3-\alpha}\Gamma(\alpha)\,G(z)}{g(z)}\right\}>0.$$

By setting $\theta(\omega)=\omega$ and $\varphi(\omega)=1$.

It can be easily observed that $\theta(z),\varphi(z)$ are analytic in $\mathbb{C}$. We let

$$Q(z)=zq'(z)\,\varphi(z)=zq'(z)$$

$$h(z)=\theta(q(z))+Q(z)=q(z)+zq'(z).$$

By the assumptions of the theorem $Q(z)$ is starlike univalent in U and that

$$\mathbb{R}\left\{\frac{zh'(z)}{Q(z)}\right\}=\mathbb{R}\left\{2+\frac{zq''(z)}{q'(z)}\right\}=\mathbb{R}\left\{2+\frac{zG'(z)}{G(z)}+\frac{(\mu-1)\,z^{3-\alpha}\Gamma(\alpha)\,G(z)}{g(z)}\right\}>0.$$

Now we will show that

$$p(z)+zp'(z)\prec q(z)+zq'(z)$$

$$p(z)+zp'(z)=$$

$$\left(\frac{z^{\alpha-1}J_{\alpha}f}{\Gamma(2+\alpha)}\right)^{\mu}(z)\left\{1+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\right\}\prec\left(\frac{z^{\alpha-1}J_{\alpha}g}{\Gamma(2+\alpha)}\right)^{\mu}(z)\left\{1+\mu\left(\frac{zg'(z)}{g(z)}-1\right)\right\}$$

$$=q(z)+zq'(z).$$

So in view of lemma 3.1 ,$p(z)\prec q(z)$ and q is the best dominant.

Theorem 3.3. Let f,g be analytic in U such that $f(0)=g(0)=0$,$\left(\frac{z^{\alpha}J_{\alpha}g}{\Gamma(2+\alpha)}\right)^{\beta}(z)$ be convex univalent in U and $\left(\frac{z^{\alpha}J_{\alpha}f}{\Gamma(2+\alpha)}\right)^{\beta}(z)\in H[0,1]\cap Q$.

Assume that

$$\left(\frac{z^{\alpha}J_{\alpha}f}{\Gamma(2+\alpha)}\right)^{\beta}(z)\left[1+\beta\gamma\frac{zf'(z)}{f(z)}\right]$$

is univalent in U where $\mathbb{R}\{\overline{\gamma}\} > 0$. If

$\left(\frac{z^{\alpha} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\beta}(z) \in A$

and the subordination

$\left(\frac{z^{\alpha} J_{\alpha} g}{\Gamma(2+\alpha)}\right)^{\beta}(z)\left[1+\beta\gamma\frac{zg^{'}(z)}{g(z)}\right] \prec \left(\frac{z^{\alpha} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\beta}(z)\left[1+\beta\gamma\frac{zf^{'}(z)}{f(z)}\right], \beta \geqslant 1$

holds. Then:

$\left(\frac{z^{\alpha} J_{\alpha} g}{\Gamma(2+\alpha)}\right)^{\beta}(z) \prec \left(\frac{z^{\alpha} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\beta}(z)$ and $\left(\frac{z^{\alpha} J_{\alpha} g}{\Gamma(2+\alpha)}\right)^{\beta}(z)$ is the best subordinant.

Proof. We will apply Lemma 3.4. setting

$$p(z) = \left(\frac{z^{\alpha} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\beta}(z) = \left(I_z^{\alpha} f\right)^{\beta}$$

and

$$q(z) = \left(\frac{z^{\alpha} J_{\alpha} g}{\Gamma(2+\alpha)}\right)^{\beta}(z) = \left(I_z^{\alpha} g\right)^{\beta}$$

$$q(z) + \gamma z q^{'}(z) = \left(\frac{z^{\alpha} J_{\alpha} g}{\Gamma(2+\alpha)}\right)^{\beta}(z)\left[1+\beta\gamma\frac{zg^{'}(z)}{g(z)}\right]$$

$\prec \left(\frac{z^{\alpha} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\beta}(z)\left[1+\beta\gamma\frac{zf^{'}(z)}{f(z)}\right] = p(z) + \gamma z p^{'}(z).$

So in view of Lemma 3.4 ,$q(z) \prec p(z)$and $q(z)$ is best subordinant.

Theorem 3.4. Let f, g be analytic in U such that $f(0) = g(0) = 0$, $\left(\frac{z^{\alpha-1} J_{\alpha} g}{\Gamma(2+\alpha)}\right)^{\mu}(z)$ be convex univalent in U. Let

$z\left[\left(\frac{z^{\alpha-1} J_{\alpha} g}{\Gamma(2+\alpha)}\right)^{\mu}(z)\right]^{'}$ be starlike univalent function in U,

$\left(\frac{z^{\alpha-1} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\mu}(z)\left\{1+\mu\left(\frac{zf^{'}(z)}{f(z)}-1\right)\right\}$ be univalent in U and

$\left(\frac{z^{\alpha-1} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\mu}(z) \in H \cap Q$ with

$\mu(\alpha-2) \succ 0$ $\left(\frac{z^{\alpha-1} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\mu}(z) \in A$

and the subordination

$\left(\frac{z^{\alpha-1} J_{\alpha} g}{\Gamma(2+\alpha)}\right)^{\mu}(z)\left\{1+\mu\left(\frac{zg^{'}(z)}{g(z)}-1\right)\right\} \prec \left(\frac{z^{\alpha-1} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\mu}(z)\left\{1+\mu\left(\frac{zf^{'}(z)}{f(z)}-1\right)\right\}.$

Holds then

$$\left(\frac{z^{\alpha-1} J_{\alpha} g}{\Gamma(2+\alpha)}\right)^{\mu}(z) \prec \left(\frac{z^{\alpha-1} J_{\alpha} f}{\Gamma(2+\alpha)}\right)^{\mu}(z)$$

,

where $\mu \geqslant 1$,and

$\left(\frac{z^{\alpha-1}J_\alpha g}{\Gamma(2+\alpha)}\right)^\mu(z)$ is the best subordinant.

Proof. Our aim is to apply Lemma 3.3. Setting

$$p(z)=\left(\frac{z^{\alpha-1}J_\alpha f}{\Gamma(2+\alpha)}\right)^\mu(z)=\left(\frac{I_z^\alpha f}{z}\right)^\mu(z)$$

And

$$q(z)=\left(\frac{z^{\alpha-1}J_\alpha g}{\Gamma(2+\alpha)}\right)^\mu(z)=\left(\frac{I_z^\alpha g}{z}\right)^\mu(z).$$

By taking $\vartheta(\omega)=\omega$ and $\phi(\omega)=1$.

It can easily observed that $\vartheta(z),\phi(z)$ are analytic in $\mathbb{C}$ .Thus

$$\mathbb{R}\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\}=1>0.$$

Now we will show that

$$q(z)+zq'(z)\prec p(z)+zp'(z)$$

.

$$q(z)+zq'(z)$$

$$=\left(\frac{z^{\alpha-1}J_\alpha g}{\Gamma(2+\alpha)}\right)^\mu(z)\left\{1+\mu\left(\frac{zg'(z)}{g(z)}-1\right)\right\}\prec\left(\frac{z^{\alpha-1}J_\alpha f}{\Gamma(2+\alpha)}\right)^\mu(z)\left\{1+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\right\}$$

$$=p(z)+zp'(z).$$

Thus view to Lemma 3.3 , $q(z)\prec p(z)$and $q(z)$ is the best subordinant.

Theorem 3.5. Let f,g_1,g_2 be analytic functions in U such that

$f(0)=g_1(0)=g_2(0),g_2'(0)=1$, and $\left(\frac{z^\alpha J_\alpha g_1}{\Gamma(2+\alpha)}\right)^\beta(z),\left(\frac{z^\alpha J_\alpha g_2}{\Gamma(2+\alpha)}\right)^\beta(z)$ be convex univalent in U satisfying

$$\mathbb{R}\left\{1+\frac{zg_2''(z)}{g_2'(z)}+\frac{z(\beta-1)g_2'(z)}{g_2(z)}+\frac{z}{g_2'(z)}+\frac{1}{\gamma}\right\}>0,\beta>1.$$

If $\left(\frac{z^\alpha J_\alpha f}{\Gamma(2+\alpha)}\right)^\beta(z)\in H\cap Q$, $\left(\frac{z^\alpha J_\alpha f}{\Gamma(2+\alpha)}\right)^\beta(z)\left[1+\beta\gamma\frac{zf'(z)}{f(z)}\right]$ is univalent in U and satisfies

$$\left(\frac{z^\alpha J_\alpha g_1}{\Gamma(2+\alpha)}\right)^\beta(z)\left[1+\beta\gamma\frac{zg_1'(z)}{g_1(z)}\right]\prec\left(\frac{z^\alpha J_\alpha f}{\Gamma(2+\alpha)}\right)^\beta(z)\left[1+\beta\gamma\frac{zf'(z)}{f(z)}\right]\prec\left(\frac{z^\alpha J_\alpha g_2}{\Gamma(2+\alpha)}\right)^\beta(z)\left[1+\beta\gamma\frac{zg_2'(z)}{g_2(z)}\right]$$

$\beta\geqslant1,\gamma\in\mathbb{C},\mathbb{R}\{\overline{\gamma}\}>0$, then

$\left(\frac{z^\alpha J_\alpha g_1}{\Gamma(2+\alpha)}\right)^\beta(z)\prec\left(\frac{z^\alpha J_\alpha f}{\Gamma(2+\alpha)}\right)^\beta(z)\prec\left(\frac{z^\alpha J_\alpha g_2}{\Gamma(2+\alpha)}\right)^\beta(z)$ such that $\left(\frac{z^\alpha J_\alpha g_1}{\Gamma(2+\alpha)}\right)^\beta(z)$ is the best subordinant and $\left(\frac{z^{\alpha}J_{\alpha}g_2}{\Gamma(2+\alpha)}\right)^{\beta}(z)$ is the best dominant.

Proof. Combining Theorems 3.1 and 3.3 the result follows.

Theorem 3.6. Let f, g_1, g_2 be analytic functions in U such that

$f(0) = g_1(0) = g_2(0) = 0, g_2^{'}(0) = 1$ be convex univalent in U.

Denote $G_2(z) = \frac{z^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{g_2^{'}(z)}{z} - \frac{g_2(z)}{z^2}\right]$ such that

$\mathbb{R}\left\{2 + \frac{zG_2^{'}(z)}{G_2(z)} + \frac{(\mu-1)z^{3-\alpha}\Gamma(\alpha)G_2(z)}{g_2(z)}\right\} > 0$. Also let

$z\left[\left(\frac{z^{\alpha-1}J_{\alpha}g_1}{\Gamma(2+\alpha)}\right)^{\mu}(z)\right]^{'}, z\left[\left(\frac{z^{\alpha-1}J_{\alpha}g_2}{\Gamma(2+\alpha)}\right)^{\mu}(z)\right]^{'}$ be starlike univalent in U and

$\left(\frac{z^{\alpha-1}J_{\alpha}f}{\Gamma(2+\alpha)}\right)^{\mu}(z)\left\{1 + \mu\left(\frac{zf^{'}(z)}{f(z)} - 1\right)\right\}$ be univalent in U and

$\left(\frac{z^{\alpha-1}J_{\alpha}f}{\Gamma(2+\alpha)}\right)^{\mu}(z) \in H[0,1] \cap Q$ with $\mu(\alpha-2) > 0$.

If $\left(\frac{z^{\alpha-1}J_{\alpha}f}{\Gamma(2+\alpha)}\right)^{\mu}(z) \in A$ and the subordination

$$\left(\frac{z^{\alpha-1}J_{\alpha}g_1}{\Gamma(2+\alpha)}\right)^{\mu}(z)\left\{1 + \mu\left(\frac{zg_1^{'}(z)}{g_1(z)} - 1\right)\right\} \prec \left(\frac{z^{\alpha-1}J_{\alpha}f}{\Gamma(2+\alpha)}\right)^{\mu}(z)\left\{1 + \mu\left(\frac{zf^{'}(z)}{f(z)} - 1\right)\right\}$$

$\prec \left(\frac{z^{\alpha-1}J_{\alpha}g_2}{\Gamma(2+\alpha)}\right)^{\mu}(z)\left\{1 + \mu\left(\frac{zg_2^{'}(z)}{g_2(z)} - 1\right)\right\}$ holds, then

$\left(\frac{z^{\alpha-1}J_{\alpha}g_1}{\Gamma(2+\alpha)}\right)^{\mu}(z) \prec \left(\frac{z^{\alpha-1}J_{\alpha}f}{\Gamma(2+\alpha)}\right)^{\mu}(z) \prec \left(\frac{z^{\alpha-1}J_{\alpha}g_2}{\Gamma(2+\alpha)}\right)^{\mu}(z), \mu \geqslant 1$ and $\left(\frac{z^{\alpha-1}J_{\alpha}g_1}{\Gamma(2+\alpha)}\right)^{\mu}(z)$ is the best subordinant and $\left(\frac{z^{\alpha-1}J_{\alpha}g_2}{\Gamma(2+\alpha)}\right)^{\mu}(z)$ is the best dominant.

Proof. Combining Theorems 3.2 and 3.4 the results follows

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