Regularization Method for Nonlinear Singularly Perturbed Systems of Fractional Order

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ABSTRACT---- *The paper is devoted to construction of an asymptotic solution of a weakly nonlinear singularly perturbed differential system of fractional order. To construct the asymptotic solution we use ideas of normal differential forms method of V.F. Safonov. In the environment of the computer mathematical system Maple, approximate solutions are calculated, and corresponding solution schedules for various values of a small parameter are constructed.*

Keywords--- Nonlinear differential equation of fractional order, regularization, asymptotic, iteration problems, normal and unique solvability, computer mathematics system Maple.

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1. INTRODUCTION

We consider a nonlinear problem of the form

$$\varepsilon y^{(1/2)}(t,\varepsilon) = A(t)y + \varepsilon f(t,y) + h(t), \quad y(0,\varepsilon) = y^0, \quad t \in [0,T], \tag{1}$$

where $y = \{y_1,\ldots,y_n\}$ unknown vector function, $A(t)$ known $n \times n$ matrix-function, $f(t,y) = \{f_1,\ldots,f_n\}$ known vector-function, $y^0 = \{y_1^0,\ldots,y_n^0\} \in \Box^n$ known constant vector, $\varepsilon > 0$ small parameter. We look for an asymptotic solution of the problem (1) on the following bounds:

1) $A(t) \in C^\infty([0,T],\Box^n), h(t) \in C^\infty([0,T],\Box^n);$

2)   spectrum $\{\lambda_j(t)\} \equiv \sigma(A(t))$ of the matrix-function $A(t)$ satisfies the requirements:

i) $\lambda_j(t) \neq 0 \quad \forall t \in [0,T], \quad j = \overline{1,n};$

ii) $\lambda_i(t) \neq \lambda_j(t) \quad \forall t \in [0,T], \quad i \neq j, \quad i,j = \overline{1,n};$

iii) $Re\lambda_j(t) \leq 0 \quad \forall t \in [0,T], \quad j = \overline{1,n}.$

The problem (1) will be considered in general for non-linearity $f(t,y)$, analytical with respect to y, i.e. the function $f(t,y)$ belongs to a class Ω, which is described as follows:

Definition 1. They say that the vector functions $f(t,y) = \{f_1, f_2,\ldots,f_n\}$ belong to the class Ω, if its each component $f_i(t,y)$ is expanded into the series:

$$f_i(t,y) = \sum_{|m|\geq 0} f_i^{(m)}(t)y^m \equiv \sum_{m_1+\ldots+m_n=0}^\infty f_i^{(m_1,\ldots,m_n)}(t)y_1^{m_1}\ldots y_n^{m_n}, \tag{2}$$

where all coefficients $f_i^{(m)}(t) \in C^\infty([0,T],\Box^1)$ converge absolutely and uniformly with respect to $t \in [0,T]$ in the polycylinder $\Pi = \{y : |y_i| < R, j = \overline{1,n}\}$ (here: $m = (m_1,\ldots,m_n)$ multiindex, $|m| = m_1 + \ldots + m_n$, $y^m = y_1^{m_1}\ldots y_n^{m_n}$, $R > 0$ some constant.

Wenotethat absolute and uniform convergence of the series (2) in the domain Π (or another similar series) means that the series $\sum_{|m|\geq 0} |f_i^{(m)}(t)||y^m|$ converges uniformly with respect to $t \in [0,T]$ at each fixed $y \in \Pi$.

2. REGULARIZATION OF PROBLEM

Regularization of the problem (1) is performed using the vector $u = \{u_1, \ldots, u_n\}$ of regularizing variables

$$u_j = e^{\frac{1}{\varepsilon}\int_0^t \frac{\lambda_j(s)}{\sqrt{s}}ds} \equiv e^{\frac{\varphi_j(t)}{\varepsilon}}, \quad j = \overline{1,n}.$$

These variables satisfy the linear differential system $\varepsilon\dot{u} = \Lambda(t)u, \ u(0,\varepsilon) = \overline{1} \equiv \{1,\ldots,1\}$ (linear normal form). For the function $\tilde{y}(t,u,\varepsilon)$ such that

$$\tilde{y}(t,u,\varepsilon)\big|_{u=u(t,\varepsilon)} \equiv y(t,\varepsilon),$$

where $u = u(t,\varepsilon) \equiv \left\{ e^{\frac{\varphi_1(t)}{\varepsilon}}, \ldots, e^{\frac{\varphi_n(t)}{\varepsilon}} \right\}$, $y(t,\varepsilon)$ exact solution of the system (1), it is naturally to put the following problem:

$$\varepsilon\sqrt{t}\frac{\partial\tilde{y}}{\partial t} + \frac{\partial\tilde{y}}{\partial u}\Lambda(t)u - A(t)\tilde{y} - \varepsilon f(t,\tilde{y}) = h(t), \quad \tilde{y}(0,\overline{1}) = y^0. \tag{3}$$

Solution of the "extended" problem (3) is defined as a series

$$\tilde{y}(t,u,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(t,u) \tag{4}$$

by non negative powers of the parameter ε. Putting this series into the problem (3), and equating coefficients at the same degrees ε, we obtain the following iteration problems:

$$Ly_0(t,\tau) \equiv \frac{\partial y_0}{\partial u}\Lambda(t)u - A(t)y_0 = h(t), \qquad y_0(0,\overline{1}) = y^0; \tag{5$_0$}$$

$$Ly_1(t,\tau) = -\sqrt{t}\frac{\partial y_0}{\partial t} + f(t,y_0) \qquad y_1(0,\overline{1}) = 0; \tag{5$_1$}$$

$$Ly_2(t,u) = -\sqrt{t}\frac{\partial y_1}{\partial t} + \frac{\partial f(t,y_0)}{\partial y}y_1, \qquad y_2(0,\overline{1}) = 0; \tag{5$_2$}$$

$$Ly_k(t,u) = -\sqrt{t}\frac{\partial y_{k-1}}{\partial t} + P_k(t,y_0,y_1,\ldots,y_{k-1}), \quad y_{k-1}(0,\overline{1}) = 0, \ k \geq 3, \tag{5$_\kappa$}$$

where $P_k(t,y_0,\ldots,y_{k-1})$ some polynomial of $y_1,\ldots,y_{k-1}$ with coefficients, that depend on partial derivatives of functions $f(t,y)$ at the point $y = y_0(t,u)$, moreover, $P_k(t,y_0,\ldots,y_{k-1})$ is linear with respect to the last argument y_{k-1}.

Iteration problems (5_κ) will be solved in the space U of vector functions $w(t,u) = \{w_1(t,u), w_2(t,u), \ldots, w_n(t,u)\}$, each component of which is represent table by the series (2) with coefficients $w_i(t,u) \in C^\infty([0,T], \square^1)$, converging absolutely and uniformly with respect to $t \in [0,T]$ in the domain $G = \{u : |u_i| < 1 + \delta, \ j = \overline{1,n}\}$, where $\delta > 0$ some (small) constant.

3. SOLVABLE OF FIRST ITERATION PROBLEM

The iteration problem (5_0) in the space U has the following solution:

$$y_0(t,u) = \xi_1^{(0)}(t)c_1(t)u_1 + \xi_2^{(0)}(t)c_2(t)u_2 + \ldots + \xi_n^{(0)}(t)c_n(t)u_n -$$
$$- A^{-1}(t)h(t) \equiv C(t)diag\left(\xi_1^{(0)}(t),\xi_2^{(0)}(t),\ldots,\xi_n^{(0)}(t)\right)u - A^{-1}(t)h(t), \tag{6}$$

where $\xi_j^{(0)}(t) \in C^\infty([0,T], \square^1)$ arbitrary functions, $c_j(t)$ eigenvectors of the matrix $A(t)$ corresponding to eigenvalues $\lambda_j(t), j = \overline{1,n}, C(t) \equiv (c_1(t), c_2(t), \ldots, c_n(t))$. Subjecting this solution to the initial condition $y_0(0,\overline{1}) = y^0$, we will have

$$\xi_1^{(0)}(0)c_1(0) + \xi_2^{(0)}(0)c_2(0) + \ldots + \xi_n^{(0)}(0)c_n(0) = A^{-1}(t)h(t) + y^0,$$

where we find the values $\xi_j^{(0)}(0)$:

$$\xi_j^{(0)}(0) = (A^{-1}(0)h(0) + y^0, d_j(0)), \quad j = \overline{1,n}, \tag{7}$$

where $d_j(t)$ eigenvectors of the matrix $A^*(t)$, corresponding to eigenvalues $\bar{\lambda}_j(t), j = \overline{1,n}$, (i.e. $A^*(t)d_j(t) \equiv \bar{\lambda}_j(t)d_j(t), j = \overline{1,n}$). If now we put the function (6) to $f(t, y_0)$, then we obtain the series:

$$f(t, y_0(t,u)) = \sum_{|m| \geq 0} f^{(m)}(t, \xi^{(0)}(t))u^m, \tag{8}$$

belonging (at some auxiliary conditions) to the class U. We now introduce the following designation: if $w(t,u)$ series $\sum_{|m| \geq 0} w^{(m)}(t)u^m$, then by $w^{(k)}(t,u)$ we will denote the sum of the members of this dimension series $|m| = k$, i.e.

$$w^{(k)}(t,u) = \sum_{|m|=k} w^{(m)}(t)u^m, \ k = 0,1,2,\ldots,$$

and by $U^{(k)} \subset U$ subspace of the class U, consisting of all sorts of such sums (assume that the element $0 \equiv \sum_{|m|=k} 0 \cdot u^m$ belongs to the space $U^{(k)}$). In the space $U^{(k)}$ we define the dot (at any $t \in [0,T]$) product:

$$< w^{(k)}(t,u), z^{(k)}(t,u) > \equiv < \sum_{|m|=k} w^{(k)}(t)u^m, z^{(k)}(t)u^m > \overset{def}{=}$$

$$\overset{def}{=} \sum_{|m|=k} (w^{(k)}(t), z^{(k)}(t)) = \sum_{|m|=k} ((w^{(k)}(t))^T \overline{z^{(k)}(t)}).$$

Highlight in (8) $f^{(1)}(t, y_0(t,u))$. Denoting $\bar{y}(t) = -A^{-1}(t)h(t)$ and expanding $f(t, y)$ into Taylor series in a neighborhood of the point $y = \bar{y}(t)$, we will have:

$$f(t, y) = f(t, \bar{y}(t)) + \frac{\partial f(t, \bar{y}(t))}{\partial u}(y - \bar{y}(t)) + \ldots$$

where dots denote the members of the dimension $|m| \geq 2$ relatively to $y - \bar{y}(t)$. Putting here $y = y_0(t,u)$ from (6), we get

$$f(t, y_0(t,u)) = f(t, \bar{y}(t)) + \frac{\partial f(t, \bar{y}(t))}{\partial u}C(t)diag(\xi_1^{(0)}(t),\ldots,\xi_n^{(0)}(t))u + \ldots,$$

where we find

$$f^{(1)}(t, y_0(t,u)) = \frac{\partial f(t, \bar{y}(t))}{\partial y}C(t)diag(\xi_1^{(0)}(t),\ldots,\xi_n^{(0)}(t))u \equiv$$
$$\equiv \sum_{j=1}^n \frac{\partial f(t, \bar{y}(t))}{\partial y}c_j(t)\xi_j^{(0)}(t)u_j. \tag{9}$$

It will be shown below that for solvability of the system (5_1) in the space U, it is necessary and sufficient the conditions:

$$< -\sqrt{t}\frac{\partial y_0^{(1)}(t,u)}{\partial t} + f^{(1)}(t,u), d_j(t)u_j > \equiv 0 \ \forall t \in [0,T], \ j = \overline{1,n}.$$

Putting here functions (6) and (9), we will have

$$< -\sum_{j=1}^n \left[\sqrt{t}\left(\xi_j^{(0)}(t)c_j(t)\right)^{\bullet}u_j + \frac{\partial f(t, \bar{y}(t))}{\partial y}c_j(t)\xi_j^{(0)}(t)u_j \right], d_j(t)u_j > \equiv 0 \ \forall t \in [0,T],$$

or

$$-\sqrt{t}\dot{\xi}_j^{(0)}(t) + \left(\frac{\partial f(t, \bar{y}(t))}{\partial y}c_j(t) - \dot{c}_j(t), d_j(t) \right)\xi_j^{(0)}(t) \equiv 0 \ \ j = \overline{1,n}.$$

Joining the initial conditions (7) to these equations, we find uniquely the functions $\xi_j^{(0)}(t)$:

$$\xi_j^{(0)}(t) = (A^{-1}(0)h(0) + y^0, d_j(0)) \cdot e^{\int_0^t \frac{\left(\frac{\partial f(s, \bar{y}(s))}{\partial y}c_j(s) - \dot{c}_j(s), d_j(s) \right)}{\sqrt{s}}ds}, \ \ j = \overline{1,n}, \tag{10}$$

thus, we construct the solution (6) of the problem (5_0) in the space U unambiguously.

Let us turn to solution of the next problems $(5_к)$ $k \geq 1$. However, for this we need information from the theory of solvability of general form systems $Ly(t,u) = h(t,u)$ in the space U. Turning to the presentation of this theory, we note that the operator L^* in the space $U^{(1)}$ will have the form $L^* = \sum_{j=1}^{n} \bar{\lambda}_j(t)u_j \frac{\partial}{\partial u_j}$ and the basis of its kernel $KerL^*$ in $U^{(1)}$ will be: $v_j(t,u) = d_j(t)u_j, (j = \overline{1,n})$ that it is easy to see directly.

4. NORMAL AND UNIQUE SOLVABILITY OF GENERAL ITERATION PROBLEM

Each of the iteration problems $(5_к)$ can be written as follows:

$$Ly(t,u) \equiv \sum_{j=1}^{n} \lambda_j(t)u_j \frac{\partial y}{\partial u_j} - A(t)y = h(t,u), \qquad y_0(0,\overline{1}) = y^*, \qquad (11)$$

where $h(t,u)$ corresponding right hand side, $y^* \in \square^n$ known constant vector. The following proposition holds.

Theorem 1. Let the right hand side $h(t,u) \in U$ and the conditions 1), 2a) – 2b) hold. Then for solvability of the system (11) in U it is necessary and sufficient that

$$< h^{(1)}(t), v_j(t,u) > \equiv 0 \quad \forall t \in [0,T, \quad j = \overline{1,n}. \qquad (12)$$

Proof. Let $h(t,u) \in U$ have the form $h(t,u) = \sum_{|m| \geq 0} h^{(m)}(t)u^m$. We will define o solution of the system (11) in the form of the series

$$y(t,u) = \sum_{|m| \geq 0} y^{(m)}(t)u^m. \qquad (13)$$

Putting this series in to the system (11), we will have the formal equality:

$$\sum_{|m| \geq 0} y^{(m)}(t) \sum_{j=1}^{n} \lambda_j(t)u_j \frac{\partial u^m}{\partial u_j} - \sum_{|m| \geq 0} A(t)y^{(m)}(t)u^m = \sum_{|m| \geq 0} h^{(m)}(t)u^m.$$

Taking into account that $\frac{\partial u^m}{\partial u_j} = m_j u^{m-e_j} \equiv m_1 u_1^{m_1} \dots u_j^{m_j-1} \dots u_n^{m_n}$, we obtain:

$$\sum_{|m| \geq 0} \left(\sum_{j=1}^{n} m_j \lambda_j \right) y^{(m)}(t)u^m - \sum_{|m| \geq 0} A(t)y^{(m)}(t)u^m = \sum_{|m| \geq 0} h^{(m)}(t)u^m.$$

Equating here coefficients at the same powers u^m, we get the system

$$[(m, \lambda(t))I - A(t)]u^{(m)}(t) = h^{(m)}(t) \quad |m| \geq 0. \qquad (14)$$

Defining solution of this system as $y^{(m)}(t) = C(t)\xi^{(m)}(t)$, we obtain the equations systems:

$$[(m, \lambda(t))I - \Lambda(t)]\xi^{(m)}(t) = C^{-1}(t)h^{(m)}(t) \quad |m| \geq 0,$$

which can be rewritten in the scalar form:

$$[(m, \lambda(t)) - \lambda_j(t)]\xi_j^{(m)}(t) = (h^{(m)}(t), d_j(t)), \quad j = \overline{1,n}, \quad |m| \geq 0. \qquad (15)$$

At $|m| = 0$ we find:

$$\xi_j^{(m)}(t) = \frac{(h^{(0)}(t), d_j(t))}{-\lambda_j(t)}, \quad j = \overline{1,n}, \qquad (16)$$

and when $|m| \geq 2$ we calculate:

$$\xi_j^{(m)}(t) = \frac{(h^{(m)}(t), d_j(t))}{(m, \lambda(t)) - \lambda_j(t)}, \quad j = \overline{1,n}. \qquad (17)$$

When $|m| = 1$ we have the system $(e_k = \{0,\dots,1,\dots,0\}, \quad k-\text{th ort in } \square^n)$

$$(\lambda_k(t) - \lambda_j(t))\xi_j^{e_k}(t) = (h_j^{e_k}(t), d_j(t)), \quad j = \overline{1,n},$$

($k \in \{1, 2, \ldots, n\}$ fixed). For solvability of this system in the space $C^\infty([0,T], \square^n)$ it is necessary and sufficient that

$$(h_j^{e_k}(t), d_k(t)) \equiv 0 \quad \forall t \in [0,T]. \tag{18}$$

Moreover, $\xi_k^{e_k}(t) \in C^\infty([0,T], \square^1) \setminus$ arbitrary function, and $\xi_j^{e_k}(t)$ at $k \neq j$ can be calculated uniquely:

$$\xi_j^{e_k}(t) = \frac{(h^{e_k}(t), d_j(t))}{\lambda_k(t) - \lambda_j(t)}, \quad j = \overline{1,n}, \ j \neq k,$$

therefore

$$y^{e_k}(t) = C(t)\xi^{e_k} = \xi_j^{e_k}(t)c_k(t) + \sum_{j \neq k, j=1}^{n} \frac{(h^{e_k}(t), d_j(t))}{\lambda_k(t) - \lambda_j(t)} c_j(t).$$

Hence, for solvability of the system (11) in the class of formal power series (13) it is necessary and sufficient that conditions (18) are simultaneously satisfied for all $k = \overline{1,n}$, which are equivalent to conditions (12). In this case, the formal solution (13) has the form:

$$y(t,u) = \sum_{k=1}^{n} y^{e_k}(t)u^{e_k} + \sum_{|m| \geq 0, |m| \neq 1} y^{(m)}(t)u^m = \sum_{k=1}^{n} \Big[\xi_k(t)c_k(t) +$$

$$+ \sum_{j \neq k, j=1}^{n} \frac{(h^{e_k}(t), d_j(t))}{\lambda_k(t) - \lambda_j(t)} c_j(t) \Big] u_k + \sum_{|m| \geq 0, |m| \neq 1} \Big(\sum_{j=1}^{n} \frac{(h^{(m)}(t), d_j(t))}{(m, \lambda(t)) - \lambda_j(t)} c_j(t) \Big) u^m, \tag{19}$$

where $\xi_k(t) \equiv \xi_k^{e_k}(t) \in C^\infty([0,T], \square^1)$ arbitrary scalar functions, $k = \overline{1,n}$. Let us now show absolute and uniform convergence of the series (19).

First, we require that there exists a straight line (π), passing through zero $\lambda = 0$ of the complex plane λ, such that all $\lambda_j(t)$ (for all $t \in [0,T]$) lies in the open half-plane bounded by this straight line. It follows that

$$\inf_{|m| \geq 2, t \in [0,T]} \big| (m, \lambda(t) - \lambda_j(t) \big| \equiv p_j > 0, \quad j = \overline{1,n}. \tag{20}$$

And then, denoting by $p = min\{p_1, p_2, \ldots, p_n\} > 0$, we have

$$\big| y_s^{(m)}(t) \big| = \Big| \sum_{j=1}^{n} \frac{(h^{(m)}(t), d_j(t))}{(m, \lambda(t)) - \lambda_j(t)} c_{sj}(t) \Big| \leq \sum_{j=1}^{n} \frac{\big| (h^{(m)}(t), d_j(t)) \big|}{\big| (m, \lambda(t)) - \lambda_j(t) \big|} \big| c_{sj}(t) \big| \leq$$

$$\leq \frac{1}{p} \sum_{j=1}^{n} \sum_{k=1}^{n} \big| h_k^{(m)}(t) \big| \big| d_{kj}(t) \big| \big| c_{sj}(t) \big| \leq \sum_{j=1}^{n} \big| h_k^{(m)}(t) \big|, \tag{21}$$

(here c_{sj} $s-$ th component of the vector c_j, d_{kj}, $h_k^{(m)}$ $k-$ th components of the vectors d_j and $h^{(m)}$ respectively, $c = \sum_{j=1}^{n} \max_{t \in [0,T]} \big| d_{kj}(t) \big| \cdot \big| c_{sj}(t) \big|$. Since the series $\sum_{|m| \geq 2} \big| h_k^{(m)}(t) \big| |u|^m$ converge uniformly with respect to $t \in [0,T]$ ($h(t,u) \in U$), then the series $\sum_{|m| \geq 2} \big| y_s^{(m)}(t) \big| |u|^m$ also has the same property (due to the inequalities (21)).

Therefore, the function (19) belongs to the space U. Theorem 1 is proved.

Thus, in the conditions (12) the system (11) has countless solutions (19), where $\xi_k(t)-$ arbitrary scalar functions, $k = \overline{1,n}$. To calculate them, it is necessary to impose additional restrictions on the solution $y(t,u)$ of system (11).

Theorem 2. Let the conditions 1), 2a) – 2b) hold, and the right hand side $h(t,u) \in U$ satisfies conditions (12). Then the problem (11) with additional restrictions:

$$< -\sqrt{t} \frac{\partial y^{(1)}(t,u)}{\partial t} + B(t)y^{(1)}(t,u) + \upsilon^{(1)}(t,u), \nu_j(t,u) > \equiv 0 \ \forall t \in [0,T, \ j = \overline{1,n}, \tag{22}$$

where $B(t) \in C^\infty([0,T], \square^n)$ known $n \times n$ matrix, $\upsilon^{(1)}(t,u) \in U^{(1)}$ known vector function, uniquely solvable in the space U.

Proof. Subjecting the solution (19) to the initial condition $y(0,\overline{1})$; we will have:

$$\sum_{k=1}^{n}\xi_k(0)c_k(0) = y^* - \sum_{k=1}^{n}\sum_{j\neq k,j=1}^{n}\frac{(h^{e_k}(0),d_j())}{\lambda_k(0)-\lambda_j(0)}c_j(0) + \sum_{|m|\geq 0,|m|\neq 1}\left(\sum_{j=1}^{n}\frac{(h^{(m)}(0),d_j(0))}{(m,\lambda(0))-\lambda_j(0)}c_j(0)\right) \equiv y_*,$$

(here we denote the well-known right-hand side by y_*). Multiplying this equality by $d_j(0)$ and taking into account biortormality of the systems $\{c_j(t)\}$ and $\{d_k(t)\}$, we find $\xi_j(0) = (y_*,d_j(0)),\ \ j=\overline{1,n}.$ Nowsubject the solution (19) to the condition (22). Calculatefirst

$$-\sqrt{t}\frac{\partial y^{(1)}(t,u)}{\partial t}+B(t)y^{(1)}(t,u)+\upsilon^{(1)}(t,u) = \sum_{k=1}^{n}[-\sqrt{t}\dot{y}^{e_k}(t)+B(t)y^{e_k}(t)]u_k+\sum_{k=1}^{n}\upsilon^{e_k}(t)u_k.$$

Multiplying this function by $d_s(t))u_s$, we write the condition (22) in the form:

$$-\dot{\xi}_s(t)-\left[\left(\dot{c}_s(t),d_s(t)\right)-\left(Bc_s(t),d_s(t)\right)\right]\xi_s(t)=l_s(t),\ \ s=\overline{1,n},$$

where $l_s(t)$ known function in the class $C^{\infty}([0,T],\square^{1})$. Joining to these equations the initial conditions $\xi_s(0) = (y_*,d_s(0)),\ s=\overline{1,n},$ found earlier, we calculate uniquely the functions $\xi_s(t)$:

$$\xi_s(t) = (y_*,d_s(0))e^{\int_0^t\frac{[B(s)c_s(s)-\dot{c}_s(s)]}{\sqrt{s}}ds} - \int_0^t e^{\int_0^t\frac{[B(s)c_s(s)-\dot{c}_s(s)]}{\sqrt{s}}ds}l_s(x)dx,\ \ s=\overline{1,n},$$

and thus, uniquely we find the solution (19) of the problem (11). Theorem 2 is proved.

Example 1. Let in the system

$$\varepsilon y^{(1/2)} = \lambda_1 y - \varepsilon\varphi'(t),\qquad y(0,\varepsilon)=y^0,$$
$$\varepsilon z^{(1/2)} = \lambda_2 z + \varepsilon y^2 - \varepsilon g'(t),\quad z(0,\varepsilon)=z^0. \tag{23}$$

eigennumbers λ_1 and λ_2 are constant and related by $\lambda_2 = 2\lambda_1,\ \lambda_1\neq\lambda_2,\ \ Re\lambda_i\leq 0,\ \ \lambda_i\neq 0,\ i=1,2.$ Equivalent problem has the following form:

$$\varepsilon\sqrt{t}\,y' = \lambda_1 y - \varepsilon\varphi'(t),\qquad y(0,\varepsilon)=y^0,$$
$$\varepsilon\sqrt{t}\,z' = \lambda_2 z + \varepsilon y^2 - \varepsilon g'(t),\quad z(0,\varepsilon)=z^0.$$

Introduce regularizing variables:

$$\tau_1 = -\frac{2\lambda_1}{\varepsilon}\sqrt{t} \equiv \varphi_1(t,\varepsilon),\qquad \tau_2 = \frac{2\lambda_2}{\varepsilon}\sqrt{t} \equiv \varphi_1(t,\varepsilon).$$

For extended functions $\tilde{w}\equiv\{\tilde{y}(t,\tau,\varepsilon),\tilde{z}(t,\tau,\varepsilon)\}$ we obtain the following problem:

$$\varepsilon\sqrt{t}\frac{\partial\tilde{y}}{\partial t}+\lambda_1\frac{\partial\tilde{y}}{\partial\tau_1}+\lambda_2\frac{\partial\tilde{y}}{\partial\tau_2}-\lambda_1\tilde{y}=-\varepsilon\varphi'(t),\qquad \tilde{y}(0,0,0,\varepsilon)=y^0,$$

$$\varepsilon\sqrt{t}\frac{\partial\tilde{z}}{\partial t}+\lambda_1\frac{\partial\tilde{z}}{\partial\tau_1}+\lambda_2\frac{\partial\tilde{z}}{\partial\tau_2}-\lambda_2\tilde{z}=\varepsilon\tilde{y}^2-\varepsilon g'(t),\quad z(0,0,0,\varepsilon)=z^0.$$

Defining a solution of this problem in the form of series

$$\tilde{y}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k y_k(t,\tau),\quad \tilde{z}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k z_k(t,\tau).,$$

we get the following iteration systems:

$$L_0 y_0(t,\tau) \equiv \lambda_1\frac{\partial y_0}{\partial\tau_1}+\lambda_2\frac{\partial y_0}{\partial\tau_2}-\lambda_1 y_0=0,\qquad y_0(0,0,0)=y^0,$$

$$L_0 z_0(t,\tau) \equiv \lambda_1\frac{\partial z_0}{\partial\tau_1}+\lambda_2\frac{\partial z_0}{\partial\tau_2}-\lambda_2 z_0=0,\qquad z_0(0,0,0)=y^0, \tag{24}$$

$$L_0 y_1(t,\tau) = -\sqrt{t}\frac{\partial y_0}{\partial t}-\varphi'(t),\qquad y_1(0,0,0)=0,$$

$$L_0 z_1(t,\tau) = -\sqrt{t}\frac{\partial z_0}{\partial t}+y_0^2-g'(t),\ \ z_1(0,0,0)=0, \tag{25}$$

$\square\quad\square\quad\square\quad\square\quad\square$

The problem (24) in the space U has a solution

$$y_0(t,\tau) = \alpha_1(t)e^{\tau_1}, \qquad z_0(t,\tau) = \alpha_2(t)e^{\tau_2},$$

where $\alpha_k(t)$ arbitrary functions, $k=1,2.$ We find initial data for the functions $\alpha_i(t)$: $\alpha_1(0) = y^0$, $\alpha_2(0) = z^0$.

Now we construct the problem (25). It will have the form:

$$L_0 y_1(t,\tau) = -\sqrt{t}\alpha_1'(t)e^{t_1} - \varphi'(t), \qquad y_1(0,0,0) = 0,$$

$$L_0 z_1(t,\tau) = -\sqrt{t}\alpha_2'(t)e^{t_2} + \alpha_1^2(t)e^{2t_1} - g'(t), \; z_1(0,0,0) = 0. \tag{27}$$

For solvability of (27) in the space U we need the following system of equations: $-\sqrt{t}\alpha_1' = 0$, $\alpha_1(0) = y^0$, $-\sqrt{t}\alpha_2' = -\alpha_1^2$, $\alpha_2(0) = z^0$, where we find $\alpha_1(t) = y^0$, $\alpha_2(t) = z^0 + 2y_0^2\sqrt{t}$, and thus, the functions $y_0(t,\tau)$ and $z_0(t,\tau)$ will be:

$$y_0(t,\tau) = y^0 e^{\tau_1}, \; z_0(t,\tau) = z^0 e^{\tau_2} + 2y_0^2\sqrt{t}e^{\tau_2}.$$

Example 2. Find approximate solutions and build graphs of the system solution.

$$\varepsilon\begin{pmatrix} y^{(1/2)}(t,\varepsilon) \\ z^{(1/2)}(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} 0,2 & 0 \\ 0,1 & 1 \end{pmatrix}\begin{pmatrix} y(t,\varepsilon) \\ z(t,\varepsilon) \end{pmatrix} + \begin{pmatrix} 0,1 \cdot y \cdot z \\ 0,1 \cdot z^2 \end{pmatrix}, \quad \begin{matrix} y(0,1) = 1, \\ z(0,1) = 1, \end{matrix}$$

for different values ε in the environment of the computer mathematical system Maple [7-8].
>restart; cond:=y(0.1)=1,z(0.1)=1;

$$cond := \mathrm{y}(0.1) = 1, \mathrm{z}(0.1) = 1$$

>sys:=diff(0.1*y(t),t)=(sqrt(2)*y(t)+0.1*y(t)*z(t))/sqrt(t),(diff(0.1*z(t),t)=(z(t)+0.1*
(y(t)+z(t)*z(t)))/sqrt(t));

$$sys := 0.1\left(\frac{d}{dt}\mathrm{y}(t)\right) = \frac{\sqrt{2}\,\mathrm{y}(t) + 0.1\,\mathrm{y}(t)\,\mathrm{z}(t)}{\sqrt{t}}, 0.1\left(\frac{d}{dt}\mathrm{z}(t)\right) = \frac{\mathrm{z}(t) + 0.1\,\mathrm{y}(t) + 0.1\,\mathrm{z}(t)^2}{\sqrt{t}}$$

$$sys := 0.2\left(\frac{d}{dt}\mathrm{y}(t)\right) = \frac{\sqrt{2}\,\mathrm{y}(t) + 0.2\,\mathrm{y}(t)\,\mathrm{z}(t)}{\sqrt{t}}, 0.2\left(\frac{d}{dt}\mathrm{z}(t)\right) = \frac{\mathrm{z}(t) + 0.2\,\mathrm{y}(t) + 0.2\,\mathrm{z}(t)^2}{\sqrt{t}}$$

>F:=dsolve({sys,cond},[y(t),z(t)],numeric):
>with(plots):
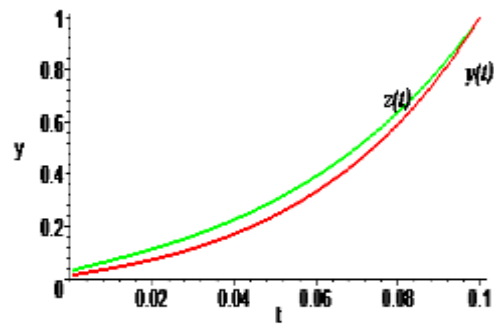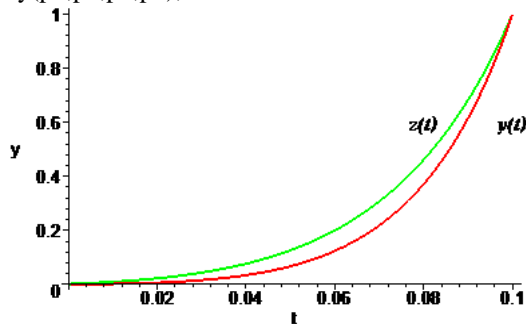>p1:=odeplot(F,[t,y(t)],0..0.1, color=red,thickness=2,linestyle=3):
>p2:=odeplot(F,[t,z(t)],0..0.1,color=green,thickness=2):
>p3:=textplot([0.1,0.6,"y(t)"], font=[TIMES,ITALIC, 12]):
>p4:=textplot([0.08,0.6,"z(t)"], font=[TIMES,ITALIC, 12]):
>display(p1,p2,p3,p4);

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