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On the Growth of Meromorphic Solutions of Higher Order Linear Differential Equations

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ABSTRACT—*In this paper, we investigate the growth of solutions of the linear differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where $A_j(z)$ (j=0,1,⋯, $k-1$) are meromorphic functions. Assume that there exists $l \in \{1,2,\cdots,k-1\}$ such that $A_l(z)$ have a finite deficient value, then we shall give some conditions on other coefficients which can guarantee that every solution $f(\neq 0)$ of the equation is of infinite order. More specifically, we estimate the lower bound of hyper-order of f if every solution $f(\neq 0)$ of the equation is of infinite order.

Keywords—Deficient value, Complex differential equations, Meromorphic function, Hyper-order, Infinite order

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1. INTRODUCTION AND MAIN RESULTS

We shall assume that reader is familiar with the fundamental results and the standard notations of Nevanlinna theory of meromorphic function (see [10], [14] or [18]). In addition, for a meromorphic function $f(z)$ in the complex plane, we will use the notation $\rho(f)$ and $\mu(f)$ to denote its order and the lower order respectively; and use the notation $\lambda(f)$ and to $\lambda(\frac{1}{f})$ denote the exponent of convergence of its zero-sequences and pole-sequences respectively.

In order to express the rate of growth of meromorphic function of infinite order, we recall the following definition (see, [20]).

Definition A.([20]) Let f be a meromorphic function, then we define the hyper-order $\rho_2(f)$ of $f(z)$,

$$\rho_2(f) = \overline{\lim}_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

For the second order linear differential equation

$$f'' + Af' + Bf = 0, \qquad (1.1)$$

where $A(z)$ and $B(z)$ are entire functions. Many authors have investigated the growth of solutions of the equation (1.1). It is well known that if $A(z)$ is entire function and $B(z)(\neq 0)$ is transcendental entire function, and f_1, f_2 are two linearly independent solutions of the equation (1.1), then at least one of f_1, f_2 must have infinite order. On the other hand, there are some equations of the form (1.1) that possess a solution $f(\neq 0)$ of finite order; for example, $f(z) = e^z$ satisfies $f'' + e^{-z} f' - (e^{-z} + 1) f = 0$. Thus a natural question is: what conditions on $A(z)$ and $B(z)$ can guarantee that every solution $f(\neq 0)$ of the equation (1.1) has infinite order? From the works of Gundersen (see [6]), Hellerstein, Miles and Rossi (see [9]), we know that if $A(z)$ and $B(z)$ are entire functions with $\rho(A) < \rho(B)$; or if $A(z)$ is a polynomial, and $B(z)$ is transcendental; or if $\rho(B) < \rho(A) \le \frac{1}{2}$, then every solution $f(\neq 0)$ of the equation (1.1) has infinite order. More results can be found in [11] and [16]. Then for a great deal of solutions with infinite order, more precise estimates for their rate of growth is a very important aspect. There are many authors investigated the hyper-order $\rho_2(f)$ of solutions of the equation (1.1) (see, [3] and [12-13]). Ki-Ho Kwon investigated the problem and obtained the following result in [12].

Theorem B.([12]) Let $A(z)$ and $B(z)$ be entire functions such that $\rho(\mathrm{A})<\rho(\mathrm{B})$ or $\rho(\mathrm{B})<\rho(\mathrm{A})\le\frac{1}{2}$. Then every solution $f(\neq 0)$ of the equation (1.1) satisfies $\rho_2(f)\ge\max\{\rho(\mathrm{A}),\rho(\mathrm{B})\}$.

It seems that there are few work done on the equation (1.1), where $A(z)$ and $B(z)$ are meromorphic functions, but the growth of meromorphic solution of the equation (1.1) is a very important aspect (see [2-3] and [5]). It would be interesting to get some relations between the equation (1.1) and some deep results in value distribution theory of meromorphic functions. Thus, we shall introduce the deficient value into the studies of the equation (1.1). It is well-know that deficient value plays fundamental role in the theory of value distribution of meromorphic function and many important work done on this aspect (see, e.g. [19]). There are some results on the value distribution theory of the solutions of the equation (1.1) having connections with deficient values (see [11]). Furthermore, we mention that the author consider the equation (1.1) and obtain the following result.

Theorem C.([15]) Let $A(z)$ be an meromorphic function having a finite deficient value, and let $B(z)$ be a transcendental meromorphic function with $\mu(B)<\frac{1}{2}$ and $\delta(\infty,B)=1$. Then every solution $f(\neq 0)$ of the equation (1.1) is of infinite order.

In this paper, we shall consider the higher order linear differential equation

$$f^{(k)}+A_{k-1}(\mathrm{z})\mathrm{f}^{(k-1)}+\cdots+A_1(\mathrm{z})\mathrm{f}'+\mathrm{A}_0(\mathrm{z})\mathrm{f}=0, \tag{1. 2}$$

where $A_j(\mathrm{z})$ (j=$0,1,\cdots,\ k-1$) are meromorphic functions. Many authors have investigated the growth of solutions of the equation (1.2), and obtained lots of results on order and hyper-order of solutions of the equation (1.2) (see [4] and [17]). Here we shall introduce the deficient value into the studies of the equation (1.2). The main results in the paper are the following.

Theorem 1.1. Let $A_j(\mathrm{z})$ (j= $0,1,\cdots,\ k-1$) be meromorphic functions. Suppose that there exists an integer $l\in\{1,2,\cdots,\mathrm{k}-1\}$ such that A_l have a finite deficient value. Suppose that $A_0(\mathrm{z})$ is a transcendental meromorphic function with $\mu(\mathrm{A}_0)<\frac{1}{2}$ and $\delta(\infty,A_0)=1$, and $\rho(\mathrm{A_i})<\mu(\mathrm{A}_0)$ for $i\neq l$ ($1\le i\le k-1$). Then every solution $f(\neq 0)$ of the equation (1.2) satisfies $\rho(\mathrm{f})=\infty$ and $\rho_2(\mathrm{f})\ge\mu(\mathrm{A}_0)$

Using the same method of the proof of Theorem 1.1, we can easily obtain the following results.

Theorem 1.2. Let $A_j(\mathrm{z})$ (j= $0,1,\cdots,\ k-1$) be meromorphic functions. Suppose that there exists an integer $l\in\{1,2,\cdots,\mathrm{k}-1\}$ such that A_l have a finite deficient value. Suppose that $A_0(\mathrm{z})$ is a transcendental meromorphic function with $\rho(\mathrm{A}_0)<\frac{1}{2}$ and $\delta(\infty,A_0)=1$, and $\rho(\mathrm{A_i})<\rho(\mathrm{A}_0)$ for $i\neq l$ ($1\le i\le k-1$). Then every solution $f(\neq 0)$ of the equation (1.2) satisfies $\rho(\mathrm{f})=\infty$ and $\rho_2(\mathrm{f})\ge\rho(\mathrm{A}_0)$.

Theorem 1.3. Let $A_j(\mathrm{z})$ (j= $0,1,\cdots,\ k-1$) be meromorphic functions. Suppose that there exists an integer $l\in\{1,2,\cdots,\mathrm{k}-1\}$, such that A_l have a finite deficient value. Suppose that there exist two constants $\alpha>0$ and $\beta>0$, for any given $\varepsilon>0$, two finite set of real numbers $\{\phi_k\}$ and $\{\theta_k\}$ that satisfy $\phi_1<\theta_1<\phi_2<\theta_2<\cdots<\phi_m<\theta_m<\phi_{m+1}$ ($\phi_{m+1}=\phi_1+2\pi$) and

$$\sum_{k=1}^{m}(\phi_{k+1}-\theta_k)<\varepsilon, \tag{1. 3}$$

such that

$$|A_0(z)| \geq \exp\{(1+o(1))\alpha|z|^{\beta}\} \qquad (1.4)$$

and

$$|A_i(z)| \leq \exp\{o(1))|z|^{\beta}\}, \text{ for } i \neq l \ (1 \leq i \leq k-1) \qquad (1.5)$$

as $z \to \infty$ in $\phi_k \leq \arg z \leq \theta_k$ ($k = 1,2,\cdots,m$). Then every solution $f(\neq 0)$ of the equation (1.2) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \beta$.

Theorem 1.4. Let $A_j(z)$ (j= $0,1,\cdots,$ $k-1$) be meromorphic functions. Suppose that there exists an integer $l \in \{1,2,\cdots,k-1\}$, such that A_l have a finite deficient value. Suppose that $A_0(z)$ is a transcendental meromorphic function with $\lambda(\frac{1}{A_0}) < \rho(A_0) < \frac{1}{2}$, and $\rho(A_i) < \rho(A_0)$ for $i \neq l \ (1 \leq i \leq k-1)$. Then every solution $f(\neq 0)$ of the equation (1.2) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \rho(A_0)$.

Theorem 1.5. Let $A_j(z)$ (j= $0,1,\cdots,$ $k-1$) be meromorphic functions. Suppose that there exists an integer $l \in \{1,2,\cdots,k-1\}$, such that A_l have a finite deficient value. Suppose that $A_0(z)$ is a transcendental meromorphic function with $\lambda(\frac{1}{A_0}) < \mu(A_0) < \frac{1}{2}$, and $\rho(A_i) < \mu(A_0)$ for $i \neq l \ (1 \leq i \leq k-1)$. Then every solution $f(\neq 0)$ of the equation (1.2) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \mu(A_0)$.

The paper is organized as the following. In Section 2, we shall state and prove some lemmas related with our theorems. In Section 3, we shall prove Theorem 1.1-1.5.

2. LEMMAS

For the proofs of our theorems, we need the following lemmas.

Lemma 2.1. ([7]) Let (f, Γ) denote a pair that consists of a transcendental meromorphic function f and a finite set $\Gamma = \{(k_1, j_1), (k_2, j_2), \cdots, (k_q, j_q)\}$ of distinct pairs of integers that satisfies $k_i > j_i \geq 0$ for $i = 1,2,\cdots,q$. Let $\alpha > 1$ and $\varepsilon > 0$ be given real constants. Then the following three statements hold.

(i) There exists a set $E_1 \subset [0,2\pi)$ that has linear measure zero, and there exists a constant $c > 0$ that depend only on α and Γ, such that if $\psi_0 \in [0,2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0)$>0 such that for all z satisfying $\arg z = \psi_0$ and $|z| = r \geq R_0$, and for all (k, j) $\in \Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c(\frac{T(\alpha r, f)}{r} \log^{\alpha} r \log T(\alpha r, f))^{k-j}. \qquad (2.1)$$

In particular, if f has finite order $\rho(f)$, then (2.1) be replaced (2.2)

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}. \qquad (2.2)$$

(ii) There exists a set $E_2 \subset (1,\infty)$ that has finite logarithmic measure, and there exists a constant $c > 0$ that depend only on α and Γ, such that for all z satisfying $|z| \notin E_2 \cup [0,1]$ and for all (k, j) $\in \Gamma$, the inequality (2.1) holds.

In particular, if f has finite order $\rho(\mathrm{f})$, then inequality (2.2) holds.

(iii) There exists a set $E_3 \subset [0,\infty)$ that has finite linear measure, and there exists a constant $c>0$ that depend only on α and Γ, such that for all z satisfying $|z| = r \notin E_3$ and for all $(k, j) \in \Gamma$,

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le c(\mathrm{T}(\alpha\,\mathrm{r},\mathrm{f})\, r^{\varepsilon} \log T(\alpha\,\mathrm{r},\mathrm{f}))^{\mathrm{k}-\mathrm{j}}. \tag{2. 3}$$

In particular, if f has finite order $\rho(\mathrm{f})$, then (2.3) be replaced (2.4)

$$\backslash\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(\mathrm{k}-\mathrm{j})(\rho(f)+\varepsilon)}. \tag{2. 4}$$

To state the following lemma, we define the upper and lower logarithmic density of $E \subset [1,\infty)$ respectively by

$$\overline{\log dens}\ E = \overline{\lim}_{r\to\infty} \frac{m_l(\mathrm{E} \cap [1,\mathrm{r}])}{\log \mathrm{r}},$$

and

$$\underline{\log densE} = \underline{\lim}_{r\to\infty} \frac{m_l(\mathrm{E} \cap [1,\mathrm{r}])}{\log \mathrm{r}},$$

where $m_l(\mathrm{E} \cap [1,\mathrm{r}]) = \int_{E\cap[1,r]} \frac{dt}{t}$.

Lemma 2.2. ([8]) Suppose that $g(z)$ is a transcendental and meromorphic in the complex plane, of lower order $\mu(\mathrm{g}) = \mu < \alpha < 1$, and define $E = \{\mathrm{r} > 1 : \log \mathrm{L}(\mathrm{r},\mathrm{g}) > \gamma(\cos\pi\alpha + \delta(\infty,\mathrm{g}) - 1)\mathrm{T}(\mathrm{r},\mathrm{g})\}$ and $L(\mathrm{r},\mathrm{g}) = \min\{|\mathrm{g(z)}| : |\mathrm{z}| = \mathrm{r}\}$, $\gamma = \dfrac{\pi\alpha}{\sin\pi\alpha}$. Then E has upper logarithmic density at least $1 - \dfrac{\mu}{\alpha}$.

Remark 1. In Lemma 2.2, if the order of $g(z)$ is less than $\dfrac{1}{2}$, then, for every $\alpha \in (\rho(\mathrm{g}), \dfrac{1}{2})$, there exists a set $E \subset [1,\infty)$, such that $\overline{\log dens}\ E \ge 1 - \dfrac{\rho(\mathrm{g})}{\alpha}$, where

$$E = \{\mathrm{r} > 1 : \log \mathrm{L}(\mathrm{r},\mathrm{g}) > \gamma(\cos\pi\alpha + \delta(\infty,\mathrm{g}) - 1)\mathrm{T}(\mathrm{r},\mathrm{g})\},\ \gamma = \frac{\pi\alpha}{\sin\pi\alpha},\ L(\mathrm{r},\mathrm{g}) = \min\{|\mathrm{g(z)}| : |\mathrm{z}| = \mathrm{r}\}.$$

Lemma 2.3. ([15]) Let $g(z)$ be a transcendental meromorphic function with $0 < \mu(g) < \dfrac{1}{2}$ and $\delta(\infty,\mathrm{g}) = 1$, and let $A(\mathrm{z})$ be a meromorphic function with $\rho(\mathrm{A}) < \infty$. If $A(\mathrm{z})$ has a finite deficient value a with deficiency $\delta = \delta(\mathrm{a},\mathrm{A})$, then for any given constant $\varepsilon > 0$, there exists a sequence $\{\mathrm{R_n}\}$ with $R_n \le R_{n+1}$ and $\mathrm{R_n} \to \infty$ as $\mathrm{n} \to \infty$, such that the following two inequalities

$$|g(\mathrm{R_n e^{i\varphi}})| > \exp\{\mathrm{R_n}^{\mu(\mathrm{g})-\varepsilon}\},\ \varphi \in [0,2\pi),$$

and

$$mes(\mathrm{F_n}) = \mathrm{mes}\{\theta \in [0,2\pi) : \log|\mathrm{A}(\mathrm{R_n e^{i\theta}}) - \mathrm{a}| \le -\frac{\delta}{4}\mathrm{T}(\mathrm{R_n},\mathrm{A})\} \ge \mathrm{d} > 0$$

hold for all sufficiently large n where d is a constant depending only on $\rho(\mathrm{A}), \mu(\mathrm{g})$ and δ.

Remark 2. If g is a transcendental meromorphic function with $\mu(g)=0$ and $\delta(\infty,g)=1$, according to Lemma 2.2, we only need to give an appropriate modification for method of proof of Lemma 2.3 and Lemma 2.3 still holds.

Remark 3. If g is a transcendental meromorphic function with $\rho(g)<\frac{1}{2}$ and $\delta(\infty,g)=1$, according to Remark 1, we only need to give an appropriate modification for method of proof of Lemma 2.3 and Lemma 2.3 still holds.

Lemma 2.4. ([5]) Suppose that $w(z)$ is a meromorphic function with $\rho(w)=\rho<\infty$. Then for any given $\varepsilon>0$, there is a set $E_4\subset(1,\infty)$ that has finite linear measure and finite logrithmic measure, such that

$$|w(z)|\le\exp\{r^{\rho+\varepsilon}\}$$

holds for $|z|=r\notin[0,1]\cup E_4,\ r\to\infty$.

Lemma 2.5. ([1]) Let $g(z)$ be an entire function with $0\le\mu(g)<1$. Then, for every $\alpha\in(\mu(g),1)$, there exists a set $E_5\subset[0,\infty)$ such that $\overline{\log dens}\ E_5\ge 1-\frac{\mu(g)}{\alpha}$, where $E_5=\{r\in[0,\infty):m(r)>M(r)\cos\pi\alpha\}$, $m(r)=\inf_{|z|=r}\log|g(z)|$, $M(r)=\sup_{|z|=r}\log|g(z)|$.

Remark 4. In Lemma 2.5, if the order of $g(z)$ is less than $\frac{1}{2}$, then, for every $\alpha\in(\rho(g),1)$, there exists a set $E_6\subset[0,\infty)$ such that $\overline{\log dens}\ E_6\ge 1-\frac{\rho(g)}{\alpha}$, where $E_6=\{r\in[0,\infty):m(r)>M(r)\cos\pi\alpha\}$.

3. PROOF OF THEOREMS

In this section, we prove theorems.

Proof of Theorem 1.1. (I). First we prove every solution $f(\neq 0)$ of the equation (1.2) is of infinite order. Suppose that $f(\neq 0)$ is a solution of the equation (1.2) with $\rho(f)<\infty$. We shall seek a contradiction. Let a be a finite deficient value of A_l with deficiency $\delta=\delta(a,A_1)$. From equation (1.2), we have the following inequality

$$|A_0(z)|\le\left|\frac{f^{(k)}(z)}{f(z)}\right|+|A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+\cdots+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{3. 1}$$

By Lemma 2.1, there exists a set $E_1\subset[1,\infty)$ with $m_l(E_1)<\infty$ such that the following inequality

$$\left|\frac{f^{(n)}(z)}{f(z)}\right|\le|z|^{k\rho(f)},\ n=1,2,\cdots,k, \tag{3. 2}$$

holds for all z with $|z|=r\notin E_1\cup[0,r_0],r_0>1$.

Since $\rho(A_i)<\mu(A_0)$ ($i\neq l,1\le i\le k-1$), from Lemma 2.4, then we have for any given constant α with $\rho(A_i)<\alpha<\mu(A_0)$, there exist a set $E_2\subset(1,\infty)$ that has finite linear measure and finite logrithmic measure and a constant $r_1>0$, such that for all z with $|z|=r\notin([0,1]\cup E_2\cup[0,r_1])$, we have

$$|A_i(z)|\le\exp\{r^{\alpha}\}. \tag{3. 3}$$

In the following, we treat two cases.

Case 1. $0 < \mu(\mathrm{A}_0) < \frac{1}{2}$. By using Lemma 2.3 to $A_l(\mathrm{z})$ and $A_0(\mathrm{z})$, for any given $\varepsilon < \frac{\mu(\mathrm{A}_0) - \alpha}{2}$, there exists a sequence $\{\mathrm{R_n}\}$ with $R_n \le R_{n+1}$ and $\mathrm{R_n} \to \infty$ as $\mathrm{n} \to \infty$, such that for every n, we have

$$mes(\mathrm{F_n}) = \mathrm{mes}\{\varphi \in [0, 2\pi) : \log|\mathrm{A}_l(\mathrm{R_n e^{i\varphi}}) - \mathrm{a}| \le -\frac{\delta}{4}\mathrm{T}(\mathrm{R_n}, \mathrm{A}_l)\} \ge \mathrm{d} > 0, \tag{3. 4}$$

and

$$|A_0(\mathrm{R_n e^{i\varphi}})| > \exp\{\mathrm{R_n}^{\mu(\mathrm{A}_0)-\varepsilon}\}, \ \varphi \in [0,2\pi), \tag{3. 5}$$

where d is a constant depending only on $\rho(\mathrm{A}_l)$, $\mu(\mathrm{A}_0)$ and δ.

For every $n \ge n_0$, we choose $\theta_n \in F_n$. From (3.1), (3.2), (3.3) and (3.4), we get

$$\begin{aligned} |A_0(\mathrm{R_n e^{i\theta_n}})| &\le R_n^{k\rho(\mathrm{f})}(1 + |\mathrm{A_{k-1}}(\mathrm{R_n e^{i\theta_n}})| + \cdots + |A_l(\mathrm{R_n e^{i\theta_n}}) - \mathrm{a}| + |\mathrm{a}| + \cdots + |\mathrm{A_1}(\mathrm{R_n e^{i\theta_n}})|) \\ &\le R_n^{k\rho(\mathrm{f})}(1 + \exp(\mathrm{R_n}^{\alpha}) + \cdots + \exp\{-\frac{\delta}{4}\mathrm{T}(\mathrm{R_n}, \mathrm{A}_l)\} + |a| + \cdots + \exp(\mathrm{R_n}^{\alpha})). \end{aligned} \tag{3. 6}$$

By using (3.5) and (3.6), we have

$$\exp\{\mathrm{R_n}^{\mu(\mathrm{A}_0)-\varepsilon}\} < R_n^{k\rho(\mathrm{f})}(1 + (\mathrm{k}-2)\exp(\mathrm{R_n}^{\alpha}) + \cdots + \exp\{-\frac{\delta}{4}\mathrm{T}(\mathrm{R_n}, \mathrm{A}_l)\} + |a|), \ n \ge n_0.$$

Obviously, when n is sufficiently large, this is a contradiction.

Case 2: $\mu(\mathrm{A}_0) = 0$. By using Lemma 2.2, there exists a set $E_3 \subset [1, \infty)$ with $\overline{\log dens}\ E_3$=1 such that for all all z satisfying $|z| = r \in E_3$, we have

$$\log|A_0(\mathrm{z})| > \frac{\pi}{4}T(\mathrm{r}, \mathrm{A}_0). \tag{3. 7}$$

It follows from the Remark 2 of Lemma 2.3, there exists a sequence $\{\mathrm{R_n}\}$ with $R_n \le R_{n+1}$ and $\mathrm{R_n} \to \infty$ as $\mathrm{n} \to \infty$ such that (3.4) and (3.7) hold. From (3.1), (3.2), (3.3) and (3.4), we can obtain (3.6). Hence, from (3.6) and (3.7), we get

$$\exp\{\frac{\pi}{4}\mathrm{T}(\mathrm{R_n}, \mathrm{A}_0)\} < R_n^{k\rho(\mathrm{f})}(1 + (\mathrm{k}-2)\exp(\mathrm{R_n}^{\alpha}) + \cdots + \exp\{-\frac{\delta}{4}\mathrm{T}(\mathrm{R_n}, \mathrm{A}_l)\} + |a|) \ (n \ge n_0). \tag{3. 8}$$

But $A_0(\mathrm{z})$ is a transcendental meromorphic function, so we have

$$\underline{\lim}_{r\to\infty} \frac{T(\mathrm{r}, \mathrm{A}_0)}{\log \mathrm{r}} = +\infty. \tag{3. 9}$$

Thus, we can easily obtain a contradiction from (3.8) and (3.9). So we have that every solution $f(\ne 0)$ of the equation (1.2) satisfies $\rho(\mathrm{f}) = \infty$.

(II) The second step, we prove $\rho_2(\mathrm{f}) \ge \mu(\mathrm{A}_0)$.

By using Lemma 2.1 that there exist a set $E_4 \subset [0, 2\pi)$ that has linear measure zero and constant $B > 0$, such that if $\psi_0 \in [0,2\pi) - E_4$, then there is a constant $r_2 = r_2(\psi_0) > 0$ such that for all z satisfying $\arg z = \psi_0$ and $|z| = r \ge r_2$, we have

$$\left|\frac{f^{(\mathrm{n})}(\mathrm{z})}{\mathrm{f}(\mathrm{z})}\right| \le BT(2\mathrm{r}, \mathrm{f})^{2\mathrm{k}} \tag{3. 10}$$

holds for $n = 1, 2, \cdots, k$.

In the following, we treat two cases.

Case 1. $0 < \mu(\mathrm{A}_0) < \frac{1}{2}$, by using Lemma 2.3 for $A_l(\mathrm{z})$ and $A_0(\mathrm{z})$, we have for any given constant $\varepsilon > 0$, there exists a sequence $\{\mathrm{R}_\mathrm{n}\}$ with $R_n \le R_{n+1}$ and $\mathrm{R}_\mathrm{n} \to \infty$ as $\mathrm{n} \to \infty$, such that for every n, we have (3.4) and (3.5) hold.

For every $n \ge n_0$, there must exist real numbers $\theta_n \in ([0, 2\pi) \bigcap \mathrm{F_n} - \mathrm{E}_4)$, $r_3 > \max\{\mathrm{r_1}, \mathrm{r_2}\}$, such that for all z satisfying $\arg z = \theta_n$ and $|z| = r \notin (\mathrm{E}_2 \bigcup [0, \mathrm{r}_3])$, we have (3.3), (3.4) and (3.5) hold.

Hence, calculating at the points $z_n = R_n e^{i\theta_n}$ with $R_n \notin (\mathrm{E}_2 \bigcup [0, \mathrm{r}_3])$, we get from (3.1), (3.3), (3.4), (3.5) and (3.10) that

$$\exp\{\mathrm{R_n}^{\mu(\mathrm{A}_0)-\varepsilon}\} < BT(2\mathrm{R_n}, \mathrm{f})^{2\mathrm{k}}(1 + (\mathrm{k}-2)\exp(\mathrm{R_n}^{\alpha}) + \cdots + \exp\{-\frac{\delta}{4}\mathrm{T}(\mathrm{R_n}, \mathrm{A}_l)\} + |a|) \ (n \ge n_0).$$

Thus,

$$\lim_{n\to\infty} \frac{\log^+ \log^+ T(\mathrm{R_n}, \mathrm{f})}{\log \mathrm{R_n}} \ge \mu(\mathrm{A}_0).$$

Therefore, $\rho_2(\mathrm{f}) \ge \mu(\mathrm{A}_0)$.

Case 2. $\mu(\mathrm{A}_0) = 0$. Obviously, we have $\rho_2(\mathrm{f}) \ge \mu(\mathrm{A}_0)$. The proof of theorem is completed.

Proof of Theorem 1.2. It follows from the Remark 3 of Lemma 2.3 and same method of proof of Theorem 1.1, we easy to obtain every solution $f(\ne 0)$ of the equation (1.2) satisfy $\rho(f) = \infty$ and $\rho_2(\mathrm{f}) \ge \rho(\mathrm{A}_0)$. Here we omit detail of proof.

Proof of Theorem 1.3. First we prove every solution $f(\ne 0)$ of the equation (1.2) is of infinite order. Suppose that $f(\ne 0)$ is a solution of the equation (1.2) with $\rho(f) < \infty$. We shall seek a contradiction. Let a be a finite deficient value of A_l with deficiency $\delta = \delta(\mathrm{a}, \mathrm{A}_l)$. By Lemma 2.1, there exists a set $E_5 \subset (1, \infty)$ with $m_l(\mathrm{E}_5) < \infty$ such that the inequality (3.2) holds for all z with $|z| = r \notin (\mathrm{E}_5 \bigcup [0, \mathrm{r}_0])\,(r_0 > 1)$. From the proof of Lemma 2.3, there still exists a sequence $\{\mathrm{R_n}\}$ that satisfy (3.4) and $R_n \notin (\mathrm{E}_5 \bigcup [0, \mathrm{r}_0])$. Let $0 < \varepsilon < \frac{d}{2}$. Then for every integer n, we choose $\varphi_n \in F_n \bigcap (\bigcup_{k=1}^{\mathrm{m}} [\phi_\mathrm{k}, \theta_\mathrm{k}])$. Thus there exists an integer $k \in \{1, 2, \cdots, \mathrm{m}\}$ such that for every integer n satisfies $\varphi_n \in F_n \bigcap [\phi_\mathrm{k}, \theta_\mathrm{k}]$ (otherwise we use the subsequence φ_{n_j} instead of φ_n. Thus, from our hypothesis, we have

$$|A_0(\mathrm{R_n e^{i\varphi_n}})| \ge \exp\{(1 + \mathrm{o}(1))\alpha R_n{}^{\beta}\} \quad (3.11)$$

and

$$|A_i(\mathrm{R_n e^{i\varphi_n}})| \le \exp\{\mathrm{o}(1))\mathrm{R_n}{}^{\beta}\}, \text{ for } i \ne l\ (1 \le i \le k-1), \quad (3.12)$$

hold as $n \to \infty$. From (3.1), (3.2) and (3.12), we get

$$|A_0(\mathrm{R_n e^{i\varphi_n}})| < R_n{}^{k\rho(\mathrm{f})}(1 + (\mathrm{k}-2)\exp\{\mathrm{o}(1)\mathrm{R_n}{}^{\beta}\} + \cdots + \exp\{-\frac{\delta}{4}\mathrm{T}(\mathrm{R_n}, \mathrm{A}_l)\} + |a|). \quad (3.13)$$

Obviously, from (3.11) and (3.13), we can lead a contradiction for sufficiently large n.

In the second step, we prove $\rho_2(\mathrm{f}) \ge \beta$.

By using Lemma 2.1 that there exist a set $E_6 \subset [0, 2\pi)$ that has linear measure zero and constant $B > 0$, such that if $\psi_0 \in [0,2\pi) - E_6$, then there is a constant $r_4 = r_4(\psi_0) > 0$ such that for all z satisfying $\arg z = \psi_0$ and $|z| = r \geq r_4$, we have

$$\left|\frac{f^{(n)}(z)}{f(z)}\right| \leq BT(2r, f)^{2k} \tag{3. 14}$$

holds for $n = 1, 2, \cdots, k$.

From the proof of Lemma 2.3, there still exists a sequence $\{R_n\}$ that satisfy (3.4) and $R_n \notin [0, r_4]$. For any given $0 < \varepsilon < \frac{d}{2}$, let $G_m = \bigcup_{k=1}^{m}[\phi_k, \theta_k]$, from (1.3) and (3.4), we have $mes(G = F_n \cap G_m) > 0$. Thus for every integer n, we choose $\varphi_n \in G - E_6$, such that for all z satisfying $\arg z = \varphi_n$ and $|z| = R_n > r_4$, we have

$$|A_0(R_n e^{i\varphi_n})| \geq \exp\{(1 + o(1))\alpha R_n{}^{\beta}\} \tag{3. 15}$$

and

$$|A_i(R_n e^{i\varphi_n})| \leq \exp\{o(1))R_n{}^{\beta}\}, \text{ for } i \neq l \ (1 \leq i \leq k-1). \tag{3. 16}$$

Hence, calculating at the points $z_n = R_n e^{i\varphi_n}$, we get from (3.1), (3.4), (3.14), (3.15) and (3.16) that

$$\exp\{(1 + o(1))\alpha R_n{}^{\beta}\} \leq BT(2R_n, f)^{2k}(1 + (k-2)\exp(o(1)R_n{}^{\beta}) + \cdots + \exp\{-\frac{\delta}{4}T(R_n, A_1)\} + |a|). \tag{3.17}$$

Thus,

$$\lim_{n\to\infty} \frac{\log^+ \log^+ T(R_n, f)}{\log R_n} \geq \beta. \tag{3. 18}$$

Therefore, $\rho_2(f) \geq \beta$. The proof of theorem is completed.

Proof of Theorem 1.4. By using conditions of Theorem 1.4, let $A_0(z) = \frac{g(z)}{h(z)}$, where $g(z)$ is an entire function and $h(z)$ is the canonical product of the poles of $A_0(z)$ with $\lambda(h) = \rho(h)$. Then it is clear that $\lambda(\frac{1}{A_0}) = \rho(h)$. Since $T(r, A_0) \leq T(r, g) + T(r, h) + O(1)$ and $\rho(h) < \rho(A_0)$, for any ε ($0 < 2\varepsilon < \rho(A_0) - \rho(h)$), there exists a sequence $\{R_n\}$ with $R_n \to \infty$ as $n \to \infty$, such that $T(R_n, A_0) > R_n^{\rho(A_0)-\varepsilon}$ and $T(R_n, h) < R_n^{\rho(h)+\varepsilon}$ hold for sufficiently large n. Hence, we get $\rho(A_0) \leq \rho(g)$. On the other hand, since $T(r, g) \leq T(r, A_0) + T(r, h)$ and $\rho(h) < \rho(A_0)$, we deduce that $\rho(g) \leq \rho(A_0)$. Thus $\rho(A_0) = \rho(g)$.

Let $\varepsilon_0 = \frac{1}{4}(\rho(g) - \lambda(\frac{1}{A_0}))$. It follows from remark 4 of Lemma 2.5 that there exist a set $E_7 = E_7(\varepsilon_0, \rho(g)) \subset [0, \infty)$ with $\overline{\log dens}\ E_7 \geq 1 - \frac{\rho(g)}{\alpha_0}$, where $\alpha_0 = \frac{\rho(g) + \frac{1}{2}}{2}$

$E_7 = \{r \in [0, \infty) : m(r) > M(r)\cos\pi\alpha_0 > r^{\frac{\rho(g) - \varepsilon_0}{2}}\}$, $m(r) = \inf_{|z|=r} \log|g(z)|$, $M(r) = \sup_{|z|=r} \log|g(z)|$.

Furthermore, there exists a constant $r_5 > 0$ such that for all $r > r_5$, we have

$$|h(z)| \le \exp\{r^{\rho(h)+\varepsilon_0}\}.$$

Thus for all $r \in E = E_7 - [0, r_5]$, we get

$$|A_0(z)| \ge \frac{\exp\{r^{\rho(g)-\frac{1}{2}\varepsilon_0}\}}{\exp\{r^{\rho(h)+\varepsilon_0}\}} > \exp\{r^{\rho(g)-\varepsilon_0}\}. \quad (3.19)$$

Let a be a finite deficient value of $A_l(z)$ with deficiency $\delta = \delta(a, A_1)$. By using Lemma 2.3 and the same methods of proof Theorem 1.1, we deduce that every solution $f(\ne 0)$ of the equation (1.2) is of infinite order and $\rho_2(f) \ge \rho(A_0)$.

Proof of Theorem 1.5. By using conditions of Theorem 1.5, let $A_0(z) = \frac{g(z)}{h(z)}$, where $g(z)$ is an entire function and $h(z)$ is the canonical product of the poles of $A_0(z)$ with $\lambda(h) = \rho(h)$, then $\lambda(\frac{1}{A_0}) = \lambda(h)$. Similarly to the case of Theorem 1.4, we deduce that $\mu(A_0) = \mu(g)$. Let $\varepsilon_1 = \frac{1}{4}(\mu(g) - \lambda(\frac{1}{A_0}))$. By using Lemma 2.5 to $g(z)$, there exists a set $E_8 = E_8(\varepsilon_1, \mu(g)) \subset [0, \infty)$ with $\overline{\log dens}\ E_8 \ge 1 - \frac{\mu(g)}{\alpha_1}$, where $\alpha_1 = \frac{\mu(g) + \frac{1}{2}}{2}$ $E_8 = \{r \in [0, \infty) : m(r) > M(r)\cos\pi\alpha_1\}$, $m(r)$, $M(r)$ are the same as case Theorem 1.4. Thus, Using the method of Theorem 1.4, we can obtain a set E with $\overline{\log dens}\ E \ge 1 - \frac{\mu(g)}{\alpha_1}$, such that for all $|z| = r \in E$, we get

$$|A_0(z)| \ge \exp\{r^{\mu(g)-\varepsilon_1}\}. \quad (3.20)$$

Using the same method of proof of in Theorem 1.1, we deduce that every solution $f(\ne 0)$ of the equation (1.2) is of infinite order and $\rho_2(f) \ge \mu(A_0)$. The proof of theorem is completed.

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