Probability Functions of Order Statistics from Discrete Uniform Distribution

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ABSTRACT---- *In this paper, we firstly give basic definitions and theorems for order statistics. Later, we show that r. probability function of order statistics from discrete uniform distribution can be obtained in another form.*

Keywords--- Order statistics, Distribution function, Discrete Uniform Distribution, Moments, Probability Function.

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1. INTRODUCTION

In literature, there are many papers about order statistics. [1] gave systematic account of some theory of ordered statistics for discrete case. [2] studied order statistics of discrete distribution. [3] gave simple expression for the joint probability function and applied some of our results to theory of order statistics for the discrete uniform distribution. [4] mentioned applications, characterization, approaches for the statistical properties of order statistics of discrete distribution. [5] studied on Markov property bearing structure and mentioned conditional Markov property of order statistics of discrete distribution. [6] were given the distribution of the sample range of discrete order statistics. [7] found the first two moments of discrete order statistics with different method. [8] gave algebraic expression values using probability function of sample maximum from discrete order statistics. [9] obtained sample extreme moments of discrete uniform distribution order statistics. [10] obtained m. moments of discrete uniform distribution order statistics. [11] found given m. moments of sample extreme from discrete uniform distribution order statistics. In this study the *r*. probability function of order statistics from discrete uniform distribution is obtained. [12] obtained mth raw moments of sample extremes of order statistics from discrete uniform distribution and given numerical values are shown in table form.

2. METERIAL AND METHOD

2.1. Order Statistics

Let $X_1, X_2,..., X_n$ be random variables possibly dependent a necessarily identically distributed. By rearranging them in no decreasing order of magnitude, we obtain order statistics $X_{1:n} \le X_{2:n} \le,..., \le X_{n:n}$. Thus, $X_{1:n} = \min (X_1, X_2,..., X_n)$, $X_{2:n}$ the second smallest observation among $X_1, X_2,..., X_n$ and finally $X_{1:n} = \max (X_1, X_2,..., X_n)$. Under stronger assumption that $X_1, X_2, \dots, X_n$ are independent and identically distributed random variables with an arbitrary cumulative distribution function *F*, we obtain that the cdf of $X_{r:n}$;

$$F_{r:n}(x) = \Pr(X_{r:n} \le x) = \Pr(X_1, X_2,..., X_n \text{ at least r rv's among } X_1, X_2,..., X_n \le x)$$

$$= \sum_{i=r}^{n} \Pr(X_1, X_2,..., X_n \text{ exactly j rv's among } X_1, X_2,..., X_n$$

$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i} \ , \ -\infty < x < \infty . \tag{2.1}$$

Furthermore, using the following equation

$$\sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \int_0^p \frac{n!}{(r-1)!(n-r)!} t^{r-1} (1-t)^{n-r} dt, \quad 0 < p < 1 \tag{2.2}$$

the cdf of $X_{r:n}$

$$F_{r:n}(x)=\int_{0}^{F(x)}\frac{n!}{(r-1)!(n-r)!}t^{r-1}(1-t)^{n-r}\,dt$$
$$=I_{F(x)}(r,n-r+1),-\infty<x<\infty \tag{2.3}$$

can be written as. In this place, I is incomplete beta function. This expression of $F_{r:n}(x)$ want to get discrete or continuous any main mass is provided for. In discrete distribution, for probability mass function of $X_{r:n}$ there are three approaches.

Approach 1 (Binomiaol Count)

For each possible value of $X_{r:n}$

$$f_{r:n}(x)=F_{r:n}(x)-F_{r:n}(x-)\,. \tag{2.4}$$

Therefore,

$$f_{r:n}(x)=\sum_{i=r}^{n}\binom{n}{i}\{[F(x)]^{i}[1-F(x)]^{n-i}-[F(x-)]^{i}[1-F(x-)]^{n-i}\} \tag{2.5}$$

is written.

Approach 2 (Beta Integral Form)

From (2.3) and (2.4), using expression of $F_{r:n}(x)$, probability mass function of $X_{r:n}$ is written;

$$f_{r:n}(x)=\int_{F(x-)}^{F(x)}\frac{n!}{(r-1)!(n-r)!}t^{r-1}(1-t)^{n-r}\,dt\,. \tag{2.6}$$

Approach 3 (Multiple Argument)

For an observation value X, let us consider the following three different events; $\{X<x\},\{X=x\},\{X>x\}$,respectively. The probability of this events are $F(x-),F(x)$ and $1-F(x)$. $\{X_{r:n}=x\}$ event can occur in $r(n-r+1)$ different ways. $i=0,1,...,r-1$ and $s=0,1,...,n-r$ including $(r-1-i)$ units observation value is less than x, $(n-r-s)$ observation value is greater than x and in the remaining equal to x. Then it can be written

$$f_{r:n}(x)=\sum_{i=0}^{r-1}\sum_{s=0}^{n-r}\frac{n![F(x-)]^{r-1-i}[1-F(x)]^{n-r-s}[f(x)]^{s+i+1}}{(r-1-i)!(n-r-s)!(s+i+1)!} \tag{2.7}$$

Here, if $x=0$, then $F(x-)=0$.

3. RESULTS

Let probability mass function be $f(x)=1/k$ and cumulative distribution function be $F(x)=x/k$, $x=1,2,...,k$ of $X_1,X_2,...,X_n$ which are n unit independent and identically distributed random variables. From (2.1), cumulative distribution function of $X_{r:n}$;

$$F_{r:n}(x)=\sum_{i=r}^{n}\binom{n}{i}\left(\frac{x}{k}\right)^{i}\left(1-\frac{x}{k}\right)^{n-i},\quad x=1,2,...,k \tag{3.1}$$

and from (2.6), probability mass function of $X_{r:n}$ can be written as

$$f_{r:n}(x)=\int_{F(x-)}^{F(x)}C(r:n)u^{r-1}(1-u)^{n-r}\,du$$
$$=\int_{(x-1)/k}^{x/k}C(r:n)u^{r-1}(1-u)^{n-r}\,du \tag{3.2}$$

Theorem

Let $X_1, X_2, ..., X_n$ be random sample of size n from discrete uniform distribution and $X_{r:n}$ is r th order statistics. Probability function of $X_{r:n}$;

$$f_{r:n}(x) = \sum_{i=1}^{r} \frac{n!}{(i-1)!(n-i+1)!}\left[\left(\frac{x-1}{k}\right)^{i-1}\left(\frac{k-x+1}{k}\right)^{n-i+1} - \left(\frac{x}{k}\right)^{i-1}\left(\frac{k-x}{k}\right)^{n-i+1}\right]$$

and

$$f_{r:n}(x) = \sum_{i=r}^{n} \frac{n!}{i!(n-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{n-i} - \left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{n-i}\right]$$

can be found with any of expression.

Proof

In (3.2), if we integrate for $r = 1$,

$$f_{1:n}(x) = \left(\frac{k-x+1}{k}\right)^{n} - \left(\frac{k-x}{k}\right)^{n} \tag{3.3}$$

if we integrate for $r = 2$,

$$f_{2:n}(x) = n\left[\left(\frac{x-1}{k}\right)\left(\frac{k-x+1}{k}\right)^{n-1} - \left(\frac{x}{k}\right)\left(\frac{k-x}{k}\right)^{n-1}\right] + f_{1:n}(x) \tag{3.4}$$

if we integrate for $r = 3$,

$$f_{3:n}(x) = \frac{n(n-1)}{2!}\left[\left(\frac{x-1}{k}\right)^{2}\left(\frac{k-x+1}{k}\right)^{n-2} - \left(\frac{x}{k}\right)^{2}\left(\frac{k-x}{k}\right)^{n-2}\right] + f_{2:n}(x) \tag{3.5}$$

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is obtained. Therefore, we conclude that

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r+1)!}\left[\left(\frac{x-1}{k}\right)^{r-1}\left(\frac{k-x+1}{k}\right)^{n-r+1} - \left(\frac{x}{k}\right)^{r-1}\left(\frac{k-x}{k}\right)^{n-r+1}\right] + f_{r-1:n}(x). \tag{3.6}$$

From (3.3), (3.4), (3.5) and (3.6),

$$f_{r:n}(x) = \sum_{i=1}^{r} \frac{n!}{(i-1)!(n-i+1)!}\left[\left(\frac{x-1}{k}\right)^{i-1}\left(\frac{k-x+1}{k}\right)^{n-i+1} - \left(\frac{x}{k}\right)^{i-1}\left(\frac{k-x}{k}\right)^{n-i+1}\right] \tag{3.7}$$

is obtained. Similarly, in (9) if we integrate for $r = n$,

$$f_{n:n}(x) = \left(\frac{x}{k}\right)^{n} - \left(\frac{x-1}{k}\right)^{n} \tag{3.8}$$

if we integrate for $r = n-1$,

$$f_{n-1:n}(x) = n\left[\left(\frac{x}{k}\right)^{n-1}\left(\frac{k-x}{k}\right) - \left(\frac{x-1}{k}\right)^{n-1}\left(\frac{k-x+1}{k}\right)\right] + f_{n:n}(x) \tag{3.9}$$

and if we integrate for $r = n-2$,

$$f_{n-2:n}(x) = \frac{n(n-1)}{2!}\left[\left(\frac{x}{k}\right)^{n-2}\left(\frac{k-x}{k}\right)^{2} - \left(\frac{x-1}{k}\right)^{n-2}\left(\frac{k-x+1}{k}\right)^{2}\right] + f_{n-1:n}(x) \tag{3.10}$$

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is obtained. Hence, we have

$$f_{r:n}(x) = \frac{n'}{r!(n-r)!}\left[\left(\frac{x}{k}\right)^{r}\left(\frac{k-x}{k}\right)^{n-r} - \left(\frac{x-1}{k}\right)^{r}\left(\frac{k-x+1}{k}\right)^{n-r}\right] + f_{r+1:n}(x). \tag{3.12}$$

Using (3.8), (3.9), (3.10) and (3.11), we obtain

$$f_{r:n}(x)=\sum_{i=r}^{n}\frac{n!}{i!(n-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{n-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{n-i}\right]. \quad (3.13)$$

The following statement shows that (3.7) and (3.12) are equal.

$$\sum_{i=1}^{r}\frac{n!}{(i-1)!(n-i+1)!}\left[\left(\frac{x-1}{k}\right)^{i-1}\left(\frac{k-x+1}{k}\right)^{n-i+1}-\left(\frac{x}{k}\right)^{i-1}\left(\frac{k-x}{k}\right)^{n-i+1}\right]=\sum_{i=r}^{n}\frac{n!}{i!(n-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{n-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{n-i}\right]$$

Now, let us prove the correctness of the above equality with the induction method. For $r=1$, we have

$$\left[\left(\frac{k-x+1}{k}\right)^{n}-\left(\frac{k-x}{k}\right)^{n}\right]=\sum_{i=1}^{n}\frac{n!}{i!(n-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{n-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{n-i}\right].$$

if $n=1$, then

$$\left[\left(\frac{k-x+1}{k}\right)-\left(\frac{k-x}{k}\right)\right]=\sum_{i=1}^{1}\frac{1!}{i!(1-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{1-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{1-i}\right],$$

for $n=2$,

$$\frac{1}{k}=\frac{1}{k}$$

$$\left[\left(\frac{k-x+1}{k}\right)^{2}-\left(\frac{k-x}{k}\right)^{2}\right]=\sum_{i=1}^{2}\frac{2!}{i!(2-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{2-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{2-i}\right]$$

$$\left(\frac{2k-2x+1}{k^{2}}\right)=\left(\frac{2k-2x+1}{k^{2}}\right),$$

and for $n=3$,

$$\left[\left(\frac{k-x+1}{k}\right)^{3}-\left(\frac{k-x}{k}\right)^{3}\right]=\sum_{i=1}^{3}\frac{3!}{i!(3-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{3-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{3-i}\right]$$

$$\left(\frac{3k^{2}+3x^{2}-6kx+3k-3x+1}{k^{3}}\right)=\left(\frac{3k^{2}+3x^{2}-6kx+3k-3x+1}{k^{3}}\right).$$

For $r=m$, let us assume that the equality holds, that is,

$$f_{m:n}(x)=\sum_{i=1}^{m}\frac{n!}{(i-1)!(n-i+1)!}\left[\left(\frac{x-1}{k}\right)^{i-1}\left(\frac{k-x+1}{k}\right)^{n-i+1}-\left(\frac{x}{k}\right)^{i-1}\left(\frac{k-x}{k}\right)^{n-i+1}\right]$$

$$=\sum_{i=m}^{n}\frac{n!}{i!(n-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{n-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{n-i}\right]. \quad (3.13)$$

For $r=m+1$, we should show the accuracy of the equality as follows:

$$f_{m+1:n}(x)=\sum_{i=1}^{m+1}\frac{n!}{(i-1)!(n-i+1)!}\left[\left(\frac{x-1}{k}\right)^{i-1}\left(\frac{k-x+1}{k}\right)^{n-i+1}-\left(\frac{x}{k}\right)^{i-1}\left(\frac{k-x}{k}\right)^{n-i+1}\right]$$

$$=\sum_{i=m+1}^{n}\frac{n!}{i!(n-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{n-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{n-i}\right].$$

If we add the term $\frac{n!}{m!(n-m)!}\left[\left(\frac{x-1}{k}\right)^{m}\left(\frac{k-x+1}{k}\right)^{n-m}-\left(\frac{x}{k}\right)^{m}\left(\frac{k-x}{k}\right)^{n-m}\right]$ to the both sides of (3.13),

$$\sum_{i=1}^{m+1}\frac{n!}{(i-1)!(n-i+1)!}\left[\left(\frac{x-1}{k}\right)^{i-1}\left(\frac{k-x+1}{k}\right)^{n-i+1}-\left(\frac{x}{k}\right)^{i-1}\left(\frac{k-x}{k}\right)^{n-i+1}\right]$$

is obtained. If we open the sum on the right side of the equation for $i=m$, then we get

$$\sum_{i=1}^{m+1}\frac{n!}{(i-1)!(n-i+1)!}\left[\left(\frac{x-1}{k}\right)^{i-1}\left(\frac{k-x+1}{k}\right)^{n-i+1}-\left(\frac{x}{k}\right)^{i-1}\left(\frac{k-x}{k}\right)^{n-i+1}\right]$$

$$=\sum_{i=m+1}^{n}\frac{n!}{i!(n-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{n-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{n-i}\right]$$

$$+\frac{n!}{m!(n-m)!}\left[\left(\frac{x-1}{k}\right)^{m}\left(\frac{k-x+1}{k}\right)^{n-m}-\left(\frac{x}{k}\right)^{m}\left(\frac{k-x}{k}\right)^{n-m}\right]-\frac{n!}{m!(n-m)!}\left[\left(\frac{x-1}{k}\right)^{m}\left(\frac{k-x+1}{k}\right)^{n-m}-\left(\frac{x}{k}\right)^{m}\left(\frac{k-x}{k}\right)^{n-m}\right]$$

$$=\sum_{i=m+1}^{n}\frac{n!}{i!(n-i)!}\left[\left(\frac{x}{k}\right)^{i}\left(\frac{k-x}{k}\right)^{n-i}-\left(\frac{x-1}{k}\right)^{i}\left(\frac{k-x+1}{k}\right)^{n-i}\right].$$

This completes the proof.

4. REFERENCES

[1] Khatri, C. G. (1962). Distributions of order statistics for discrete case. Annals of the Institute of Statistical Mathematics, 14(1), 167-171.
[2] Das Gupta, S. & Sarkar, S.K. (1984). On TP2 and log-concavity. In Tong, Y.L. (ed.), Inequalities in statistics and probability, London: Institute of Mathematical, pp. 54–58.
[3] Kabe, D. G. (1969). Some distribution problems of order statistics from discrete populations. Annals of the Institute of Statistical Mathematics, 21(1), 551-556.
[4] David, H. A., & Nagaraja, H. N. (1981). Order statistics. New York: John Willey & Sons. David2Order Statistics1981.
[5] Nagaraja, H. N. (1982). Some nondegenerate limit laws for the selection differential. Annals of Statistics, 10, 1306-1310.
[6] Arnold, B. C., Balakrishnan, N., & Nagaraja, H. N. (1992). A first course in order statistics (Vol. 54). Siam.
[7] Ahsanullah, M., & Nevzorov, V. B. (2001). Ordered Random Variables, Nova Sci. *Publ., Huntington, NY, 200*(1).
[8] Çalik, S., & Güngör, M. (2004). On the expected values of the sample maximum of order statistics from a discrete uniform distribution. Applied mathematics and computation, 157(3), 695-700.
[9] Turan, A. (2008) Kesikli Düzgün Dağılımdaki Sıra İstatistiklerin Örnek Ekstremlerinin Mometleri, Fırat University, Science Institute, Elazığ.
[10] Calik, S., et al., On The Moments of Order Statistics from Discrete Distributions, Pak. J. Statist., 26 (2010), 2, pp. 417-426.
[11] Buğatekin Turan, A.,. Generalized Moments of Sample Extremes of Order Statistics from Discrete Uniform Distribution, Fırat University, Institute of Science 2011.
[12] Calik, S., & Bugatekin, A. T. (2018). Moments of Sample Extremes of Order Statistics from Discrete Uniforms Dıstrıbutıon and Numerıcal Results. Thermal Science.