Strict Fuzzy Triangular and Trigonometric Exponential Truncated Distributions

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ABSTRACT— ***In this paper, based on the definition of strict fuzzy probability introduced by Abed, et. al. (2016) we will show how we can construct strict fuzzy membership function from classical fuzzy membership function introduced by Zadeh (1965). This technique will be applied to triangular and trigonometric fuzzy memberships. Once we reach that result we will introduce strict fuzzy exponential triangular and trigonometric distributions and some of their properties.***

1. INTRODUCTION

Consider a measurable space (Ω, Ɓ,P), where Ω is defined to be R^n, Ɓ is Borel sigma algebra and P is a probability measure on Ɓ such that P: Ɓ → [0,1] also consider $x \in R^n$. Then, strict fuzzy probability was presented by Abed, et. al. (2016) for a subset A (an event) by Lesbuge-Stejles integral as

$$P(A) = \int_{R^n} I_A(x)\mu_\Omega(x)\, dP, \tag{1}$$

where $\mu_\Omega(x)$ is measurable membership function defined on the set Ω.

Based on Zadeh's (1968) definition of fuzzy probability, Yung, et. al. (2006) introduced exponential fuzzy probability distribution for triangular and trigonometric fuzzy memberships, they also introduced the expected value and the variance of this distribution. While, Agahi and Ghezelayagh (2009) studied the truncated fuzzy normal distribution.

In the following sections of the paper we will show how to construct triangular and trigonometric membership function on universal set, present what will be called truncated strict fuzzy triangular and trigonometric exponential probability distributions and their statistical properties and finally we will draw some conclusions out of this study.

2. DEFINITIONS

Strict fuzzy random variable: Let $\Omega = R^n$ be sample space, X to be a random variable defined on Ω and x to be a value of the random variable X. Now consider that, each element x belongs to Ω by membership function $\mu_\Omega(x): \Omega \to [0,1]$, also consider that the a probability distribution of X, say f(x). Then, X is called strict fuzzy random variable.

Cumulative strict fuzzy probability distribution: If X is a strict fuzzy random variable, then it's cumulative distribution is

$$\tilde{F}(x) = \int_{-\infty}^{x} \mu_\Omega(x) f(x) dx. \tag{2}$$

Strict fuzzy R^{th} central moment: The R^{th} central moment of strict fuzzy random variable X is given by

$$\tilde{E}[x^r] = \int_{-\infty}^{\infty} x^r \mu_\Omega(x) f(x) dx, \tag{3}$$

where, $\tilde{E}$ denotes strict fuzzy expectation. Note that, we can find the non-central moment using the binomial expansion and linearity property of expectation as $\tilde{E}\left[\left(x-\tilde{E}[x^r]\right)^n\right]=\sum_{r=0}^{n}\binom{n}{r}\tilde{E}[x^r]\left(-\tilde{E}[x^r]\right)^{n-r}$.

Strict fuzzy moment generating function: The moment generating function of strict fuzzy random variable X take the form of

$$\tilde{E}[e^{tx}]=\int_{-\infty}^{\infty}e^{tx}\mu_{\Omega}(x)f(x)dx, \tag{4}$$

by differentiating equation (4) we get the fact of $\frac{d^r}{dx^r}\tilde{E}[e^{tx}]_{t=0}=\tilde{E}[x^r]$.

Strict fuzzy characteristic function: For strict fuzzy random variable X a characteristic function of

$$\tilde{E}[e^{itx}]=\int_{-\infty}^{\infty}e^{itx}\mu_{\Omega}(x)f(x)dx. \tag{5}$$

If strict fuzzy random variable X has moments up to R^{th} order, then the characteristic function $\tilde{E}[e^{itx}]$ is r times continuously differentiable on the entire real line. In this case $\tilde{E}[x^r]=(-i)^{-r}\frac{d^r}{dx^r}\tilde{E}[e^{itx}]_{t=0}$.

3. TRIANGULAR AND TRIGONOMETRIC STRICT FUZZY MEMBERSHIP FUNCTION

Consider we have three classical sets $A=\{x;x\in[a_1,a_5]\}$, $B=\{x;x\in[a_3,a_7]\}$ and the $C=A\cap B=\{x;x\in[a_1,a_5]\cap[a_3,a_7]\}=\{x;x\in[a_3,a_5]\}$ which is the intersection of A and B, while $a_j\in R\ \forall j=1,2,\dots,7$ and $a_1\le a_2\le a_3\le a_4\le a_5\le a_6\le a_7$. Now, let $\Omega=A\cup B$ or the sets A and B exhaust the universal set, if we fuzzified both sets A and B by defining a triangular membership functions for these sets, then the classical representation of the membership function is shown by

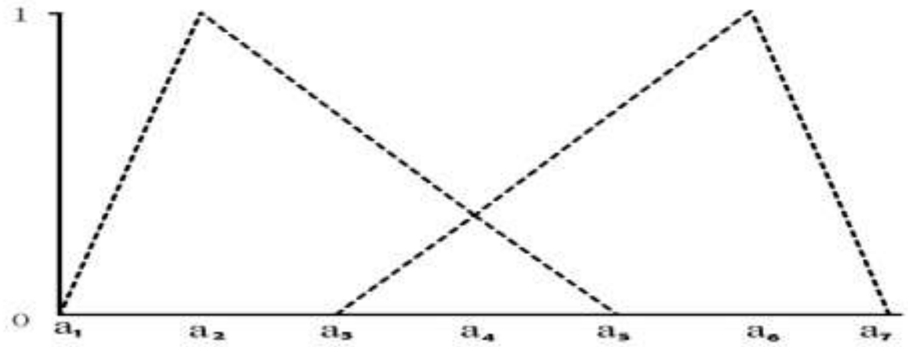

Figure (1) shows the triangular membership function over the set Ω (but not membership function of Ω, as this figure is clear to not be a function).

While, the union is defined to be

$$\mu_{A\cup B}(x)=Max\{\mu_A(x),\mu_B(x)\},$$

which has the graph of

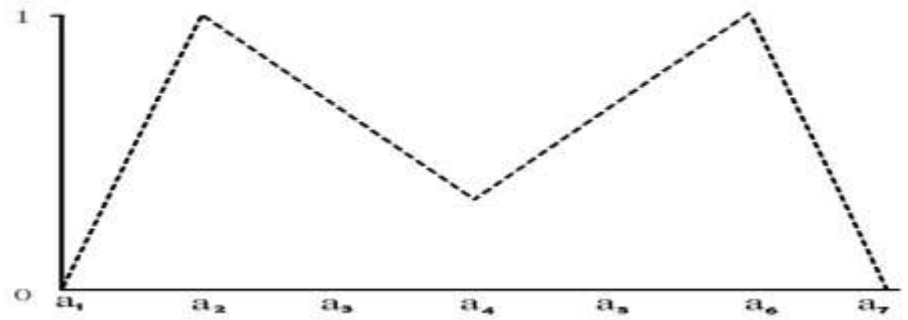

Figure (2) show the triangular membership function for the union of fuzzy sets A and B.

Similarly, the intersection corresponds to

$$\mu_{A\cap B}(x) = Min\{\mu_A(x), \mu_B(x)\}. \quad (6)$$

This, yields the figure of

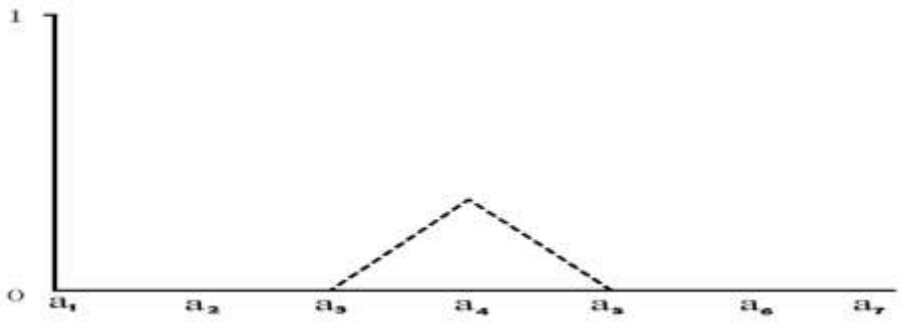

Figure (3) illustrates the triangular membership function for the intersection of A and B.

To apply the definition of strict fuzzy membership proposed by Abed, et. al. (2016) we need to define membership function on Ω which can be reached by manipulating figure (1) to get

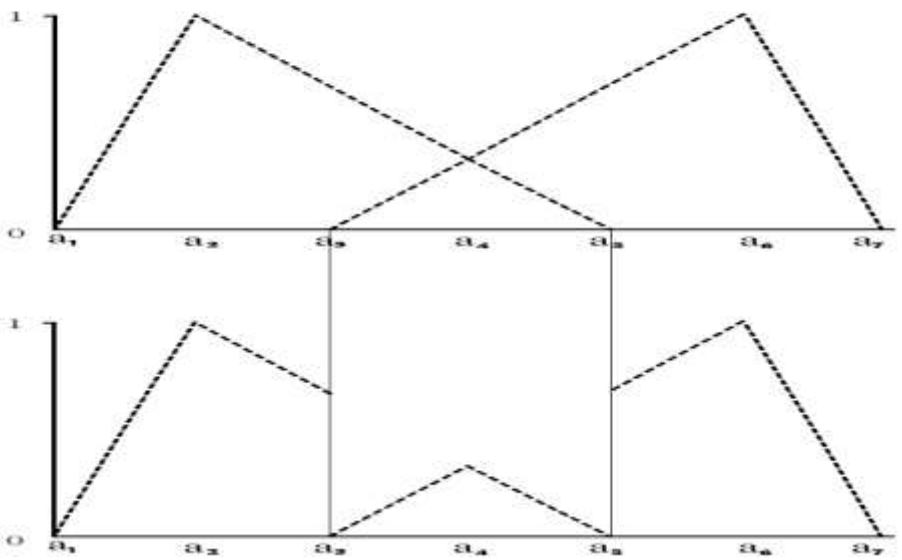

Figure (4) shows the triangular membership function of Ω.

It is considerable to note that, this presentation preserve the shape of A∼C, B∼C and also preserve the minimum of memberships as in equation (6) for the set C = A ∩ B.

To clarify the concept we can consider temperature as a variable. Let the set A∼C represent cold temperature, the set B∼C represent warm temperature and the set C = A ∩ B represent cool temperature. The new presentation gives the same membership values for cold, warm temperature, while it gives the value of the minimum of memberships for cool temperature.

Mathematically the triangular membership of Ω, can be written as

$$\mu_{\Omega}(x) = \begin{cases} 0 & x < a_1, x > a_7 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \le x \le a_2 \\ \dfrac{a_5 - x}{a_5 - a_2} & a_2 < x \le a_3 \\ \dfrac{x - a_3}{a_6 - a_3} & a_3 < x \le a_4 \\ \dfrac{a_5 - x}{a_5 - a_2} & a_4 < x \le a_5 \\ \dfrac{x - a_3}{a_6 - a_3} & a_5 < x \le a_6 \\ \dfrac{a_7 - x}{a_7 - a_6} & a_6 < x \le a_7 \end{cases} \quad (7)$$

The concept of trigonometric membership function on universal set is no different from triangular one. Classical trigonometric membership function for the sets A, B and C is presented as

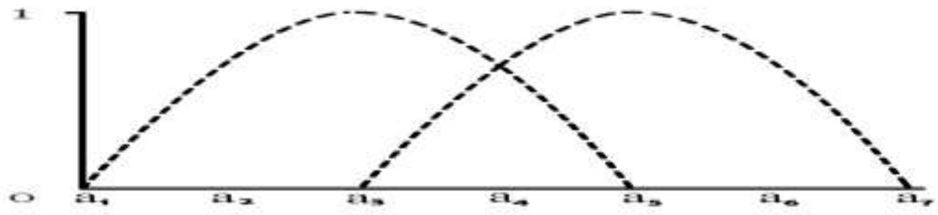

Figure (5) shows the trigonometric membership function defined over Ω.

Using the same manipulation we can transform figure (5) to a membership function defined on Ω as

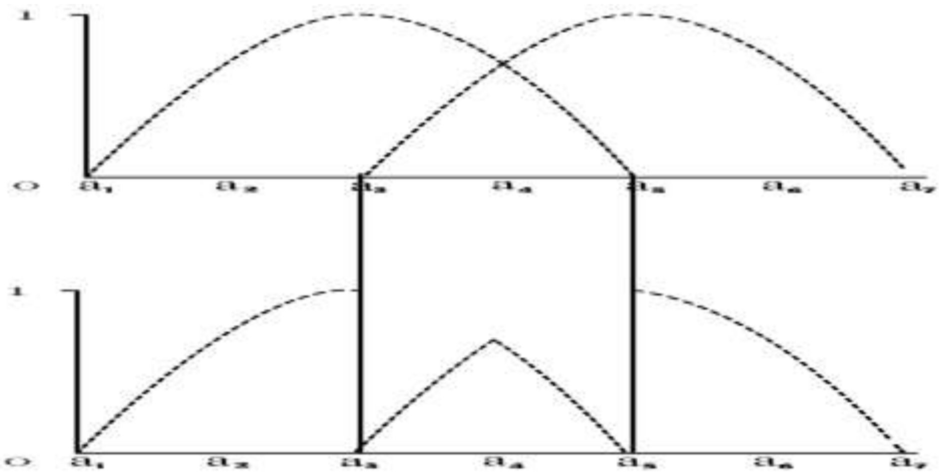

Figure (6) shows the trigonometric membership function of Ω.

Which takes the mathematical form of

$$\mu_{\Omega}(x) = \begin{cases} 0 & x < a_1, x > a_7 \\ sin(\dfrac{\pi}{a_5 - a_1}(x - a_1)) & a_1 \le x \le a_3 \\ sin(\dfrac{\pi}{a_7 - a_3}(x - a_3)) & a_3 < x \le a_4 \\ sin(\dfrac{\pi}{a_5 - a_1}(x - a_1)) & a_4 < x \le a_5 \\ sin(\dfrac{\pi}{a_7 - a_3}(x - a_3)) & a_5 < x \le a_7 \end{cases}$$

This technique can be generally adopted to any functional form of membership.

4. STRICT FUZZY TRIANGULAR EXPONENTIAL DISTRIBUTION

Suppose that our sample space Ω is somehow restricted to have the domain of $[a_1, a_7]$, we define a random variable $X \in [a_1, a_7]$. If X has an exponential distribution, then the truncated exponential distribution will be in the form of

$$f(x) = \frac{\lambda e^{-\lambda x}}{F(a_7) - F(a_1)} \quad a_1 \le x \le a_7, \tag{8}$$

where, $F(a_7)$ and $F(a_1)$ are the value of the classical cumulative density function of exponential distribution.

4.1 Strict Fuzzy Triangular Exponential Probability Density Function

If we let the elements in our sample space Ω to have a triangular membership function $\mu_\Omega(x)$ defined in equation (7). Using the definition of strict fuzzy probability in equation (1) we can write the strict fuzzy triangular exponential distribution as

$$\begin{gathered}\tilde{f}(x) = \left[I_{[a_1,a_2]}(x)\frac{x-a_1}{a_2-a_1} + I_{(a_2,a_3]}(x)\frac{a_5-x}{a_5-a_2}\right.\\ +I_{(a_3,a_4]}(x)\frac{x-a_3}{a_6-a_3} + I_{(a_4,a_5]}(x)\frac{a_5-x}{a_5-a_2} + I_{(a_5,a_6]}(x)\frac{x-a_3}{a_6-a_3}\\ \left.+I_{(a_6,a_7]}(x)\frac{a_7-x}{a_7-a_6}\right]\frac{\lambda e^{-\lambda x}}{F(a_7)-F(a_1)}.\end{gathered} \tag{9}$$

In previous equation in each term we can split the numerator into two terms. This will result of having twelve coefficients corresponding to the six terms. For the sake of simplicity we will define each of the twelve coefficients as

$$\begin{gathered}c_1 = \frac{1}{a_2-a_1}, c_2 = -\frac{a_1}{a_2-a_1}, c_3 = \frac{a_5}{a_5-a_2}, c_4 = -\frac{1}{a_5-a_2}\\ c_5 = \frac{1}{a_6-a_3}, c_6 = -\frac{a_3}{a_6-a_3}, c_7 = \frac{a_5}{a_5-a_2}, c_8 = -\frac{1}{a_5-a_2}\\ c_9 = \frac{1}{a_6-a_3}, c_{10} = -\frac{a_3}{a_6-a_3}, c_{11} = \frac{a_7}{a_7-a_6},\ c_{12} = -\frac{1}{a_7-a_6}\end{gathered}$$

Now, what we want is to illustrate the strict fuzzy probability distribution described in equation (9), we will fix the values of the upper bound and lower bound of each interval and we will vary the some values of the parameter λ.

Below figure show the strict fuzzy triangular exponential probability distribution by setting the value of the parameters $a_1 = 0, a_2 = 5, a_3 = 10, a_4 = 15, a_5 = 20, a_6 = 25$ and $a_7 = 30$ for various values of $\lambda = 0.1, \lambda = 0.05$ and $\lambda = 0.005$. Any graph later on in the paper will be shown based on these values

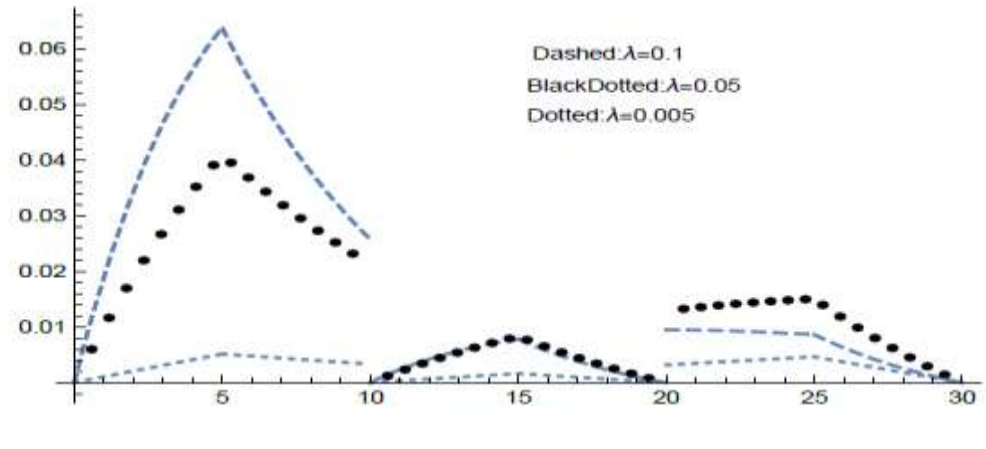

Figure (7) strict fuzzy triangular exponential probability density function.

4.2 Strict Fuzzy Triangular Exponential Cumulative Distribution

If a random variable X has a strict fuzzy triangular distribution, then cumulative distribution can be expressed as the following

$$\tilde{F}(x) = \frac{\lambda e^{-\lambda x}}{F(a_7) - F(a_1)} \left\{ \sum_{j=0}^{2} \left[(c_{4j+1} \frac{(\frac{1}{\lambda} + x)}{\lambda} + c_{4j+2}(\frac{-1}{\lambda}) \right]_{a_{2j+1}}^{a_{2j+2} I_{[x > a_{2j+2}]}(x) + x I_{[a_{2j+1} \leq x \leq a_{2j+2}]}(x)} + \sum_{j=1}^{3} \left[(c_{4j-1}(\frac{-1}{\lambda}) + c_{4j} \frac{(\frac{1}{\lambda} + x)}{\lambda} \right]_{a_{2j}}^{a_{2j+1} I_{[x > a_{2j+2}]}(x) + x I_{[a_{2j} \leq x \leq a_{2j+2}]}(x)} \right\}. \tag{10}$$

Proof:

Using the definition of cumulative strict fuzzy distribution in equation (2) we have

$$\tilde{F}(x) = \int_{a_1}^{x} \Big[I_{[a_1,a_2]}(x) \frac{x - a_1}{a_2 - a_1} + I_{(a_2,a_3]}(x) \frac{a_5 - x}{a_5 - a_2} + I_{(a_3,a_4]}(x) \frac{x - a_3}{a_6 - a_3} + I_{(a_4,a_5]}(x) \frac{a_5 - x}{a_5 - a_2} + I_{(a_5,a_6]}(x) \frac{x - a_3}{a_6 - a_3} + I_{(a_6,a_7\}}(x) \frac{a_7 - x}{a_7 - a_6} \Big] \frac{\lambda e^{-\lambda x}}{F(a_7) - F(a_1)} dx, \tag{11}$$

in each term in previous equation we have to evaluate one of two integrals. It is easy to show that the integrals have the forms of

$$\int x e^{-\lambda x} dx = \frac{(\frac{1}{\lambda} + x)}{\lambda} e^{-\lambda x}. \tag{12}$$

And the other integral has the form of

$$\int e^{-\lambda x} dx = \frac{-1}{\lambda} e^{-\lambda x}. \tag{13}$$

It is easy to see that by substituting equations (12) and (13) in equation (11) we reach the statement of equation (10).

The cumulative strict fuzzy triangular exponential distribution defined in equation (10) for various values of λ can be figured as

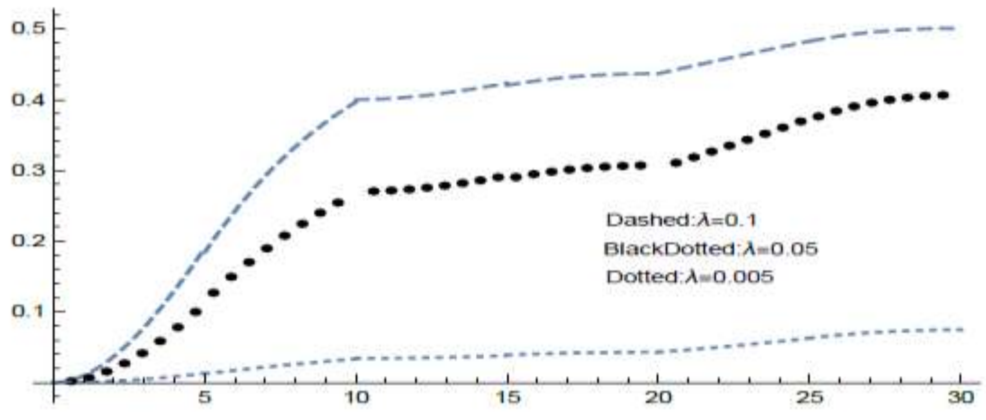

Figure (8) cumulative strict fuzzy triangular exponential density function.

It is important to note that, the strict fuzzy cumulative distribution does not reach the value of one like classical cumulative distribution. However, if needed, a normalizing constant can be added to attain the unity of the area under strict fuzzy probability distribution. Also, it is clear that, from figure (8) that, the ending values of the cumulative distribution depends on the value of λ.

4.3 Moments Strict Fuzzy Triangular Exponential Distribution

If X has a strict fuzzy triangular distribution, then the R^{th} central moment can be expressed as some linear combination of the R^{th} and R^{th}+1 classical moments as

$$\tilde{E}[x^r] = \sum_{j=0}^{2}\left[(c_{4j+1}E[x^{r+1}] + c_{4j+2}E[x^r]\right]_{a_{2j+1}}^{a_{2j+2}} + \sum_{j=1}^{3}\left[(c_{4j-1}E[x^r] + c_{4j}E[x^{r+1}]\right]_{a_{2j}}^{a_{2j+1}},$$

where $E[x^r]$ is the classical R^{th} moment of truncated exponential distribution, which is given by

$$E[x^r] = \frac{\lambda}{F(a_7) - F(a_1)}\int x^r e^{-\lambda x}dx,$$

while, $E[x^{r+1}]$ is the same as the previous expression, but we replace r by $r+1$.

Proof:

The technique that will be used in this proof is that we will first evaluate the classical expected value of the R^{th} moment and the R^{th}+1 moment as follows, for the R^{th} moment we get

$$E[x^r] = \frac{\lambda}{F(a_7) - F(a_1)}\int x^r e^{-\lambda x}dx$$
$$= \frac{\lambda e^{-\lambda x}}{F(a_7) - F(a_1)}\left[\sum_{m=0}^{r}(-1)^m \frac{r!\,x^{r-m}}{(r-m)!\,(-\lambda)^{m+1}}\right], \quad (14)$$

while the R^{th}+1 classical moment is found by simply replacing r in equation (14) by $r+1$, so it can be expressed as

$$E[x^{r+1}] = \frac{\lambda e^{-\lambda x}}{F(a_7) - F(a_1)}\left[\sum_{m=0}^{r+1}(-1)^m \frac{(r+1)!\,x^{r+1-m}}{(r+1-m)!\,(-\lambda)^{m+1}}\right]. \quad (15)$$

Using equations (14) and (15) in the definition of strict fuzzy expectation in equation (3) we can write reach the result of

$$\tilde{E}[x^r] = \sum_{j=0}^{2}\left[(c_{4j+1}E[x^{r+1}] + c_{4j+2}E[x^r]\right]_{a_{2j+1}}^{a_{2j+2}} + \sum_{j=1}^{3}\left[(c_{4j-1}E[x^r] + c_{4j}E[x^{r+1}]\right]_{a_{2j}}^{a_{2j+1}}$$

which is needed.

4.4 Moment Generating Function for Strict Fuzzy Triangular Exponential Distribution

The moment generating function for strict triangular fuzzy distribution is given by

$$\tilde{E}[e^{tx}] = \sum_{j=0}^{2}\left[(c_{4j+1}E[xe^{tx}] + c_{4j+2}E[e^{tx}]\right]_{a_{2j+1}}^{a_{2j+2}} + \sum_{j=1}^{3}\left[(c_{4j-1}E[e^{tx}] + c_{4j}E[xe^{tx}]\right]_{a_{2j}}^{a_{2j+1}}.$$

Proof:

To proof this theorem, we need to evaluate two expectations. It is easy to see that

$$E[xe^{tx}] = \int xe^{tx}f(x)dx = \lambda\left[\frac{e^{(t-\lambda)x}}{t-\lambda}\left(\frac{-1}{t-\lambda} + x\right)\right] \quad t < \lambda, \quad (16)$$

also it can be shown that

$$E[e^{tx}] = \int e^{tx}f(x)dx = \frac{\lambda}{t-\lambda}e^{(t-\lambda)x} \quad t < \lambda. \quad (17)$$

Now, using both the expectations in equation (16) and (17) and plunging them into the definition of strict fuzzy moment generating function in equation (4), It is clear that we reached our result

4.5 Characteristic Function for Strict Fuzzy Triangular Exponential Distribution

For strict fuzzy random variable X. The characteristic function of strict triangular fuzzy exponential distribution can be expressed as

$$\tilde{E}[e^{itx}] = \sum_{j=1}^{6}\Bigg[c_{2j-1}\left[\frac{-\lambda e^{-\lambda x}}{(\lambda^2+t^2)^2}(\lambda + x(\lambda^2+t^2))(\lambda cos\, tx + tsin\, tx) - t(-t\cos tx + \lambda sin\, tx)\right]$$
$$+c_{j+1}\left[\frac{\lambda e^{-\lambda x}}{\lambda^2+t^2}(-\lambda\cos tx + t\sin tx)\right] + i\Bigg\{c_{2j-1}\frac{-\lambda e^{-\lambda x}}{(\lambda^2+t^2)^2}\Big[t(\lambda cos\, tx + tsin\, tx)$$
$$+(-t\cos tx + \lambda sin\, tx)(\lambda + x(\lambda^2+t^2))\Big] - c_{j+1}\frac{\lambda e^{-\lambda x}}{\lambda^2+t^2}(\lambda\sin tx + t\cos tx)\Bigg\}\Bigg]_{a_j}^{a_{j+1}}.$$

Proof:

The method of the proof of this theorem is not much different from previous theorems, we first need to get

$$E[xe^{itx}] = \int xe^{itx}f(x)dx = \lambda\int xe^{(it-\lambda)x}dx$$
$$= \frac{-\lambda e^{(it-\lambda)x}}{\lambda - it}\left(\frac{1}{\lambda - it} + x\right),$$

by simple manipulation previous equation becomes

$$\frac{-\lambda e^{-\lambda x}}{(\lambda^2+t^2)^2}e^{itx}(\lambda - it)\big(\lambda + x(\lambda^2+t^2)\big) + it),$$

using Euler identity we can replace complex exponential by sine and cosine as

$$\frac{-\lambda e^{-\lambda x}}{(\lambda^2+t^2)^2}(cos\, tx + isin\, tx)(\lambda - it)\big(\lambda + x(\lambda^2+t^2)\big) + it),$$

by grouping real and imaginary parts we get

$$\frac{-\lambda e^{-\lambda x}}{(\lambda^2+t^2)^2}\{(\lambda + x(\lambda^2+t^2))(\lambda cos\, tx + tsin\, tx)$$
$$-t(-t\cos tx + \lambda sin\, tx) + i[t(\lambda cos\, tx + tsin\, tx) \tag{18}$$
$$(-t\cos tx + \lambda sin\, tx)\big(\lambda + x(\lambda^2+t^2)\big)]\}.$$

Secondly we need to evaluate the following integral

$$E[e^{itx}] = \int e^{itx}f(x)dx = \frac{\lambda e^{-\lambda x}}{\lambda^2+t^2}(-\lambda\cos tx + t\sin tx) - i\frac{\lambda e^{-\lambda x}}{\lambda^2+t^2}(\lambda\sin tx + t\cos tx). \tag{19}$$

It is easy to see that, by using both equations (18) and (19) in the definition of strict fuzzy characteristic function in equation (5) we can reach our proof.

5. STRICT FUZZY TRIGONOMETRIC EXPONENTIAL DISTRIBUTION

In this section, we will introduce what will be called strict fuzzy trigonometric exponential distribution, we will use both the definition of strict fuzzy trigonometric membership function in equation (7) and the definition of truncated exponential distribution in equation (8). Also here, we assume that our domain is restricted on $[a_1, a_7]$. For simplicity we will define

$$h_1 = \frac{\pi}{a_5 - a_1}, h_2 = \frac{\pi}{a_7 - a_3}, h_3 = \frac{\pi}{a_5 - a_1}, h_4 = \frac{\pi}{a_7 - a_3}$$

$$l_1 = l_3 = -a_1, l_2 = -a_3, l_4 = -a_3$$

$$q_1 = a_1, q_2 = a_3, q_3 = a_4, q_4 = a_5, q_5 = a_7.$$

5.1 Strict Fuzzy Trigonometric Exponential Probability Density Function

The probability density function of strict fuzzy triangular exponential distribution is given by the multiplication of both the membership function and corresponding probability function as

$$\tilde{f}(x) = \sum_{j=1}^{4} I_{[q_j+q_{j+1}]}(x) \sin\left(h_j\left(x + l_j\right)\right) \frac{\lambda e^{-\lambda x}}{F(a_7) - F(a_1)}. \tag{20}$$

Figure (9) illustrates some possible shapes of strict fuzzy trigonometric exponential distribution in equation (20) as

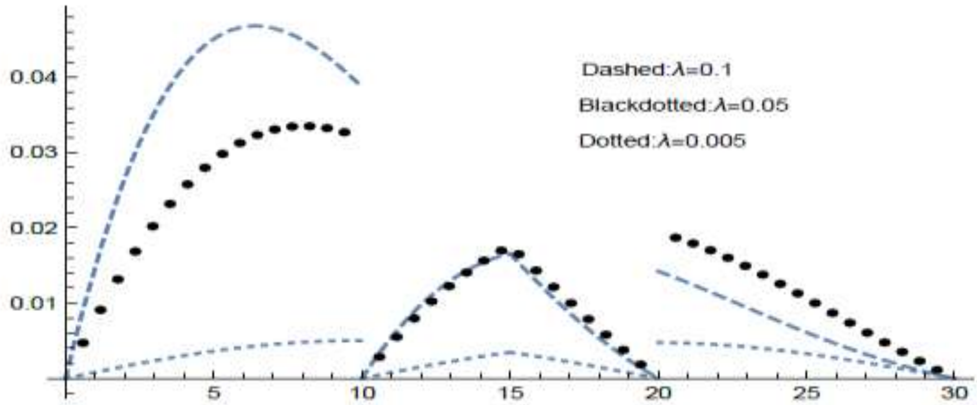

Figure (9) strict fuzzy trigonometric exponential density function.

5.2 Strict Fuzzy Trigonometric Cumulative Distribution

If X is a strict fuzzy random variable, then, the cumulative density function for trigonometric exponential distribution is given by

$$\tilde{F}(x) = \frac{\lambda}{(F(a_7) - F(a_1))} \sum_{j=1}^{4} \left[\frac{-e^{-\lambda x}}{(h_j^2 + \lambda^2)} [h_j(\cos\left(h_j x + l_j h_j\right) + \lambda \sin(h_j x + l_j h_j)) - h_j] \right]_{q_j}^{q_{j+1} I_{[x > q_{j+1}]}(x) + x I_{[q_j \le x \le q_{j+1}]}(x)}$$

Proof :

What we need is to apply the definition of strict fuzzy cumulative function in equation (2). In other words, we need to find is the value of the integral of

$$\tilde{F}(x) = \int_{q_j}^{x} \sum_{j=1}^{4} I_{[q_j+q_{j+1}]}(x) \sin(h_j\left(x + l_j\right)) \frac{\lambda e^{-\lambda x}}{F(a_7) - F(a_1)} dx,$$

we can simply evaluate the j^{th} integral regardless of the indicator function, so we can express the j^{th} integral as follows

$$\frac{\lambda}{F(a_7)-F(a_1)}\int_{q_j}^{x} sin(h_j(x+l_j))e^{-\lambda x}dx,$$

replacing sine by complex exponentials, we have

$$\frac{\lambda}{2i(F(a_7)-F(a_1))}\int_{q_j}^{x} e^{i(h_j(x+l_j))}e^{-\lambda x}-e^{-i(h_j(x+l_j))}e^{-\lambda x}dx,$$

then, above integral yields

$$\left[\frac{\lambda e^{-\lambda x}}{2(F(a_7)-F(a_1))}(\frac{e^{(l_jh_j+h_jx)i}}{-h_j-i\lambda}+\frac{e^{(-l_jh_j-h_jx)i}}{-h_j+i\lambda})\right]_{q_j}^{x},$$

here, we replace the complex exponentials by sines and cosines we get

$$\Bigg[\frac{\lambda e^{-\lambda x}}{2(F(a_7)-F(a_1))}[\frac{-h_j+i\lambda}{h_j^2+\lambda^2}(cos(h_jx+l_jh_j)$$

$$+i\,sin(h_jx+l_jh_j))+\frac{-h_j-i\lambda}{h_j^2+\lambda^2}(cos(-l_jh_j-h_jx)+i\,sin(-l_jh_j-h_jx))]\Bigg]_{q_j}^{x}$$

$$=\frac{\lambda e^{-\lambda x}}{2(h_j^2+\lambda^2)(F(a_7)-F(a_1))}\Bigg[[-h_j(cos(h_jx+l_jh_j) \tag{21}$$

$$-\lambda\,sin(h_jx+l_jh_j))-h_j\,(cos(-l_jh_j-h_jx)+\lambda\,sin(-l_jh_j-h_jx))]\ +i[\lambda(cos(h_jx+l_jh_j)$$

$$-h_j\,sin(h_jx+l_jh_j))-\lambda\,(cos(-l_jh_j-h_jx)]-h_j\,sin(-l_jh_j-h_jx))]\Bigg]_{q_j}^{x}.$$

It is clear that the cumulative distribution is a real function. So, the imaginary part in equation (21) is equal zero. By, placing the indicator function an taking the sum over j we complete our proof.

The graph of the cumulative strict fuzzy trigonometric exponential distribution for some possible values of λ can be seen as

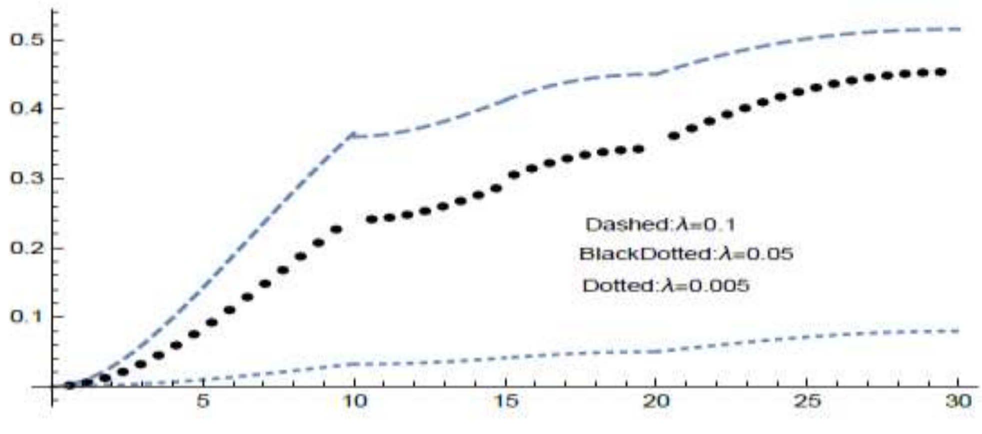

Figure (10) strict fuzzy trigonometric exponential density function.

5.3 Moments Strict Fuzzy Trigonometric Exponential Distribution

In this section we derive the R^{th} central moment for strict fuzzy trigonometric exponential distribution. We will need the following fact in our derivation

$$\tilde{E}[x^r] = \int x^r e^{ax} dx = e^{ax} \sum_{j=0}^{r} (-1)^j \frac{r!\, x^{r-j}}{(r-j)!\,(a)^{j+1}}. \tag{22}$$

taking the j^{th} integral of the R^{th} moment as in definition in equation (3) we get

$$\frac{\lambda}{F(a_7)-F(a_1)} \int x^r sin(h_j(x+l_j)) e^{-\lambda x} dx$$

$$= \frac{\lambda}{2i(F(a_7)-F(a_1))} \int e^{i(h_j(x+l_j))} x^r e^{-\lambda x} - e^{-i(h_j(x+l_j))} x^r e^{-\lambda x} dx$$

$$= \frac{\lambda}{2i(F(a_7)-F(a_1))} \left[\frac{e^{il_jh_j}}{(ih_j-\lambda)^{r+1}} \int ((ih_j-\lambda)x)^r \right.$$

$$\left. e^{(ih_j-\lambda)x} d(ih_j-\lambda)x - \frac{e^{-il_jh_j}}{(-ih_j-\lambda)^{r+1}} \int \left((-ih_j-\lambda)x\right)^r e^{(-ih_j-\lambda)x} d(-ih_j-\lambda)x \right],$$

using equation (22) we can write previous equation as

$$\frac{\lambda}{2i(F(a_7)-F(a_1))} \left[\frac{e^{(l_jh_j+h_jx)i} e^{-\lambda x}}{(ih_j-\lambda)^{r+1}} \sum_{m=0}^{r} (-1)^m \frac{r!\,((ih_j-\lambda)x)^{r-m}}{(r-m)!\,(ih_j-\lambda)^{m+1}} - \frac{e^{(-l_jh_j-h_jx)i} e^{-\lambda x}}{(-ih_j-\lambda)^{r+1}} \right. \tag{23}$$
$$\left. \sum_{m=0}^{r} (-1)^m \frac{r!\,((-ih_j-\lambda)x)^{r-m}}{(r-m)!\,(-ih_j-\lambda)^{m+1}} \right],$$

in the first term in equation (23) we can write

$$\sum_{m=0}^{r} (-1)^m \frac{r!\,((ih_j-\lambda)x)^{r-m}}{(r-m)!\,(ih_j-\lambda)^{m+1}}$$

$$= \sum_{m=0}^{r} \frac{(-1)^m r!\, x^{r-m}}{(r-m)!} \frac{(-\lambda+ih_j)^r}{(-\lambda+ih_j)^{2m+1}}.$$

Now, what we need is to rewrite first term in equation (23) in standard complex number form. Expanding the bracket in equation (23) using binomial expansion we reach

$$(-\lambda+ih_j)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} (-\lambda)^k (ih_j)^{2m+1-k}. \tag{24}$$

Now, the sum in equation (24) can be splited into two sums the first sum is to K_1 which is a non-negative integer such that 2k is the greatest even in 2m + 1. Similarly, for the second sum we choose K_2 such that 2k + 1 is the greatest odd in 2m + 1. Then, equation (24) can be rewritten as

$$\sum_{k=0}^{K_1} \binom{2m+1}{2k} (-\lambda)^{2k} (ih_j)^{2m+1-2k} + \sum_{k=0}^{K_2} \binom{2m+1}{2k+1} (-\lambda)^{2k+1} (ih_j)^{2(m-k)},$$

previous equation reduces to

$$i\sum_{k=0}^{K_1}\binom{2m+1}{2k}(-\lambda)^{2k}{h_j}^{2(m-k)+1}-\sum_{k=0}^{K_2}\binom{2m+1}{2k+1}(-\lambda)^{2k+1}(h_j)^{2(m-k)},$$

which can be written as a complex number taking the form of

$$-b_1+ib_2. \tag{25}$$

Similarly, in first term in equation (23) we can use binomial expansion of

$$(-\lambda+ih_j)^r=\sum_{s=0}^{r}\binom{r}{s}(-\lambda)^s(ih_j)^{r-s},$$

for non-negative integers S_1 to be chosen such that $2s$ is greatest even in r. While, S_2 to be chosen such that $2s+1$ to be greatest odd in r. We can write previous equation as

$$i^r({h_j}^r+\sum_{s=1}^{S_1}\binom{r}{2s}(-\lambda)^{2s}{h_j}^{r-2s})+i^{r-1}(-\lambda r{h_j}^{r-1}-\sum_{s=1}^{S_2}\binom{r}{2s+1}(-\lambda)^{2s+1}{h_j}^{r-2s-1}),$$

which take the general complex form of

$$\begin{cases}-b_3+ib_4 & if\ r\ is\ even\\ ib_3-b_4 & if\ r\ is\ odd\end{cases}. \tag{26}$$

Applying the same manipulation to $(-\lambda+ih_j)^{r+1}$ in the first term of equation (23), we get

$$(-\lambda+ih_j)^{r+1}=i^{r+1}({h_j}^{r+1}+\sum_{s=1}^{S_1}\binom{r+1}{2s}(-\lambda)^{2s}{h_j}^{r+1-2s})$$

$$+i^r(-\lambda(r+1){h_j}^r-\sum_{s=1}^{S_2}\binom{r+1}{2s+1}(-\lambda)^{2s+1}{h_j}^{r-2s}),$$

which can be written as

$$\begin{cases}ib_5-b_6 & if\ r\ is\ even\\ -b_5+ib_6 & if\ r\ is\ odd\end{cases}. \tag{27}$$

Replacing the exponential of the first term in equation (23) by sine and cosine, we have

$$e^{(l_jh_j+h_j)i}=cos(l_jh_j+h_jx)+i\,sin(l_jh_j+h_jx)=b_7+ib_8 \tag{28}$$

For second term in equation (23) we replace h_j by $-h_j$, then we can get

$$(-\lambda-ih_j)^{2m+1}=i\sum_{k=0}^{K_1}\binom{2m+1}{2k}(-\lambda)^{2k}(-h_j)^{2(m-k)+1}$$

$$-\sum_{k=0}^{K_2}\binom{2m+1}{2k+1}(-\lambda)^{2k+1}(-h_j)^{2(m-k)},$$

in standard form we can write the previous equation as

$$-d_1+id_2. \tag{29}$$

And

$$(-\lambda-ih_j)^r=i^r((-h_j)^r+\sum_{s=1}^{S_1}\binom{r}{2s}(-\lambda)^{2s}(-h_j)^{r-2s}) \tag{30}$$

$$+i^{r-1}(-\lambda r(-h_j)^{r-1} - \sum_{s=1}^{S_2}\binom{r}{2s+1}(-\lambda)^{2s+1}(-h_j)^{r-2s-1}),$$

which yields two complex numbers below

$$\begin{cases}-d_3 + id_4 & if\ r\ is\ even\\ id_3 - d_4 & if\ r\ is\ odd\end{cases}. \tag{31}$$

Similarly we replace r in equation (30) by r + 1, we have

$$\left(-\lambda - ih_j\right)^{r+1} = i^{r+1}((-h_j)^{r+1} + \sum_{s=1}^{S_1}\binom{r+1}{2s}(-\lambda)^{2s}(-h_j)^{r+1-2s}) + i^r(-\lambda(r+1)(-h_j)^r$$
$$-\sum_{s=1}^{S_2}\binom{r+1}{2s+1}(-\lambda)^{2s+1}(-h_j)^{r-2s}),$$

which can be written as

$$\begin{cases}id_5 - d_6 & if\ r\ is\ even\\ -d_5 + id_6 & if\ r\ is\ odd\end{cases}. \tag{32}$$

Finally we can say that

$$e^{(-l_jh_j-h_jx)i} = cos\left(l_jh_j - h_jx\right) + i\,sin(l_jh_j - h_jx) = d_7 + id_8 \tag{33}$$

Now, using equations (25), (26), (27), (28), (29), (31), (32) and (33) and substituting in equation (23). Also, after rearranging real parts of the solution we get the form of the Rth moment if r is even as

$$\tilde{E}[x^r] = \frac{\lambda}{2\left(F(a_7) - F(a_1)\right)}\sum_{j=1}^{4} I_{[q_j+q_{j+1}]}(x)\left[\frac{(-b_5b_7 - b_8b_6\ )}{(b_6^2 + b_5^2)}\sum_{m=0}^{r}\frac{(-1)^m r!\,x^r(b_3b_1 + b_4b_2)}{(r-m)!\,(b_1^2 + b_2^2)}\right.$$
$$-\frac{(b_7b_6\ - b_5b_8)e^{-\lambda x}}{(b_6^2 + b_5^2)}\sum_{m=0}^{r}\frac{(-1)^m r!\,x^r(b_3b_2 - b_4b_1)}{(r-m)!\,(b_1^2 + b_2^2)}$$
$$-\frac{(-d_5d_7 - d_8d_6\ )}{(d_6^2 + d_5^2)}\sum_{m=0}^{r}\frac{(-1)^m r!\,x^r(d_3d_1 + d_4d_2)}{(r-m)!\,(d_1^2 + d_2^2)}$$
$$\left.+\frac{(d_7d_6\ - d_5d_8)e^{-\lambda x}}{(d_6^2 + d_5^2)}\sum_{m=0}^{r}\frac{(-1)^m r!\,x^r(d_3d_2 - d_4d_1)}{(r-m)!\,(d_1^2 + d_2^2)}\right]_{q_j}^{q_{j+1}}.$$

While, if r is odd we get

$$\tilde{E}[x^r] = \frac{\lambda}{2\left(F(a_7) - F(a_1)\right)}\sum_{j=1}^{4} I_{[q_j+q_{j+1}]}(x)\left[\frac{(-b_6b_7 - b_8b_5)}{b_6^2 + b_5^2}\sum_{m=0}^{r}\frac{(-1)^m r!\,x^r(b_4b_1 + b_3b_2)}{(r-m)!\,(b_1^2 + b_2^2)}\right.$$
$$-\frac{(b_7b_5 - b_6b_8)e^{-\lambda x}}{b_6^2 + b_5^2}\sum_{m=0}^{r}\frac{(-1)^m r!\,x^r(b_4b_2 - b_1b_3)}{(r-m)!\,(b_1^2 + b_2^2)}$$
$$-\frac{(-d_6d_7 - d_8d_5)}{d_6^2 + d_5^2}\sum_{m=0}^{r}\frac{(-1)^m r!\,x^r(d_4d_1 + d_3d_2)}{(r-m)!\,(d_1^2 + d_2^2)}$$
$$\left.+\frac{(d_7d_5 - d_6d_8)e^{-\lambda x}}{d_6^2 + d_5^2}\sum_{m=0}^{r}\frac{(-1)^m r!\,x^r(d_4d_2 - d_1d_3)}{(r-m)!\,(d_1^2 + d_2^2)}\right]_{q_j}^{q_{j+1}}.$$

11 Moment Generating Function for Strict Fuzzy Trigonometric Exponential Distribution

Moment generating function for strict fuzzy trigonometric exponential random variable is

$$\tilde{E}[e^{tx}]\frac{\lambda}{\left(F(a_7)-F(a_1)\right)}\sum_{j=1}^{4}I_{[q_j+q_{j+1}]}(x)\left[\frac{e^{(t-\lambda)x}}{\left({h_j}^2+(t-\lambda)^2\right)}[-h_j\,cos(h_j\,l_j+h_jx)\right.$$

$$\left.+(t-\lambda)\,sin(h_jl_j+h_jx)]\right]_{q_j}^{q_{j+1}}.$$

Proof:

Taking the j[th] integral of moment generating function defined in equation (4) we have

$$\frac{\lambda}{F(a_7)-F(a_1)}\int sin(h_j(x+l_j))e^{-\lambda x}e^{tx}dx$$

$$=\frac{\lambda}{2i(F(a_7)-F(a_1))}\left[\frac{e^{il_jh_j}}{(ih_j+t-\lambda)}e^{(ih_j+t-\lambda)x}-\frac{e^{-il_jh_j}}{(-ih_j+t-\lambda)}e^{(-ih_j+t-\lambda)x}\right],$$

expanding the expression and finding the real part, it is easy to see that

$$\frac{\lambda e^{(t-\lambda)x}}{2(F(a_7)-F(a_1))\left({h_j}^2+(t-\lambda)^2\right)}\Big\{-h_j\,cos(h_j\,l_j+h_jx)+(t-\lambda)\,sin(h_jl_j+h_jx)$$

$$-h_j\,cos(-h_j\,l_j-h_jx)-(t-\lambda)\,sin(-h_jl_j+-h_jx)+i[-(t-\lambda)\,cos(h_j\,l_j+h_jx)$$

$$-h_j\,sin(h_jl_j+h_jx)-(t-\lambda)\,cos(-h_j\,l_j-h_jx)+h_j\,sin(-h_jl_j+-h_jx)]\Big\},$$

taking the sum over j, we get our statement.

5.5 Characteristic Function for Strict Fuzzy Trigonometric Exponential Distribution

The characteristic function of strict fuzzy trigonometric exponential distribution takes the form of

$$\tilde{E}[e^{itx}]=\frac{\lambda}{2\left(F(a_7)-F(a_1)\right)}\sum_{j=1}^{4}I_{[q_j+q_{j+1}]}(x)\left[e^{(t-\lambda)x}\left\{\frac{1}{(h_j+t)^2+\lambda^2}[(-h_j-t)\right.\right.$$

$$cos(l_jh_j+h_jx)-\lambda\,sin(l_jh_j+h_jx)]\;-\frac{1}{\left(-h_j+t\right)^2+\lambda^2}$$

$$[(h_j-t)\;\;cos\left(l_jh_j+h_jx\right)+\lambda\,sin\left(l_jh_j+h_jx\right)]$$

$$+i\left[\frac{1}{(h_j+t)^2+\lambda^2}[(-h_j-t)sin(l_jh_j+h_jx)+\lambda\,cos(l_jh_j+h_jx)\right.$$

$$\left.\left.\left.-\frac{1}{\left(-h_j+t\right)^2+\lambda^2}[-(h_j-t)\,sin\left(l_jh_j+h_jx\right)+\lambda\,cos\left(l_jh_j+h_jx\right)]\right]\right\}\right]_{q_j}^{q_{j+1}}.$$

Proof:

The j[th] integral of characteristic function defined in equation (5), we get

$$\frac{\lambda}{F(a_7)-F(a_1)}\int sin(h_j(x+l_j))e^{-\lambda x}e^{itx}dx$$

$$= \frac{\lambda}{2i(F(a_7) - F(a_1))} \int_{a_1}^{x} e^{i(h_j(x+l_j))} e^{(it-\lambda)x} - e^{-i(h_j(x+l_j))} e^{(it-\lambda)x} dx$$

$$= \frac{\lambda e^{(t-\lambda)x}}{2(F(a_7) - F(a_1))} \left[\frac{e^{i(l_j h_j + h_j x)}(-(h_j + t) + i\lambda))}{(h_j + t)^2 + \lambda^2} - \frac{e^{i(-l_j h_j - h_j x)}(-(-h_j + t) + i\lambda)}{(-h_j + t)^2 + \lambda^2} \right],$$

after simple algebra, we reach the result of

$$\frac{\lambda}{F(a_7) - F(a_1)} \left[\frac{(cos(l_j h_j + h_j x) + i\, sin(l_j h_j + h_j x))\, (-(h_j + t) + i\lambda))}{(h_j + t)^2 + \lambda^2} - \frac{(cos(-l_j h_j - h_j x) + i\, sin(-l_j h_j - h_j x))(-(-h_j + t) + i\lambda)}{(-h_j + t)^2 + \lambda^2} \right],$$

after grouping real and imaginary parts and summing over j, we have the complete proof.

6. CONCLUSION

In the paper, we have shown the construction of strict fuzzy membership function from classical membership function. We developed a technique that preserves the behavior of classical membership function over universal set. We presented truncated strict fuzzy triangular and trigonometric distribution.

It can be pointed out that, there are many aspects of strict fuzzy sets and strict fuzzy probability that has to be considered. One aspect, is defining strict fuzzy number and it's arithmetic, one technique is to use confidence interval introduced by Chachi, et. al. (2012) instead of α cuts. Hence, the convexity property of classical fuzzy number is not needed. Another aspect is the estimation and testing of the parameters of the strict fuzzy density function.

The quantification of uncertainty is very important in many fields, we hope that the finding of this paper will have vast theoretical and practical applications.

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