A Note in the Topological Groups

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ABSTRACT— ***In this note, for a topological group G, we introduce a new concept as bounded topological group, that is, $E \subseteq G$ is called bounded, if for every neighborhood V of identity element of G, there is a natural number n such that $E \subseteq V^n$. We study some properties of this new concept and its relationships with other topological properties of topological groups.***

Keywords— Topological Group, Bounded Topological Groups, Group

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1. INTRODUCTION

A topological group consists of a group G equipped with a topology τ such that multiplication $(x,y) \to xy$ and inverse operation $x \to x^{-1}$ are both continuous mappings. In the papers [1, 2, 3], some authoress have been studied some properties of topological groups. In this paper, we study the boundedness of topological group. Suppose that G is a topological group and $E \subseteq G$. If G is metrizablity, we show that $E \subseteq G$ is bounded with respect to topology if and only if it is bounded with respect to metric induced by this topology. If $E \subseteq G$ is bounded and closed, then we show that E is compact. Conversely, if E is a component of e and compact, then E is bounded. We investigate some topological properties for bounded subset of G.

For a topological group G, e is an identity element of G and for $E \subseteq G$, E^- is closure of E and for every $n \in \mathrm{N}$, we define E^n as follows

$$E^n = \{x_1 x_2 x_3 \ldots x_n : x_i \in E, 1 \le i \le n\}.$$

A topological space X is $O-dimensional$ if the family of all sets that are both open and closed is open basis for the topology, for more information see chapter 2 of [2].

2. MAIN RESULTS

Definition 2.1. Let G be topological group and $E \subseteq G$. We say that E is a topological bounded, if for every neighborhood V of e, there is a natural number n such that $E \subseteq V^n$.

It is clear that if E is a topological bounded subset of G and H is a subgroup of G, then E/H is a topological bounded subset of G/H. For example P/N is bounded topological group. In this section, we use some notations, definitions and results that for more information look chapter 2 from [2].

Theorem 2.2. Let G be a topological group and metrizable with respect to a left invariant metric d. Then G is topological bounded if and only if G is bounded with respect to metric d.

Proof. Let G be a topological bounded group and $\varepsilon > 0$. Take $d([0,\varepsilon)) = U \times V$ where U and V are neighborhoods of e. Suppose that W is symmetric neighborhood of e such that $W \subseteq U \cap V$. Then there is natural number n such that $W^n = G$. Since $d(W \times W) < \varepsilon$, we show that $d(W^2 \times W^2) < 2\varepsilon$, and so $d(W^n \times W^n) < n\varepsilon$. Assume that $x, y, x', y' \in W$. Then we have

$$d(xy, x'y') \le d(xy, e) + d(e, x'y') = d(y, x^{-1}) + d(x'^{-1}, y') < 2\varepsilon.$$

It follows that $d(G\times G) = d(W^n \times W^n) < n\varepsilon$.

Conversely, suppose that G is bounded with respect to metric d. Then there is $M > 0$ such that $d(G\times G) < M$. Let U be a neighborhood of e. Choose $\varepsilon > 0$ such that $d^{-1}([0,\varepsilon)) \subseteq U\times U$. Take a natural number n such that $n\varepsilon > M$. Then we have

$$G\times G = d^{-1}([0, M)) = d^{-1}([0, n\varepsilon)) \subseteq V^n \times V^n.$$

It follows that $G = V^n$, and so that G is topological bounded.

Theorem 2.3. Let G be topological group and let H be a normal subgroup of G. If H and G/H are topological bounded, then G is topological bounded.

Proof. Let U be a neighborhood of e. Put $V = U \cap H$. Then there are natural numbers m and n such that $(U/H)^n = G/H$ and $V^m = H$. We show that $U^{n+m} = G$.

Let $x\in G$. Then if $x \in H$, we have

$$x \in V^m \subset U^m \subset U^{n+m}.$$

Now let $x \notin H$. Then $xH \in (U/H)^n$. Assume that $x_1, x_2,..., x_n \in U$ such that

$$xH = x_1x_2...x_nH.$$

Consequently there is $h \in H$ such that $xh \in U^n$, and so $x \in U^nH \subset U^nV^m \subset U^nU^m = U^{n+m}$. We conclude that $U^{n+m} = G$, and proof hold.

Theorem 2.4. If G is an infinite locally compact O-dimensional topological group, then G is unbounded.

Proof. Let U be a neighborhood of e such that U^- is compact and $U^- \neq G$. Since G is a O-dimensional topological group, U contains an open and closed neighborhood as V . Then V is a compact neighborhood of e. By apply [2, Theorem 4.10], there is a neighborhood W of e such that $WV \subset V$. Take $W_0 = W \cap V$. Then ${W_0}^2 \subset WV \subset V \subset U^-$. By finite induction, we have

$${W_0}^n \subset W_0{W_0}^{n-1} \subset WV \subset V \subset U^-,$$

for $n \in \mathrm{N}$. It follows that ${W_0}^n \not\subset G$, and so G is unbounded.

Theorem 2.5. Suppose that G is a locally compact, Hausdorff, and totally disconnected topological group. Then G is unbounded.

Proof. By using [2, Theorem 3.5] and Theorem 2.4, proof hold.

Theorem 2.6. Let G be an infinite topological group. Then we have the following assertions.

1. If $E \subseteq G$ is topological bounded, then E^- is topological bounded subset of G.
2. If G is topological bounded, then G is connected and moreover G has no proper open subgroups.

Proof. 1) Let U be a neighborhood of e and suppose that V is a neighborhood of e such that $V^- \subset U$. Since E is topological bounded subset of G, there is natural number n such that $E \subset V^n$. Then $E^- \subset (V^n)^- \subset (V^-)^n \subset U^n$. It follows that E^- is a topological bounded subset of G.

2) Since G is topological bounded, there is a natural number n such that $G = V^n$ where V is neighborhood of e. By using [2, Corollary 7.9], proof hold.

Corollary 2.7. Assume that G is a locally compact topological group. Then every topological bounded and closed subset of G is compact, moreover if $E \subseteq G$ is topological bounded, then E^- is compact.

Every topological bounded topological group G, in general, is not compact, for example P/Z is topological bounded, but is not compact.

Theorem 2.8. Let G be topological group and suppose that $E \subseteq G$ is the component of e. If E is compact, then E is topological bounded.
Proof. Since E is the component of e, by using [2, Theorem 7.4], for every neighborhood U of e, we have $E \subseteq \bigcup_{k=1}^{\infty} U^k$. Since E is compact there is natural number n such that $E \subseteq U^n$. Then E is topological bounded subset of G.

For a topological group G, in general, each compact subset E is not topological bounded and by proceeding theorem, E must be a component of e. For example, for $n \geq 1$, $Z_n = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{n}\}$ with discrete topology is not topological bounded, but it is compact.

Corollary 2.9. If G is a locally compact topological group, then the component of e is topological bounded.

Theorem 2.10. Let G and $G^{'}$ be topological group and suppose that $\pi : G \to G^{'}$ is group isomorphism. If π is continuous and $E \subseteq G$ is a topological bounded subset of G, then $\pi(E)$ is topological bounded subset of $G^{'}$.
Proof. Let $V^{'}$ be a neighborhood of $e^{'} \in G^{'}$. Then $\pi^{-1}(V^{'})$ is a neighborhood of e. Since E is a topological bounded subset of G, there is a natural number n such that $E \subseteq (\pi^{-1})^n (V^{'}) \subseteq \pi^{-1}(V^{'n})$ implies that $\pi(E) \subseteq V^{'n}$. Thus $\pi(E)$ is a topological bounded subset of $G^{'}$.

Definition 2.11. Let G and $G^{'}$ be topological group. We say that the mapping $\pi : G \to G^{'}$ is compact, if for every topological bounded subset $E \subseteq G$, $\pi(E)$ is relatively compact.
For example, every identity mapping from P/N into itself is compact operator.

Theorem 2.12. Let G and $G^{'}$ be topological group and suppose that $\pi : G \to G^{'}$ is continuous and group isomorphism. If $G^{'}$ is locally compact, then π is compact.
Proof. Let $E \subseteq G$ be topological bounded. By using Theorem 2.10, $\pi(E)$ is topological bounded subset of $G^{'}$ and by using Theorem 2.6, $\pi(E)^{-}$ is compact, it follows that π is compact.

3. REFERENCES

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