Probability Functions of Order Statistics from Discrete Uniform Distribution

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ABSTRACT --- In this paper, we firstly give basic definitions and theorems for order statistics. Later, we show that r. probability function of order statistics from discrete uniform distribution can be obtained in another form.

Keywords--- Order statistics, Distribution function, Discrete Uniform Distribution, Moments, Probability Function.

1. INTRODUCTION


2. MATERIAL AND METHOD

2.1. Order Statistics

Let \( X_1, X_2, \ldots, X_n \) be random variables possibly dependent a necessarily identically distributed. By rearranging them in no decreasing order of magnitude, we obtain order statistics \( X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} \). Thus, \( X_{1:n} = \min \{X_1, X_2, \ldots, X_n\} \), \( X_{2:n} \) the second smallest observation among \( X_1, X_2, \ldots, X_n \) and finally \( X_{n:n} = \max \{X_1, X_2, \ldots, X_n\} \). Under stronger assumption that \( X_1, X_2, \ldots, X_n \) are independent and identically distributed random variables with an arbitrary cumulative distribution function \( F \), we obtain that the cdf of \( X_{r:n} \):

\[
F_{r:n}(x) = \Pr(X_{r:n} \leq x) = \Pr(X_1, X_2, \ldots, X_n \text{ at least } r \text{ rv's among } X_1, X_2, \ldots, X_n \leq x)
\]

\[
= \sum_{i=r}^{n} \Pr(X_1, X_2, \ldots, X_n \text{ exactly } j \text{ rv's among } X_1, X_2, \ldots, X_n)
\]

\[
= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^j [1 - F(x)]^{n-j}, \quad -\infty < x < \infty. \quad (2.1)
\]

Furthermore, using the following equation

\[
\sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \int_0^p \frac{n!}{(r-1)!(n-r)!} t^{r-1} (1-t)^{n-r} dt, \quad 0 < p < 1
\]

(2.2)
the cdf of $X_{r,n}$
\[
F_{r,n}(x) = \int_0^x \frac{n!}{(r-1)!(n-r)!} t^{r-1} (1-t)^{n-r} \, dt
\]
\[
= I_{F(x)}(r,n-r+1), -\infty < x < \infty
\]  
(2.3)
can be written as. In this place, $I$ is incomplete beta function. This expression of $F_{r,n}(x)$ want to get discrete or continuous any main mass is provided for. In discrete distribution, for probability mass function of $X_{r,n}$ there are three approaches.

**Approach 1 (Binomial Count)**

For each possible value of $X_{r,n}$
\[
f_{r,n}(x) = F_{r,n}(x) - F_{r,n}(x-).
\]  
(2.4)
Therefore,
\[
f_{r,n}(x) = \sum_{i=x}^{n} \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i} - [F(x-)]^i [1-F(x-)]^{n-i}
\]  
(2.5)
is written.

**Approach 2 (Beta Integral Form)**

From (2.3) and (2.4), using expression of $F_{r,n}(x)$, probability mass function of $X_{r,n}$ is written;
\[
f_{r,n}(x) = \int_{F(x-)}^{F(x)} \frac{n!}{(r-1)!(n-r)!} t^{r-1} (1-t)^{n-r} \, dt.
\]  
(2.6)

**Approach 3 (Multiple Argument)**

For an observation value $X$, let us consider the following three different events: $\{X < x\}, \{X = x\}, \{X > x\}$ respectively. The probability of this events are $F(x-), F(x)$ and $1-F(x)$. $\{X_{r,n} = x\}$ event can occur in $r(n-r+1)$ different ways. $i = 0,1,...,r-1$ and $s = 0,1,...,n-r$ including $(r-1-i)$ units observation value is less than $x$, $(n-r-s)$ observation value is greater than $x$ and in the remaining equal to $x$. Then it can be written
\[
f_{r,n}(x) = \sum_{i=0}^{r-1} \sum_{s=0}^{n-r} \frac{n! [F(x-)]^{i+1} [1-F(x)]^{n-i}[f(x)]^{s+i+1}}{(r-1-i)!(n-r-s)!(s+i+1)!}
\]  
(2.7)
Here, if $x=0$, then $F(x-) = 0$.

3. RESULTS

Let probability mass function be $f(x) = 1/k$ and cumulative distribution function be $F(x) = x/k$, $x = 1,2,...,k$ of $X_1, X_2,..., X_n$ which are n unit independent and identically distributed random variables. From (2.1), cumulative distribution function of $X_{r,n}$:
\[
F_{r,n}(x) = \sum_{i=x}^{n} \binom{n}{i} \left(\frac{x}{k}\right)^i \left(1-\frac{x}{k}\right)^{n-i}, \quad x = 1,2,...,k
\]  
(3.1)
and from (2.6), probability mass function of $X_{r,n}$ can be written as
\[
f_{r,n}(x) = \int_{F(x-)}^{F(x)} C(r:n)u^{r-1} (1-u)^{n-r} \, du
\]
\[
= \int_{(x-1)/k}^{x/k} C(r:n)u^{r-1} (1-u)^{n-r} \, du
\]  
(3.2)

Theorem
Let \( X_1, X_2, \ldots, X_n \) be random sample of size \( n \) from discrete uniform distribution and \( X_{(r)} \) is \( r \)th order statistics. Probability function of \( X_{(r)} \):

\[
f_{r,n}(x) = \sum_{i=r}^{n} \frac{n!}{(i-1)!(n-i)!} \left[ \frac{(x-1)^{i-1}}{k} \left( \frac{k-x+1}{k} \right)^{n-i+1} \right] - \left[ \frac{x^{i-1}}{k} \left( \frac{k-x}{k} \right)^{n+1} \right]
\]

and

\[
f_{r,n}(\alpha) = \sum_{i=r}^{n} \frac{n!}{i!(n-i)!} \left[ \frac{x^i k^{n-i}}{k} \left( \frac{k-x}{k} \right)^{n-i} \right] - \left( \frac{x-1}{k} \right)^{n+1} \left( \frac{k-x}{k} \right)^{n-r+1}
\]

can be found with any of expression. 

**Proof**

In (3.2), if we integrate for \( r = 1 \),

\[
f_{1,n}(x) = \left( \frac{k-x+1}{k} \right)^n - \left( \frac{k-x}{k} \right)^n \tag{3.3}
\]

if we integrate for \( r = 2 \),

\[
f_{2,n}(x) = n \left[ \left( \frac{x-1}{k} \right)^2 \left( \frac{k-x+1}{k} \right)^{n-2} \right] - \left( \frac{x}{k} \right)^{n} \left( \frac{k-x}{k} \right)^{n-2} + f_{1,n}(x) \tag{3.4}
\]

if we integrate for \( r = 3 \),

\[
f_{3,n}(x) = \frac{n(n-1)}{2!} \left[ \left( \frac{x-1}{k} \right)^2 \left( \frac{k-x+1}{k} \right)^{n-2} \right] - \left( \frac{x}{k} \right)^{n} \left( \frac{k-x}{k} \right)^{n-2} + f_{2,n}(x) \tag{3.5}
\]

is obtained. Therefore, we conclude that

\[
f_{r,n}(x) = \frac{n!}{(r-1)!(n-r+1)!} \left[ \left( \frac{x-1}{k} \right)^{r-1} \left( \frac{k-x+1}{k} \right)^{n-r+1} \right] - \left( \frac{x}{k} \right)^{r} \left( \frac{k-x}{k} \right)^{n-r+1} + f_{r-1,n}(x). \tag{3.6}
\]

From (3.3), (3.4), (3.5) and (3.6),

\[
f_{r,n}(x) = \sum_{i=r}^{n} \frac{n!}{(i-1)!(n-i+1)!} \left[ \left( \frac{x-1}{k} \right)^{i-1} \left( \frac{k-x+1}{k} \right)^{n-i+1} \right] - \left( \frac{x}{k} \right)^{i} \left( \frac{k-x}{k} \right)^{n-i+1}
\]

is obtained. Similarly, in (9) if we integrate for \( r = n \),

\[
f_{n,n}(x) = \left( \frac{x}{k} \right)^n - \left( \frac{x-1}{k} \right)^n \tag{3.8}
\]

if we integrate for \( r = n-1 \),

\[
f_{n-1,n}(x) = n \left[ \left( \frac{x}{k} \right)^{n-1} \left( \frac{k-x}{k} \right) \right] - \left( \frac{x-1}{k} \right)^{n-1} \left( \frac{k-x+1}{k} \right) + f_{n,n}(x) \tag{3.9}
\]

and if we integrate for \( r = n-2 \),

\[
f_{n-2,n}(x) = \frac{n(n-1)}{2!} \left[ \left( \frac{x}{k} \right)^{n-2} \left( \frac{k-x}{k} \right)^2 \right] - \left( \frac{x-1}{k} \right)^{n-2} \left( \frac{k-x+1}{k} \right)^2 + f_{n-1,n}(x) \tag{3.10}
\]

is obtained. Hence, we have

\[
f_{r,n}(x) = \frac{n^r}{r!(n-r)!} \left[ \left( \frac{x}{k} \right)^{r} \left( \frac{k-x}{k} \right)^{n-r} \right] - \left( \frac{x-1}{k} \right)^{r} \left( \frac{k-x+1}{k} \right)^{n-r} + f_{r+1,n}(x). \tag{3.12}
\]

Using (3.8), (3.9), (3.10) and (3.11), we obtain
\[ f_{r,n}(x) = \sum_{i=r}^{n} \frac{n!}{i!(n-i)!} \left[ \frac{x^i}{k^i} \left( \frac{k-x}{k} \right)^{n-i} - \frac{(x-1)^i}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} \right]. \]  

(3.13)

The following statement shows that (3.7) and (3.12) are equal.

\[ \sum_{i=\frac{1}{r-1}(n-i)!}^{n!} \left[ \frac{x^i}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} - \frac{(x-1)^i}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} \right] = \sum_{i=r}^{n} \frac{n!}{i!(n-i)!} \left[ \frac{x^i}{k^i} \left( \frac{k-x}{k} \right)^{n-i} - \frac{(x-1)^i}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} \right] \]

Now, let us prove the correctness of the above equality with the induction method. For \( r = 1 \), we have

\[ \left[ \frac{(k-x+1)x^1}{k} - \frac{(k-x)x^1}{k} \right] = \sum_{i=1}^{n} \frac{n!}{i!(1-i)!} \left[ \frac{x^1}{k^i} \left( \frac{k-x}{k} \right)^{n-i} - \frac{(x-1)^1}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} \right]. \]

if \( n = 1 \), then

\[ \left[ \frac{(k-x+1)x^1}{k} - \frac{(k-x)x^1}{k} \right] = \sum_{i=1}^{1} \frac{1!}{i!(1-i)!} \left[ \frac{x^1}{k^i} \left( \frac{k-x}{k} \right)^{n-i} - \frac{(x-1)^1}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} \right]. \]

for \( n = 2 \),

\[ \frac{1}{k} = \frac{1}{k} \]

\[ \left[ \frac{(k-x+1)x^2}{k^2} - \frac{(k-x)x^2}{k^2} \right] = \sum_{i=2}^{2} \frac{2!}{i!(2-i)!} \left[ \frac{x^2}{k^i} \left( \frac{k-x}{k} \right)^{n-i} - \frac{(x-1)^2}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} \right]. \]

and for \( n = 3 \),

\[ \frac{(k-x+1)x^3}{k^3} - \frac{(k-x)x^3}{k^3} = \sum_{i=3}^{3} \frac{3!}{i!(3-i)!} \left[ \frac{x^3}{k^i} \left( \frac{k-x}{k} \right)^{n-i} - \frac{(x-1)^3}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} \right]. \]

For \( r = m \), let us assume that the equality holds, that is,

\[ f_{m,n}(x) = \sum_{i=m}^{n} \frac{n!}{i!(n-i)!} \left[ \frac{x^i}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} - \frac{(x-1)^i}{k^i} \left( \frac{k-x}{k} \right)^{n-i} \right]. \]  

For \( r = m + 1 \), we should show the accuracy of the equality as follows:

\[ f_{m+1,n}(x) = \sum_{i=m+1}^{n} \frac{n!}{i!(n-i)!} \left[ \frac{x^i}{k^i} \left( \frac{k-x+1}{k} \right)^{n-i} - \frac{(x-1)^i}{k^i} \left( \frac{k-x}{k} \right)^{n-i} \right]. \]

If we add the term \( \frac{n!}{m!(n-m)!} \left[ \frac{x^m}{k^m} \left( \frac{k-x+1}{k} \right)^{n-m} - \frac{(x-1)^m}{k^m} \left( \frac{k-x}{k} \right)^{n-m} \right] \) to the both sides of (3.13),
The sum can be written as
\[
\sum_{i=1}^{m} \frac{n!}{(i-1)!(n-i)!} \left[ \left( \frac{x-1}{k} \right) \left( \frac{k-x+1}{k} \right)^{n-i+1} \right] - \left[ \left( \frac{x}{k} \right) \left( \frac{k-x}{k} \right)^{n-i+1} \right]
\]

is obtained. If we open the sum on the right side of the equation for \( i = m \), then we get
\[
\sum_{i=1}^{m} \frac{n!}{(i-1)!(n-i)!} \left[ \left( \frac{x-1}{k} \right) \left( \frac{k-x+1}{k} \right)^{n-i+1} \right] - \left[ \left( \frac{x}{k} \right) \left( \frac{k-x}{k} \right)^{n-i+1} \right]
\]
\[= \sum_{i=m+1}^{n} \frac{n!}{(n-i)!} \left[ \left( \frac{x}{k} \right) \left( \frac{k-x}{k} \right)^{n-i+1} \right] - \left[ \left( \frac{x-1}{k} \right) \left( \frac{k-x+1}{k} \right)^{n-i+1} \right]
\]
\[= \sum_{i=m+1}^{n} \frac{n!}{(n-i)!} \left[ \left( \frac{x}{k} \right) \left( \frac{k-x}{k} \right)^{n-i+1} \right] - \left[ \left( \frac{x}{k} \right) \left( \frac{k-x}{k} \right)^{n-i+1} \right]
\]
\[= \frac{n!}{m!(n-m)!} \left[ \left( \frac{x}{k} \right) \left( \frac{k-x}{k} \right)^{n-m} \right] - \left[ \left( \frac{x}{k} \right) \left( \frac{k-x}{k} \right)^{n-m} \right]
\]
\[= \sum_{i=m+1}^{n} \frac{n!}{(n-i)!} \left[ \left( \frac{x}{k} \right) \left( \frac{k-x}{k} \right)^{n-i+1} \right] - \left[ \left( \frac{x-1}{k} \right) \left( \frac{k-x+1}{k} \right)^{n-i+1} \right]
\]

This completes the proof.

4. REFERENCES