On Intuitionistic Fuzzy G-Modules On GF(p^n)

Poonam K. Sharma

Post Graduate Department of Math-ematics
D.A.V. College, Jalandhar, Punjab (India)
Email: pksharma [AT] davjalandhar.com

Abstract. In this paper, we have constructed an intuitionistic fuzzy G-module with level cardinality $(n+1)$ on the Galois field GF(p^n), and then proved that infinite many such intuitionistic fuzzy G-modules can be constructed on it. We have also proved that each such intuitionistic fuzzy G-module, admits a sequence of k intuitionistic fuzzy G-submodules, where k is the number of divisors of n. Further, we have also discussed intuitionistic fuzzy noetherian G-module on GF(p^n).

2010 AMS Classification: 0355, 12E20

Keywords: Galois field, intuitionistic fuzzy set, intuitionistic fuzzy G-module, intuitionistic fuzzy Galois G-module, Ascending chain condition (A.C.C).

---

1. Introduction

It is well-known result that there exists finite field of order q if and only if q is of the form p^n, where p is a prime number and n is a positive integer. Such a field is called Galois field and is denoted by $GF(p^n)$. The notion of intuitionistic fuzzy G-modules and their properties are discussed by the author et.al. in [4, 5, 6, 7, 8]. In this paper, we construct an intuitionistic fuzzy G-module of level cardinality $(n+1)$. We also prove that there is a sequence of k intuitionistic fuzzy G-submodules where k is the number of divisors of n. Further, we have also discussed intuitionistic fuzzy noetherian G-module on $GF(p^n)$.

2. Preliminaries

In this section, we first discuss some important results and properties of Galois field $GF(p^n)$, G-modules, intuitionistic fuzzy set theory and intuitionistic fuzzy G-modules, which are respectively taken from [9], [3], [1, 2], [4, 5, 6].

Definition 2.1. ([9]) A field K with p^n elements is called a Galois field and is denoted by $GF(p^n)$, where p being a positive prime number.

Theorem 2.2. *([9]) Let p be a prime number and n be a positive integer. Then there exists a field with p^n elements.*

Theorem 2.3. *([9]) The multiplicative group of Galois field is cyclic.*

Theorem 2.4. *([9]) Let K' be a subfield of the Galois field $GF(p^n)$. Then there exists an integer m such that K' contains p^m elements and m divides n.*

Remark 2.5. ([9]) Any finite field having p^n elements (p is prime) has a subfield isomorphic to Z_p.

Definition 2.6. ([3]) Let G be a group and M be a vector space over a field K. Then M is called a G-module if for every $g \in G$ and $m \in M$, $\exists$ a product (called the action of G on M), $gm \in M$ satisfies the following axioms

(i): $1_G.m = m, \forall\ m \in M$ (1_G being the identity of G)
(ii): $(g \cdot h) \cdot m = g \cdot (h \cdot m), \forall\ m \in M, g, h \in G$
(iii): $g.(k_1m_1 + k_2m_2) = k_1(g.m_1) + k_2(g.m_2), \forall\ k_1, k_2 \in K; m_1, m_2 \in M$ and $g \in G$

Example 2.7. For any prime p, we have $M = (Z_p, \times_p, +_p)$, is a field. Let $G = M - \{0\}$. Then under the field operations of M, it is a G-module.

Example 2.8. For the prime 2, let M be the field having $2^4 = 16$ elements i.e., $M = \{$ zeros of the polynomial $x^{16} - x$ over $Z_2\}$. Let $M^* = \{$zeros of the polynomial $x^4 - x$ over $Z_2\}$. Then M^* is the field having $2^2 = 4$ elements. Hence by theorem (2.4) M^* is a subfield of M. Let $G^* = M^* - \{0\}$. Then M is G^*-module. Also, M has a subfield K isomorphic to Z_2. If $G^{**} = K - \{0\}$, then M is also a $G^{**}-$ module.

Example 2.9. ([4],[5]) Let $G = \{1, -1, i, -i\}$ and $M = C^n (n \geq 1)$. Then M is a vector space over C, and under the usual addition and multiplication of the elements of M, we can show that M is a G-module.

Example 2.10. Consider the Galois field $M = GF(p^n)$. Then M is a vector space over $K = GF(p) \cong Z_p$, the field of integers modulo p. Let $G = K^*$ the multiplicative group of M. Then we can show that M is a G-module.
Let the divisors of n be $1 = d_1, d_2, ..., d_k = n$ such that $1 = d_1 < d_2 < .... < d_k = n$. Let $G = Z_p - \{0\}$. Then we can show that M has "k" G-submodules $M_i = GF(p^{d_i})$ for $i = 1, 2, ..., k$.

Definition 2.11. ([3],[9]) Let M be a G-module. The G-submodules of M are said to satisfy the ascending chain condition (A.C.C) if any chain of G-submodules of $M, M_1 \subseteq M_2 \subseteq$ .... terminates. This means that there exists a positive integer k such that $M_k = M_n$ for $k \geq n$. If G-submodules of M satisfy the A.C.C. then M is said to be a Noetherian module.

Example 2.12. Every finite dimensional vector space V over a field K is Noetherian module. In particular, $M = GF(p^n)$ as G-module over $GF(p)$ is a Noetherian module, where $G = K^*$ is the multiplicative group of M.

Definition 2.13. ([1],[2]) Let X be a non-empty set. An intuitionistic fuzzy set (IFS) A of X is an object of the form $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and degree of non-membership of the element $x \in X$ respectively and for any $x \in X$, we have $\mu_A(x) + \nu_A(x) \leq 1$.

Remark 2.14.
(i) When $\mu_A(x) + \nu_A(x) = 1$, i.e., $\nu_A(x) = 1 - \mu_A(x)$, $\forall x \in X$. Then A is called a fuzzy set.

(ii) For convenience, we write the IFS $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ by $A = (\mu_A, \nu_A)$.

Definition 2.15. Let G be a group and M be a G-module over K, which is a subfield of C. Then a intuitionistic fuzzy G-module on M is an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of M such that following conditions are satisfied
(i) $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$, $\forall a, b \in K$ and $x, y \in M$ and
(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, $\forall g \in G$; $m \in M$.

Example 2.16. ([4]) Let $G = \{1, -1\}, M = R^n$ over R. Then M is a G-module. Define the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ on M by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x \neq 0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.25, & \text{if } x \neq 0 \end{cases}$$

where $x = (x_1, x_2, ..., x_n) \in R^n$. Then A is an intuitionistic fuzzy G-module on M.

Theorem 2.17. *([6]) Consider a maximal chain of submodules of G-module M over the field K*

$$M_0 \subset M_1 \subset M_2 \subset .... \subset M_n = M \ ,$$

where $\subset$ denotes proper inclusion. Then there exists an intuitionistic fuzzy G-module A of M given by

$$\mu_A(x) = \begin{cases} \alpha_0 & \text{if } x \in M_0 \\ \alpha_1 & \text{if } x \in M_1 \backslash M_0 \\ \alpha_2 & \text{if } x \in M_2 \backslash M_1 \\ .... & \\ \alpha_n & \text{if } x \in M_n \backslash M_{n-1} \end{cases} ; \nu_A(x) = \begin{cases} \beta_0 & \text{if } x \in M_0 \\ \beta_1 & \text{if } x \in M_1 \backslash M_0 \\ \beta_2 & \text{if } x \in M_2 \backslash M_1 \\ .... & \\ \beta_n & \text{if } x \in M_n \backslash M_{n-1}. \end{cases}$$

where $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq .... \geq \alpha_n$ and $\beta_0 \leq \beta_1 \leq \beta_2 \leq \beta_n; \alpha_i, \beta_i \in [0, 1]$ such that $\alpha_i + \beta_i \leq 1, \forall\ i = 0, 1, ..., n$.

Remark 2.18. ([6]) The converse of above theorem (3.5) is also true i.e., any intuitionistic fuzzy G-module A of a G-module M can be expressed in the above form.

Definition 2.19. ([6]) Let A be an intuitionistic fuzzy set of a G-module M. Put $\wedge(A) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), (\alpha_2, \beta_2), ..., (\alpha_n, \beta_n)\}$, where $\alpha_i, \beta_i \in [0, 1]$ such that $\alpha_i + \beta_i \leq 1, \forall i = 0, 1, ..., n$ then we call the chain $(\alpha_0, \beta_0) \geq (\alpha_1, \beta_1) \geq (\alpha_2, \beta_2) \geq .... \geq (\alpha_n, \beta_n)$ a double keychain if and only if $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq .... \geq \alpha_n$ and $\beta_0 \leq \beta_1 \leq \beta_2 \leq \beta_n$ and the pair (α_i, β_i) are called double pinned flags for the intuitionistic fuzzy set A. The number $| \wedge (A)| = n + 1$ is called the level cardinality of the intuitionistic fuzzy set A.

Example 2.20. ([6]) Consider the G-module $M = R(i) = C$ over the field R and let $G = \{1, -1\}$ be the group. Define an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ on M

defined by

$$\mu_A(z) = \begin{cases} 1, & \text{if } z = 0 \\ 0.5, & \text{if } z \in R - \{0\} \\ 0.25, & \text{if } z \in R(i) - R \end{cases} \quad ; \quad \nu_A(z) = \begin{cases} 0, & \text{if } z = 0 \\ 0.25, & \text{if } z \in R - \{0\} \\ 0.5, & \text{if } z \in R(i) - R. \end{cases}$$

Then A is an intuitionistic fuzzy G-module on M of level cardinality $|\wedge(A)| = 3$.

3. Intuitionistic Fuzzy Galois Module

In this section, we construct an intuitionistic fuzzy G-module A on Galois field $GF(p^n)$ and also show that infinite many such intuitionistic fuzzy G-modules can be constructed. We have also discussed intuitionistic fuzzy noetherian G-module on $GF(p^n)$.

Proposition 3.1. *Any n-dimensional G-module M over K has an intuitionistic fuzzy G-module A of level cardinality $|\wedge(A)| = n+1$.*

Proof. Let $\{m_1, m_2, ..., m_n\}$ be the basis of G-module M. Let M_i be the G-submodule of M span by $\{m_1, m_2, ..., m_i\}$. Take $M_0 = \{0\}$. Then we get a maximal chain of G-submodules of M as $M_0 \subset M_1 \subset M_2 \subset .... \subset M_n = M$. Let $\wedge(A) = \{(1,0), (1/2, 1/n+1), (1/3, 1/n), ..., (1/n+1, 1/2)\}$ be the set of double pinned flags for the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ defined by

$$\mu_A(m) = \begin{cases} 1, & \text{if } m \in M_0 = \{0\} \\ 1/2, & \text{if } m \in M_1 \backslash M_0 \\ 1/3, & \text{if } m \in M_2 \backslash M_1 \\ .., & ..... \\ 1/n, & \text{if } m \in M_{n-1} \backslash M_{n-2} \\ 1/n+1, & \text{if } m \in M_n \backslash M_{n-1} \end{cases} \quad ; \quad \nu_A(m) = \begin{cases} 0, & \text{if } m \in M_0 = \{0\} \\ 1/n+1, & \text{if } m \in M_1 \backslash M_0 \\ 1/n, & \text{if } m \in M_2 \backslash M_1 \\ .., & ..... \\ 1/3, & \text{if } m \in M_{n-1} \backslash M_{n-2} \\ 1/2, & \text{if } m \in M_n \backslash M_{n-1}. \end{cases}$$

i.e., if $m = c_1 m_1 + c_2 m_2 + ... + c_n m_n$, then

$$\mu_A(c_1 m_1 + c_2 m_2 + ... + c_n m_n) = \begin{cases} 1, & \text{if } c_i = 0 \forall i \\ 1/2, & \text{if } c_1 \neq 0, c_2 = c_3 = 0, c_n = 0 \\ 1/3, & \text{if } c_2 \neq 0, c_3 = c_4 = 0, c_n = 0 \\ ..., & .... \\ 1/n, & \text{if } c_{n-1} \neq 0, c_n = 0 \\ 1/n+1, & \text{if } c_n \neq 0 \end{cases} \quad \textit{and}$$

$$\nu_A(c_1 m_1 + c_2 m_2 + ... + c_n m_n) = \begin{cases} 0, & \text{if } c_i = 0 \forall i \\ 1/n+1, & \text{if } c_1 \neq 0, c_2 = c_3 = 0, c_n = 0 \\ 1/n, & \text{if } c_2 \neq 0, c_3 = c_4 = 0, c_n = 0 \\ ..., & .... \\ 1/3, & \text{if } c_{n-1} \neq 0, c_n = 0 \\ 1/2, & \text{if } c_n \neq 0. \end{cases}$$

Then, A is an intuitionistic fuzzy G-module of level cardinality $|\wedge(A)| = n+1$. □

Theorem 3.2. *For every prime number p and every positive integer n, there exists an intuitionistic fuzzy G-module A on $GF(p^n)$ of level cardinality $|\wedge(A)| = n+1$*

Proof. It follows from Proposition (3.1) by taking $K = GF(p^n)$. □

Proposition 3.3. *For any intuitionistic fuzzy G-module A on a G - module M and for each $r \in (0,1]$, the IFS $A_r = (\mu_A, \nu_A)$ defined by $\mu_{A_r}(x) = r\mu_A(x)$ and $\nu_{A_r}(x) = (1-r)\nu_A(x)$, $\forall\ x \in M$. is also an intuitionistic fuzzy G-module on M.*

Proof. Let $a, b \in K, x, y \in M$ be any elements, then
$\mu_{A_r}(ax+by) = r\mu_A(ax+by) \geq r(\mu_A(x) \wedge \mu_A(y)) = r\mu_A(x) \wedge r\mu_A(y) = \mu_{A_r}(x) \wedge \mu_{A_r}(y)$ and
$\nu_{A_r}(ax+by) = (1-r)\nu_A(ax+by) \leq (1-r)(\nu_A(x) \vee \nu_A(y)) = (1-r)\nu_A(x) \vee (1-r)\nu_A(y) = \nu_{A_r}(x) \vee \nu_{A_r}(y)$.
Let $g \in G$ and $x \in M$ be any elements, we have
$\mu_{A_r}(gx) = r\mu_A(gx) \geq r\mu_A(x) = \mu_{A_r}(x)$ and
$\nu_{A_r}(gx) = (1-r)\nu_A(gx) \leq (1-r)\nu_A(x) = \mu_{A_r}(x)$.
Hence A_r is an intuitionistic fuzzy G-module on M. □

Remark 3.4. It is easy to check that if in the proposition (3.4), we have $r, s \in (0,1]$ such that $r < s$ then $A_r \subset A_s$.

Theorem 3.5. *For every prime number p and every positive integer n, there exists infinite many intuitionistic fuzzy Galois G-module $A_r, r \in (0,1]$ of level cardinality $|\wedge(A_r)| = n+1$*

Proof. Follows from Theorem (3.2) and Proposition (3.4). □

Theorem 3.6. *For every prime number p and every positive integer n, any intuitionistic fuzzy G-module A on $GF(p^n)$ has a sequence of intuitionistic fuzzy G-submodules $A_j, j = 1, 2, ..., k$, where k is the number of divisors of n.*

Proof. Consider the Galois field $M = GF(p^n)$. Then M is a vector space over $K = GF(p) \cong Z_p$, the field of integers modulo p and $dim_K M = n$. Without loss of generality, we assume that A is an intuitionistic fuzzy G-module in Theorem (3.1). Let the divisors of n be $1 = d_1, d_2, ..., d_k = n$ such that $d_1 < d_2 < ..... < d_k$. Then from theorem (2.13) M has k G-submodules $M_j = GF(p^{d_j})$ for $j = 1, 2, ..., k$ such that $Z_p \cong M_1 \subset M_2 \subset ..... \subset M_k$. Clearly, M_j is a subspace of M of dimension d_j. Let $\{\alpha_1, \alpha_2, ..., \alpha_{d_j}\}$ be a basis of M_j. Then we can extend this to form a basis $\{\alpha_1, \alpha_2, ..., \alpha_{d_j}, ..., \alpha_n\}$ for M. Define an intuitionistic fuzzy set A_j on M_j by

$$\mu_{A_j}(c_1\alpha_1 + c_1\alpha_1 + ... + c_{d_j}\alpha_{d_j}) = \begin{cases} 1, & \text{if } c_i = 0 \forall i \\ 1/2, & \text{if } c_1 \neq 0, c_2 = c_3 = 0, c_{d_j} = 0 \\ 1/3, & \text{if } c_2 \neq 0, c_3 = c_4 = 0, c_{d_j} = 0 \\ ..., & .... \\ 1/d_j, & \text{if } c_{d_j - 1} \neq 0, c_{d_j} = 0 \\ 1/d_j + 1, & \text{if } c_{d_j} \neq 0 \end{cases} \quad \textit{and}$$

$$\nu_{A_j}(c_1\alpha_1 + c_1\alpha_1 + ... + c_{d_j}\alpha_{d_j}) = \begin{cases} 0, & \text{if } c_i = 0 \forall i \\ 1/d_j + 1, & \text{if } c_1 \neq 0, c_2 = c_3 = 0, c_{d_j} = 0 \\ 1/d_j, & \text{if } c_2 \neq 0, c_3 = c_4 = 0, c_{d_j} = 0 \\ ..., & .... \\ 1/3, & \text{if } c_{d_j-1} \neq 0, c_{d_j} = 0 \\ 1/2, & \text{if } c_{d_j} \neq 0. \end{cases}$$

Then for each j, A_j is an intuitionistic fuzzy G-module of M of level cardinality $| \wedge (A_j)| = d_j + 1$.
Note that $A_1 \subset A_2 \subset ..... \subset A_k$ be a sequence of k intuitionistic fuzzy G-submodules of M, where k is the number of divisors of n. □

Theorem 3.7. *Every intuitionistic fuzzy Galois G-module has an ascending chain of intuitionistic fuzzy G-submodules, which terminates.*

Proof. By theorem (3.2), for every prime number p and every positive integer n, there exists an intuitionistic fuzzy G-module A on $GF(p^n)$ of level cardinality $| \wedge (A)| = n + 1$. Also, by theorem (3.7) any intuitionistic fuzzy G-module A on $GF(p^n)$ has a sequence of intuitionistic fuzzy G-submodules $A_j, j = 1, 2, ..., k$, where k is the number of divisors of n.
Let $t_j = 1/(d_j + 1)\ for j = 1, 2, ..., k$. Then for each j, we have an IFS B_j on M_j defined by

$$\mu_{B_j}(x) = \begin{cases} \mu_{A_j}(x), & \text{if } x \in M_j \\ t_j, & \text{if } x \in M - M_j \end{cases} ; \quad \nu_{B_j}(x) = \begin{cases} \nu_{A_j}(x), & \text{if } x \in M_j \\ 1 - t_j, & \text{if } x \in M - M_j \end{cases}$$

Clearly, each B_j is an intuitionistic fuzzy G-module on M. Let $C_j = B_j|_{M_j}$, for $j = 1, 2, ..., k$. Then each C_j is an intuitionistic fuzzy G-module on M_j such that $C_1 \subseteq C_2 \subseteq$ .....terminate at k. □

Corollary 3.8. *For an intuitionistic fuzzy Galois G-module there exists infinite many chains of intuitionistic fuzzy G-submodules terminates at k.*

Proof. Follows from Theorem (3.6) and Theorem (3.8) □

4. Conclusions

In this paper, we have constructed an intuitionistic fuzzy G-module of level cardinality $(n+1)$ on the Galois field $GF(p^n)$, and then proved that infinite many such intuitionistic fuzzy G-modules can be constructed on it. We have also proved that each such an intuitionistic fuzzy G-module, admits a sequence of k intuitionistic fuzzy G-submodules A_j, where k is the number of divisors of n. We have also proved that any ascending chain of intuitionistic fuzzy Galois modules terminates at some finite stage and that there are infinitely many such terminating chains of intuitionistic fuzzy G-modules.

Acknowledgements. Author is very thankful to the university grant commission, New Delhi for providing necessary financial assistance to carry out the present work under major research project file no. F. 42-2 / 2013 (SR).

References

[1] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1) (1986) 87-96.
[2] K.T. Atanassov, Intuitionistic Fuzzy Sets Theory and Applications, Studies on Fuzziness and Soft Computing, 35, Physica-Verlag, Heidelberg (1999).
[3] C.W. Curties and I. Reiner, Representation Theory of finite groups and associated algebras, INC, (1962).
[4] P.K. Sharma and Tarandeep Kaur, Intuitionistic fuzzy G-modules, Notes on Intuitionistic Fuzzy Sets, 21(1) (2015) 6-23.
[5] P.K. Sharma and Tarandeep Kaur, On intuitionistic fuzzy representation of intuitionistic fuzzy G-modules, Annals of Fuzzy Mathematics and Informatics, 11(4) (2016) 557-569.
[6] P.K.Sharma, Reducibility and Complete Reducibility of intuitionistic fuzzy G-modules, Annals of Fuzzy Mathematics and Informatics, Vol. 11, No.6, 2016, 885-898.
[7] P.K Sharma, Semi-simple Intuitionistic Fuzzy G- Modules, International Journal of Pure and Applied Researches, Vol. 1, no. 2, 2016, pp. 101- 108.
[8] P.K. Sharma and Simpi Chopra, Injectivity of intuitionistic fuzzy G-modules, Annals of Fuzzy Mathematics and Informatics (Communicated)
[9] Gupta, Goyal, Advanced course Modern Algebra, Pragati Prakashan, Meerut, ISBN 935006611-4.
[10] L.A. Zadeh, Fuzzy Sets, Inform. Control. 8 (1965) 338-353.