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On the Globally Asymptotically Stability of Periodic Solutions of a Certain Class of Non-Linear Delay Differential Equations

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ABSTRACT--- ***The objective of this paper is to investigate and give sufficient conditions that will guarantee globally asymptotically stable periodic solutions of the non-linear differential equations with delay of the form (1.1). the Razumikhin's technique was improve upon, to enhance better results equation (1.2) was studied along side with equation (1.1). Equation (1.2) is an integro-differential equations with delay kernel. The coefficients of (1.2) are periodic, and the equation can be rewritten as in form of (3.1), where a,b and c ≥ 0 and ω-periodic continuous function on R.G ≥ 0, is a normalized kernel from equation (1.2). Equation (1.2) enable us to defined equation (3.1) as a fixed point. Since the defined operator "B" for equation (3.1) are not empty, claim (1-iv) enable us to use the fixed point theorem to investigate and established our defined properties. (Theorem 3.1 Lemma 3.1 and Theorem 3.2) was used to prove for periodic and asymptotically stability and the Liapunovis direct (second) method was used to prove our main result. See, (Theorem 3.3, 3.4 and 3.5) which established the objective of this study.***

Keyword--- Globally Asymptotically, Stability, Periodic Solution, Delay Differential Equations

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1. INTRODUCTION

The problem of stability and boundedness of solutions has been the subject of many investigations. Many Research papers and books have been devoted to the study of stability of periodic solutions with delay or non-linear differential equations. Among others see the Literature: (1) established the necessary and sufficient conditions for the periodicity and stability results for solutions of a certain third order non-linear differential equations. And on the other hand, (2) considered conditions that guarantee periodic solutions for differential equations with state-dependent and positive feedback. (3) Considered a system of delay differential equations together with a Liapunov function to established conditions that guarantees asymptotic stability when the delay is unbounded. (4) Use the frequency-domain technique to established conditions for the existence of globally exponentially stable, bounded, periodic and almost periodic for some certain fifth order non-linear differential equations. (6,7) derived necessary conditions for stability of motion of Regulated system with delay. (8) gives sufficient conditions that guarantees global boundedness for a delay differential equations. (10) established conditions that guarantees stability of a system with delay, using Liapunov direct method.

In this paper we study global asymptotic stability of periodic solutions with delay of the equations of the form. (1.1) and (1.2), and use the Liapunov's second method and the fixed point Theorem to established necessary and sufficient conditions that guarantees globally asymptotically stability of periodic solutions with delay of a certain non-linear differential equations. My approach in this study has an advantage over

(6) and the results obtained in this study generalize the results in (3) in the case when the delay was unbounded.

In this paper, we consider a single-species model with a general periodic delay.

$\frac{dx}{dt} = x(t)[a(t) - b(t)x(t) + c(t)x(t - r(t)]$ 1.1

and

$\frac{dx}{dt} = rx\left[1 - \frac{1}{k}\int_{\infty}^{t} G(t-s)x(s)dx\right].$ 1.2

Where a(t)> 0, b(t) > 0, $c(t) \geq 0$ and the delay $\tau(t \geq 0)$, are all continously differentiable in their respective argument. ω-periodic function on $[-\infty, \infty)$ if this model is considered as a population model which size is small, growth is proportional to the size and when the population size is not so small, the positive feedback is $a(t) + c(t)\,x(t - \tau(t))$. While the negative feedback is b(t) x(t) fixed point theorem and Razuminkin technique will be use to prove the main results which guarantee the existence conditions for globally asymptotically stable periodic solutions of nonlinear differential equations with delay of the equation (1.1). Equation (1.2) is an intego-differential equations, where g(t) is called the delay kernel, is a weighting factor which indicates how much emphasis should be given to the size of the population at earlier times to determine the present effect on resource availability. The normalized delay kernel of (1.2) is given by

$\int_0^{\infty} G(u)du = 1$ if $G(u)$ is the Dirac function $\sigma(r - t), where$

$\int_{\infty}^{\infty} \sigma(\tau - s)f(s)ds = f(\tau)$. and equation (1.2) reduces to

$\frac{dx}{dt} = rx(t)\left[1 - \frac{1}{k}\int_{\infty}^{t}\sigma(t - r - s)x(s)ds\right] = rx(t)\left[1 - \frac{x(t-r)}{k}\right]$

Consider the stability of the equilibrium X^* = K for equation(1.2) and we let X = X – K. then(1.2) can be written as $\frac{dx}{dt} = -\tau\int_{-\infty}^{t} G(t-s)x(s)ds + rx(t)\int_{-\infty}^{t} G(t-s)x(s)ds.$

The linear zed equation about x = k is given by

$$\frac{dx}{dt} = -\tau\int_{-\infty}^{t} G(t-s)x(s)ds \quad (1.3)$$

and the characteristics equation takes the form $\lambda + r\int_0^{\infty} G(s)e^{-\lambda s}\,ds = 0$ (1.4)

If all eigen values of the characteristic equation (1.4) have negative real part, then the solution x=0 of (1.3) that is the equilibrium X^*=K of (1.2) is asymptotically stable. Suppose that $V'_{(1)(t,x)\leq -w_{4([x(t)])}}$

(1.5)

The prototype For V is the Krasovskii functional$V(t,x) = x^T(t)\,Ax(t) + \int_{t-r(t)}^{t} x^{T(S)Bx(s)}\,ds$. Where A and B are $n \times n$ matrices. If we express the second term of V as Z $(t, x\,(\cdot))$. We notice that $\left(\frac{d}{dt}\right) 2Z\,(t,x) = x^T(t)Bx(t) - x^T(t - r(t))\,Bx(t - r(t))\,\left(1 - r'(t)\right)$ so that if B and r' are bounded then Z is lipschitz in t for any bounded function X.

If the system $X'(t) \equiv G(t, xt)$ (1.6). Where G is continuous G: $(0\,\infty)X\,C_m \to R^n$, and G takes bounded sets into bounded sets. equation (1.6) is uniformly asymptotically stable

2. NOTATION AND DEFINITIONS

The initial value equation (1.2) is

$$x(\theta) = \Phi(\theta) \geq 0,\ -\infty < \theta \leq 0 \quad (2.1)$$

Where $\boldsymbol{\Phi}(\theta)$ is continuous on $(-\infty, \infty)$. an equilibrium X^* of (1.2) is called stable if given any $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ such that $|\Phi(t) - X^*| \leq \sigma$ for t $\varepsilon(-\infty, 0)$ implies that any solution $x(t)$ of

(1.2) and (2.1)exist and satisfies $|x(t) - x^*| < \epsilon$ for all t≥ 0. If in addition there exists a constant $\delta > 0$ such that $|\Phi(t) - x^*| \leq \delta$ *on* $(-\infty, 0]$ implies $\lim_{t\to\infty} x(t) = x^*$ then x^* is called asymptotically stable

Stability

Definition 1: The origin is said to be stable in the sense of Lyapunov or simply stable if for every real number $\epsilon > 0$ and initial time $t_o > 0$ there exists a real number $\delta > 0$ depending on ϵ and in general on t such that for all initial conditions satisfying the inequality $\|x_0\| < \delta$ the motion satisfies $\|x(t)\| < \epsilon$for all t> t_0

Definition 2:

Asymptotic stability; The origin is said to be asymptotically stable if it is stable and every motion starting sufficiently close to the origin converges to the origin as t tends to infinity. i.e. $\lim_{n\to\infty}\|x(t)\| \to 0$

Definition 3:

Globally asymptotically stable: the origin is said to be globally asymptotically stable in the large if in every motion starting at any point in the state space, returns to the origin at t tends to infinity.

Definition 4:

the equilibrium state $x = 0$ is called exponentially stable if there exist three positive number $\delta_\circ, n, c$ such that $\|x(t)\| \leq n\|x(t_\circ)\|e^{-c(t-t_o)}$ hold for every perturbed motion with $\|x(t_\circ)\| < \delta_\circ$

3. PRELIMINARY NOTES

If the coefficients of equation (1.2) are periodic, it can be re-written in the form of $\frac{dx}{dt} = x(t)\left[a(t) - b(t)x(t) - c(t)\int_{-\infty}^{t} G(t-s)x(s)dx\right]$ (3.1).

Where $a > 0, b > 0,\ c \geq 0$ are ω-periodic continuous functions on R and $G \geq 0$ is a normalized kernel. let $(\omega\ (R, R))$ denote the branch space of all ω-periodic continuous `functions endowed with the usual supreme norm $\|x\| = sup|x(\epsilon)|$ for a $\epsilon\ c_\omega$ define the average of a as $(a) = \frac{1}{\omega}\int_0^\omega a(s)ds$ and a bounded funtion f is defined by $(G \times F)(t) = \int_{-\infty}^{t} G(t-s)f(s)ds$ note that the ω-periodic solution of (3.1) is a fixed point of the operator B:$\Gamma \to c_\omega$ define by $(Bx)(t) = u(t),\ t \in R,$

Where $\Gamma = \{x \in\ c_\omega : \langle a - c(G * x) > 0\ \}$ since (a)> $0, x(t) \equiv 0$ belongs to Γ. That is Γ is not empty. Define $u_{0(t)=(B_0)(t)}$

Calm I. If x_1 and x_2 belong to Γ with $x_1\ \leq x_{2,}$ then $Bx_2 \leq Bx_1$

In fact. Let $\propto i(t) = a(t) - c(t)(G * xi)(t) and\ Ui(t) = (Bxi)(t) for\ t \in R(i = 1,2)$. Then $\propto_1(t) \geq\propto_2(t)$. Since $\propto_i(t) = \mathring{u}_i(t)/Ui(t) + b(t)ui(t),$

we have $\propto_i = \langle bui\rangle$ because $ui(t)(i = 1,2)$ are periodic. Thus, we deduce $(\langle bu_1\rangle \geq \langle bu_2\rangle and\ for\ some\ to\ \in R_1 U_2(t_0) \leq U_1(t_0)$

Setting $v(t) = U_1(t) - U_2(t)$, we have

$\dot{V}(t) \geq (\propto i(t) - b(t)(u_1(t) + u_2(t))v(t),$

Which implies that $V(t) \geq 0 for\ all\ t \geq to$. by the Periodicity of $V(t)$, we have $Bx_2 \leq Bx_1$.

Claim II. If V and c belong to C_ω, then $\langle c(G * v)\rangle = \langle V(G * c)\rangle$

In fact, if we define $G(t) = 0$ for $t < 0$, we have

$$\int_{jw}^{(j+1)\omega} G(t-s)v(s)\text{ds}\ dt$$

$\langle c(G*v)\rangle = \sum_{j=-\infty}^{+\infty} \int_o^{\omega} c(t)$

$= \sum_{j=-\infty}^{+\infty} \int_o^{\omega} c(t) \quad \int_o^{\omega} G(t-s-j\omega)v(ts)ds\,dt$

$= \sum_{j=-\infty}^{+\infty} \int_o^{w} v(t) \int_{-jw}^{(1-j)w} G(t-s)c(s)ds\,dt$

$= \langle V(G*v)\rangle$.

Claim III. Let z be a bounded continuous function on R. Then

$\liminf_{t\to\infty}(G*z)(t) \geq \lim_{t\to\infty} z(t)$; $\limsup_{t\to\infty}(G*z)(t) \leq \limsup_{t\to\infty} z(t)$.

We only prove the first inequality. Let $l = \liminf_{t\to\infty} z(t)$. Choose $\in > 0$ and pick tc, such that $z(t) > t - \in$ for any $t > tc$. If $t > tc$; we have $(G*z)(t) = \int_{-\infty}^{tc} G(t-s)z(s)ds + \int_{tc}^{t} G(t-s)z(s)ds. \quad \geq$

$\mathrm{Inf}_t z(t) \int_{-\infty}^{tc} G(t-s)ds + (l-\in)\int_{t_c}^{t} G(t-s)ds$

Hence $\liminf_{t\to\infty} (G*z)(t) \geq l - \in$,

Which implies the first inequality.

Claim IV. Let $U \in \Gamma$ and let $V(t) > 0$ be the solution of (3.1). Then

$\liminf_{t\to\infty} (v(t)-u(t)) > 0 \textit{ implies } \liminf_{t\to\infty} ((Bu)(t) - v(t) > 0$

$\limsup_{t\to\infty} (v(t)-u(t)) < 0 \textit{ implies } \limsup_{t\to\infty} (Bu)(t) - v(t)) < 0$

We prove the first statement. Let $\omega(t) = (Bu)(t) t \in R$.

Then $\dot{\omega}(t)$ is a solution of $\omega(t) = a(t)\omega(t) - b(t)\omega(t)^2 - c(t)\omega(t)(G*u)(t)$

While$\dot{V}(t) = a(t)v(t) - b(t)v(t)^2 - c(t)v(t)(G*v)(t)$.

Define $\dot{Z}(t) = (a(\dot{t}) - b(t)\omega(t) - c(t)(G*u))z(t) + c(t)v(t)\,(G*(v-u))\,(t)$

$= (\dot{\omega}(t)/\omega(t) - b(t)\,v(t))\,Z(t) + c(t)\,v(t)\,(G*(v-u))\,(t)$

Let $l = \lim inf t \to \infty(v(t) - u(t)$. Because of claim III, there exist a to $\in R$, such that $\dot{Z}(t) > (\dot{\omega}(t)/\omega(t) - b(t)v(t)z(t) + (t)lc(t)v(t)/2$ for all $t > t_o$, that is

$Z(t) > Z(to)exp\left\{\int_{to}^{t} \beta(s)ds\right\} + \frac{1}{2}\int_{t_o}^{t} \beta(\theta)\,d\theta\,c(s)v(s)ds.$

where $\beta = \omega(t)/\omega(t) - b(t)v(t)$. Because $\dot{\omega}(t)/w(t)$ is periodic and its average is zero, $b(t)v(t)$ is positive and bounded, we can see that $\int_{to}^{t} \beta(sx)ds > k - tk_2$. Where $k_1 - k_2 > 0$ are constants.

Thus $z(t) > k_3 \int_{t_o}^{t} \exp\,(s-t)k)ds = (k_3/k)\,(1 - \exp((to-t)k_2))$,

Where $k_3 > 0$ is a suitable constant. Then $\lim\ inf t \to \infty z(t) \geq K_3/K$ which implies the first statement.

Below Theorems we a give sufficient conditions to the proof of the main Results.

Theorem 3.0 Let $0 \in G, f(0) = 0$ and assume a Lyapunov function $v: G \to R$ exists, for which $V^{(1)}(x) \leq 0 (\forall\ x\ t\ G)$. Then the solution $x(t) \equiv 0$ of equation (1.1) is stable in the sense of Lyapunov.

Theorem 3.1 Suppose $\langle a\rangle > 0. if\ b(t) > (G*c)(t)$ (3.2), for and $t \in (o, w)$, then equation (3.1) has a unique positive $\omega -$ periodic solution $x^*(t)$ which is globally asymptotically stable with respect to all the solutions of equation (3.1) under initial condition $x(\theta), \theta \in (-\infty, 0), \emptyset(0) > 0$.

Lemma 3.1 Let the function g: $[\propto -h, \beta] \rightarrow R$ be continuous for some $h > 0, \propto < \beta \leq \infty, and\ \dot{g}^{ur}(t) \leq 0$ for all values $t \in (\propto, \beta)$ for which $g(s) < g(t)(\forall_s \in [\propto -h, t))$. Then $g(t) \leq Max_s \in (\propto -h, \propto) g(s)(\propto \leq t \subset \beta)$.

Theorem 3.2 If $\int_0^\infty G(s)ds < \frac{1}{r}$,
then $x^* = K$ of (1.2) is asymptotically stable

Theorem 3.3. Assume the Lyapunov function $V: G \rightarrow R$ exists with $x, y, \in G, V(Y) < V(x)$ imply (grad $V(x), f(x, y)) \leq 0$ Then the solution $x(t) \equiv 0$ of equation (1.1) is stable.

Theorem 3.4; Assume, in addition to the conditions of theorem 3.3, that for any sufficiently small $p > 0$, there exist $q > p$ such that, if $V(x) = P, V(Y) < q$, then (grad $V(x), f(x, y)) < 0$
Then the solution $x(t) \equiv 0$ of equation (1.1) is asymptotically stable.

Theorem 3.5; Let $V\ and\ Z: (0, \infty) \times C_m \rightarrow (0, \infty)$ be continuous and let V be totally lipschitz in the second argument. Suppose that $Z(t_2, \emptyset) - Z(t_{1,} \emptyset) \leq K(t_2 - t_1)$ for some $K > 0$, all t_1 and t_2 satisfying $0 < t_1 < t_2 < \infty$, and all $\emptyset \in C_{m,}$ if $W(|\emptyset(0)|) + Z(t, \emptyset) \leq V(t, \emptyset) \leq \omega_1(|\emptyset(0)|) + Z(t, \emptyset), Z(t, \emptyset) \leq \omega_2(\|\emptyset\|)$, and $v^1{}_{(4)}(t, x_t) \leq -\omega_4(|x(t)|)$, then $x = 0$ is uniformly asymptotically stable of (1.6).

4. THE PROOFS OF THE MAIN RESULTS

Proof of Theorem 3.1

Since $\dot{u}_0(t)/u_0(t) = a(t) - b(t)u_0(t)$, the periodicity of $u_0(t)$ and claim II imply that $\langle a \rangle = \langle bu_0 \rangle > (c(G * u_0)]$. As $u_0 > 0$, we have $Bu_0 \leq u_0$. therefore, for any $V \in C\omega$ satisfying $0 < V \leq u_0$, we have $Bu_0 \leq BV < u_0$. Hence, the set $\Gamma_0 = \{V \in C\omega: 0 < V \leq u_o\} C\Gamma$ is invariant under B. moreover,
$Bu_0 \leq BV \leq u_0 \Rightarrow Bu_0 \leq B^2V \leq B^2_{uo} \Rightarrow B^3_{uo} \leq B^3V \leq B^2_{uo}$
And by induction
$B^{2n+1}u_o \leq B^{2n+1}V \leq B^{2n}u_o, B^{2n+1}u_o \leq B^{2n+2}V \leq B^{2n+2}u_o n = 0, 1, 2 ...$
Since $0 < B^2_o = Bu_0$, by claim I, we know that $\{B^{2n+2}u_o\}$ is increasing and $\{B^{2n}u_o\}$ is decreasing.
Define
$Un(t) = (B^n uo)(t) = (BUn - 1)(t)$.
Then
$u^-(t) = \lim_{n\rightarrow\infty} U_{2n+1}(t) and U^+(t) = \lim_{n\rightarrow\infty} Un(t)$
Exist with $0 < u^-(t) \leq u^+(t)$. If we can show that $u^-(t) = u^+(t) = u^*(t)$, it is easy to see that $u^*(t)$ is the unique fixed point of B. by the definition, we have,
$\dot{U}n(t)(a(t) - c(t)(G * Un - 1)(t)Un\ (t) - b(t)(un)^2$
By the Monotonicity and uniform boundedness of $\{Un\}$. We have the $L^2 -$ convergence of both U_{2n+1} and U_{2n} and their derivatives.
Taking the limits, we have
$\dot{u}^-(t) = (a(t) - C(t)(G * u^+)(t)u^-(t) - b(t)u^-(t)^2$
$\dot{u}^+(t) = (a(t) - C(t)(G * u^{-1})(t)u^+(t) - b(t)u^+(t)^2$
Dividing them by $u^-(t) and\ u^+(t)$ respectively, we have
$\langle a - c(G * u^+) - bu^- \rangle = \langle a - c(G * u^-) - bu^+ \rangle$
followed by the fact that Inu^+ and Inu^- are periodic. Let V(t) = U^+ (t) – $U - (t)$. Then we have {C(G*V)} = (bV). Now by claim II we have {C(G*V)} = { V{G*C)}.
Hence, {V(b – G*C)} = 0, which implies that V = 0 by the assumption (3.2)

Therefore, U* (t) is a unique periodic solution of the equation (3.1)

To prove the global stability, first, we show that any solution v(t) of equation (3.1) satisfies $\lim inf_{t \to \infty} v(t) > 0$. In fact, we have

$$\dot{V}(t) < a(t)\, v(t) - b(t)\, v(t)^2$$

and $$\lim_{t \to \infty} \sup(v(t) - (Bu)(t)) \leq 0$$

Choose $\in\, > 0$ so that u(t) = Uo(t) + $\varepsilon \;\epsilon\; \Gamma$. By claim IV we have

$$\lim inf_{t \to \infty} (v(t) - (Bu)(t)) \geq \epsilon.$$

Since (Bu)(t) is strictly positive and periodic, we have lim inf t → ∞v(t) > 0. Thus by claim III, lim inft → ∞ (Uolt) – V(t)) > 0 and by iniluction.

$\lim inf_{t \to \infty} v(t) - (B^{2n+1}Uo)(t) > 0$, $\lim_{t \to \infty} Sup(v(t) - (B^{2n}Uo)(t) <\!\!,0$

Given ∈ > 0, choose n such that

$U^*(t) - \in\; < (B^{2n+1}uo)(t) < (B^{2n}Uo)(t) < U^*(t) + \epsilon.$

Since $(B^{2n+1}uo)(t) < v(t) < (B^{2n}uo)(t)$ for large t, it follows that the sequence $\{B^j u\}$ tends to u* uniformly as j → ∞.

This complete the proof.

If a, b and c are real positive constants, then condition (3.1) becomes b > c.

Corollary 4.1. If b > c and g satisfies the above assumptions, then the positive equilibrium x* = a/(b + c) of equation (3.1) (with constant coefficient) is globally stable with respect to positive solutions of (3.1).

Proof of Theorem 3.2

Proof. Since the roots of (1.4) coincide with the zeros *of the function* $g(\lambda) = \lambda + r\int_o^{\infty} G(s)\, c^{-\lambda s}\, ds,$ *applying the argument principle to* $g(\lambda)$ *along the contour* $\Gamma = \Gamma$ (a, ε) that constitutes the boundary of the region

$\{\lambda\, /\varepsilon \leq Re\lambda \leq a,\ -a \leq lm \leq a,\ 0 < \epsilon\ a\}.$

Since the zeros of g (⋋) are isolated, me may choose a and ϵ. So that no zeros of g (⋋) lie on Γ. The argument principle now states that the number of zeros of g (⋋) contained in the region bounded by Γ is equal to the number of times g (⋋) wraps τ around the origin as ⋋ traverses Γ. (A zero of g (⋋) of multiplicity m is counted m times). Thus, it suffices to show for all small $\epsilon > 0$ and all large a > r, that g (⋋) does not on circle 0 as ⋋ traverses Γ (a, ϵ).

Along the segment of Γ given by ⋋ = a + iv, - a ≤ μ ≤ a,

we have

$$g(a+iv) = a + iv + r\int_0^{\infty} G(s)e^{-(a+iv)s}\, ds.$$

Since a > 0, it follow that

$$\left|\int_0^{\infty} G(s)e^{-(a+iv)s}\, ds.\right| \leq \int_0^{\infty} G(s)\, ds = 1$$

Because a > r, we may conclude that every real value assumed by g (⋋) along this segment must be positive. Along the segment of

Γgiven by ⋋ = μ + ia, C≤ a, we have

g(μ + ia) = μ + ia + r $\int_0^{\infty} G(s)e^{-(\mu+iv)s}$ ds.

A similar argument show g (⋋) to assume no real value along this path. In fact, lim g(μ + ia) is always negative here. Similarly, one can show that the lim g(u-ia) is negative along 0. The segment ⋋ = μ - ia, C $\leq \mu \leq$ a. By continuity g (⋏) must assume at least one positive real value (and no negative values) as ⋏ travels clock wise from ε + ia to ϵ - ia along τ.

Finally, consider the path traced out as ⋋ = ϵ + iv increase from ϵ - ia to ϵ + ia. under the assumption, 1m g(ϵ + iv) is seen to increase monotonically with v. in fact,

$$\frac{d}{dv}\ 1mg\ (\epsilon + iv) = \frac{d}{dv}\ \ (v + r \int_0^{\infty} G(s)e^{-\varepsilon s}\ Sin\ (vs)ds)$$

$$= 1 + r \int_0^{\infty} sG(s)e^{-\varepsilon s}\ Cos\ (vs)ds$$

$$\geq 1 - r \int_0^{\infty} sG(s)\ ds$$

$$> 0.$$

It follows immediately that g(⋋) assumes precisely one real value along this last segment of Γ. Since no zero of g(⋋) lies on τ, that real value is non-zero. Assuming it to be negative, g(⋋) would have wrapped Γ once about the origin, predicting exactly one zero⋋ vo of g(⋋) inside the region bounded by Γ.

Since ∝ and G are real, the zeros of g(⋋) occur in complex conjugate pairs, forcing ⋋ o to be real.

This, however, is a contradiction since the positivity of ∝ shows g(⋋) to have no real positive zeros. Thus, the real assumed by g(⋋) along this last segment must be positive.

Therefore, g(⋋) does not encircle the origin.

This completes the proof.

Proof of Theorem 3.3

For this proof, let set g(t) = v(x(t)), where x(.) is any solution of equation (1.1) with initial function ∅ sufficiently close to zero, and apply lemma 3.1. We see that even a stronger statement for such solutions holds: v(x(t)) Max $[t_o - h, t_o]V(\emptyset)$. However in general, the monotonixcity of the function $t \rightarrow V(x(t))$does not hold true, unlike the situation arising valuer conditions of Theorem 3.0. Thus the function v turns out to be not a guiding function in the restricted sense, but a "barrier" function.

It should be noted that the function t → g(t), defined in the proof of Theorem 3.3, is smooth for t $\geq$ ∝: = to. Lemma 3.1 follows easily under this additional assumption from the following particular case of A. Sard's theorem: a set of stationary values of a smooth function of a single argument has that everywhere dense complement.

Proof of Theorem 3.4

Indeed, if the solution t → x(t) of equation (1.1) is modulo sufficiently small and does not tend to zero as $t \rightarrow \infty$, then we denote P: = $\overline{\imath m\imath t}$ $t \rightarrow \infty$ $V(x(t))$, and apply the additional condition of Theorem 3.3 which leads to the contradiction.

In particular, we obtain for n = 1, taking v (x) = $x^{2,}$ that the condition $/b/$ < - a is sufficient for the asymptotic stability of equation (1.1).

Remark: If the function v is defined on the whole $\mathbb{R}^n$ in theorem 3.4, and v(x) → ∞ as /x/→ ∞ and p > 0 can be arbitrary, then the solution of equation (1.1) tends to zero as $t \rightarrow \infty$ for arbitrary initial function.

Proof of Theorem 3.5

We first show uniform stability. Let $\bar{\varepsilon}$ > 0 be given.

We will find $\bar{\sigma}$ > 0 such that ($t_0 \geq 0$, $\|\emptyset\| < \sigma$, and t $\geq t_0$)

Imply 1 x (t, ∅)) < $\bar{\varepsilon}$. There exists δ > o such that W_1 (σ) + W_2 (σ) < W(ε). Thus, if $\|\emptyset\| < \delta$ and x (t) = x (t, ∅), then W (1x (t) 1) ≤ V(t, x_t) ≤ V(to, ∅) ≤ w_1 (1 ∅(0) + w_2 (‖∅‖) $< W$ (ε) so that $| x(t) | < \varepsilon \; if \; t > to,$ yielding uniform stability.

To complete the proof we must find η > 0 such that for any ϵ > 0 there exists T for which (t_0 ≥ 0, ‖∅‖< η, and t ≥ t_0 + T) imply $|x$ (t, ∅) | < ϵ. Pick the $\bar{\delta}$ of uniform stability when $\bar{\epsilon}$ = M. select a η = $\bar{\delta}$. Now, let ϵ > 0 be given and let to ≥ 0, ‖∅‖ < η, and let x (t) = x (t, ∅).

For the given ϵ > 0, find the δ of uniform stability as in the above proof. By that proof we see that on each interval of length h either /x(t) / ≥ δ for some t in the interval, or / x(t) < ϵ for all subsequent t. thus, let to ≥ 0 be arbitrary and suppose there is a sequence (t_n) with to ≤ t_1 ≤ t_0 + h ≤ t_2 ≤ t_0 + 2h ≤ t_3 ≤ … ≤ t_0 + (n – 1) h ≤ t_n≤ t_n + nh … with / x (t_i) / ≥ δ. We will show that n may not exceed a fixed integer. As in Theorem 3.0 without proof, we use the right Lipchitz condition on 2 to find k > 0 and P > 0 with P ≤ h and / x(t) / >, K if ti – P ≤ t ≤ t_1. We select only the t_1 with even indices so that the intervals over which we now integrate do not overlap. For t ≥ t_{2n}, we have V (t, x_t) ≤ V (to, ∅) −

$$\sum_{i=1}^{n} \int_{t2i-p}^{t_{2i}} W_4\,(k)\,dt \;\leq\; W_1\,(\eta) + W_2(\eta) - nP\,W_4\,(k). \;\; Choose\; N$$

>$[W_1(\eta) + W_2\,(\eta)]\;/\;P\,W_4\,(k)$ and select T = 2Nh. This yields uniform asymptotic stability of equation (1.6). This complete the proof.

Remark 4.1, Obviously, the authors in (2 – 10) considered existence of periodic solutions with delay of non linear differential equations of various order. Author in (3) considered a system of differential equations with delay and use Liapunov's function to established asymptotic stability where the delay was unbounded. Hence, the results obtained in (2 – 10) are not the same in this paper which implies that the results of this paper are essentially new. Theorem 3.1, lemma 3.1, Theorem 3. And the inequality i-iv established the conditions for equation (1.2) to be globally asymptotically stability of periodic solutions with delay. And theorem 3.3,3.4 and 3.5 with Lipunov's second method, prove properties that satisfied globally asymptotically stability of non-linear differential equation.

5. CONCLUSION

The globally asymptotically stability of periodic solution of a certain class of delay differential equation have been proved in this paper. This paper improved on the paper in Author (3) where all are properties satisfied the existence and unique of solutions of the form (1.1) and (1.2) that enable us established the stability of the periodic solutions.

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