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Direct Sum of Intuitionistic Fuzzy Submodules of G-module

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Abstract. In this paper, we introduce the notion of direct sum of intuitionistic fuzzy submodules of G-module M and discuss some related properties. If B and C are intuitionistic fuzzy submodules of G-module M and G-module N respectively, then we extend the definition of intuitionistic fuzzy submodules B and C to formulate intuitionistic fuzzy submodules of the direct sum $M \oplus N$. It is proved that the support of the direct sum of intuitionistic fuzzy submodules B and C is equal to the direct sum of supports of intuitionistic fuzzy submodules B and C. Also we analyze the direct sum of arbitrary family of intuitionistic fuzzy submodules and obtained some results. We also introduce the notion of decomposability and indecompositity of intuitionistic fuzzy G-modules and prove some results.

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1. Introduction

The notion of a fuzzy set in a set was introduced by Zadeh [23], and since then this concept has been applied to many mathematical branches. Rosenfeld [11] applied the notion of fuzzy sets to algebra and introduced the notion of fuzzy subgroups. The literature of various fuzzy algebraic concepts has been growing very rapidly. In particular, Negoita and Ralescu [10] introduced and examined the notion of fuzzy submodule of a module. Since then different types of fuzzy submodules were investigated in the last two decades. Shery Fernadez introduced and studied the notion of fuzzy G-modules in [8].

One of the interesting generalizations of the theory of fuzzy sets is the theory of intuitionistic fuzzy sets introduced by Atanassov [1, 2, 3]. Biswas [5] was the first one to introduce the notion of intuitionistic fuzzy subgroup of a group. Using the Atanassov's idea, Davvaz et al. [7] established the intuitionistic fuzzification of the concept of submodule in a module and introduced the notion of intuitionistic fuzzy submodule of a module which was further studied by many authors (for example see [4, 9, 12, 13, 14]). The notion of intuitionistic fuzzy G-modules was introduced by the author et al. in [15]. Many properties like representation, reducibility, complete reducibility, semi-simplicity, fundamental theorems of isomorphisms, injectivity and projectivity of intuitionistic fuzzy G-modules have been discussed in [16, 17, 18, 19, 20, 21]. Here in this paper, we developed the intuitionistic fuzzification of one of the most useful concept, direct sum of submodules of a module. Some related results like decomposability and indecomposibility of intuitionistic fuzzy G-modules has also been discussed.

2. Preliminaries

For the sake of convenience we set our the former concepts which will be used in this paper are mainly taken from [2], [4], [5], [6], [15], [17] and [19]. Throughout the paper, M will always be a G-module over the field K (a subfield of the field of complex numbers).

Definition 2.1. ([2]) Let X be a non-empty set. An intuitionistic fuzzy set (IFS) A of X is an object of the form $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and degree of non-membership of the element $x \in X$ respectively and for any $x \in X$, we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Remark 2.2.
(i) when $\mu_A(x)+\nu_A(x) = 1$, i.e., $\nu_A(x) = 1-\mu_A(x)$ $\forall x \in X$, then A is called a fuzzy set.
(ii) For convenience, we write the IFS $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ by $A = (\mu_A, \nu_A)$.

Definition 2.3. ([2]) Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two IFSs of X, then
(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ $\forall$ $x \in X$;
(ii) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ $\forall$ $x \in X$;
(iii) $A^c = (\mu_{A^c}, \nu_{A^c})$, where $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$ $\forall$ $x \in X$;
(iv) $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B})$, where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ and $\nu_{A\cap B}(x) = \mu_A(x) \vee \mu_B(x)$;
(v) $A\cup B = (\mu_{A\cup B}, \nu_{A\cup B})$, where $\mu_{A\cup B}(x) = \mu_A(x)\vee\mu_B(x)$ and $\nu_{A\cup B}(x) = \nu_A(x)\wedge \nu_B(x)$.

Definition 2.4. [15] Let $(X, +)$ be a groupoid and A, B be two IFSs of X. Then the intuitionistic fuzzy sum of A and B are denoted by $A + B = (\mu_{A+B}, \nu_{A+B})$ and is defined as $\mu_{A+B}(x) = \vee_{x=y+z}\{\mu_A(y) \wedge \mu_B(z)\}; \nu_{A+B}(x) = \vee_{x=y+z}\{\nu_A(y) \vee \nu_B(z)\}$; $\forall$ $x \in X$.

Definition 2.5. ([15]) Let $A = (\mu_A, \nu_A)$ be an IFS of a universe set X, then support of A in X is denoted by A^* and is defined as
$A^* = \{x \in X : \mu_A(x) > 0$ and $\nu_A(x) < 1\}$.

Definition 2.6. ([6]) Let G be a group and M be a vector space over a field K. Then M is called a G-module if for every $g \in G$ and $m \in M$, $\exists$ a product (called the action of G on M), $gm \in M$ satisfies the following axioms

(i): $1_G.m = m$, $\forall$ $m \in M$ (1_G being the identity of G)
(ii): $(g \cdot h) \cdot m = g \cdot (h \cdot m)$, $\forall$ $m \in M, g, h \in G$
(iii): $g.(k_1m_1 + k_2m_2) = k_1(g.m_1) + k_2(g.m_2)$, $\forall$ $k_1, k_2 \in K; m_1, m_2 \in M$ and $g \in G$.

Since G acts on M on the left hand side, M may be called a left G-module. In a similar way, we can define a right G-module. But here we shall consider only left G-modules. A parallel study is possible using right G-modules also.

Definition 2.7. ([6]) Let M be a G-module. A vector subspace N of M is a G-submodule if N is also a G-module under the same action of G.

Definition 2.8. ([6]) Let M and M^* be G-modules. A mapping f : M $\rightarrow$ M* is a G-module homomorphism if
(i) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$
(ii)$f(gm) = gf(m), \forall k_1, k_2 \in K; m, m_1, m_2 \in$M and $g \in$G.

Definition 2.9. ([15]) Let G be a group and M be a G-module over K, which is a subfield of C. Then an intuitionistic fuzzy G-module on M is an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of M such that following conditions are satisfied
(i) $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$, $\forall a, b \in K$ and $x, y \in M$ and
(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\mu_A(gm) \leq \mu_A(m)$, $\forall\ g \in G; m \in M$.

Example 2.10. ([15]) Let $G = \{1, -1\}$, $M = R^n$ is a vector space over R. Then M is a G-module. Define the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ on M by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x \neq 0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.25, & \text{if } x \neq 0. \end{cases}$$

where $x = (x_1, x_2, ..., x_n) \in R^n$. Then A is an intuitionistic fuzzy G-module on M.

Proposition 2.11. *Let M be a G-module and $N(\neq \{0\}, M)$ be a G-submodule of M. Then the characteristic intuitionistic fuzzy set χ_N on N is defined by*

$$\chi_N(x) = \begin{cases} (1, 0), & \text{if } x \in N \\ (0, 1), & \text{otherwise.} \end{cases}$$

is an intuitionistic fuzzy G-module of M.

Remark 2.12. Clearly, $(\chi_N)^* = N$.

Theorem 2.13. *([15]) Let M be a G-module over the field K and A be an intuitionistic fuzzy G-module of M, then A^* is a G-submodule of M. But converse is not true.*

Theorem 2.14. *([17]) Consider a maximal chain of submodules of G-module M*

$$M_0 \subset M_1 \subset M_2 \subset .... \subset M_n = M \ ,$$

where $\subset$ denotes proper inclusion. Then there exists an intuitionistic fuzzy G-module A of M given by

$$\mu_A(x) = \begin{cases} \alpha_0 & \textit{if } x \in M_0 \\ \alpha_1 & \textit{if } x \in M_1 \backslash M_0 \\ \alpha_2 & \textit{if } x \in M_2 \backslash M_1 \\ \dots\dots\dots & \\ \alpha_n & \textit{if } x \in M_n \backslash M_{n-1} \end{cases} ; \nu_A(x) = \begin{cases} \beta_0 & \textit{if } x \in M_0 \\ \beta_1 & \textit{if } x \in M_1 \backslash M_0 \\ \beta_2 & \textit{if } x \in M_2 \backslash M_1 \\ \dots\dots\dots & \\ \beta_n & \textit{if } x \in M_n \backslash M_{n-1} \end{cases}$$

where $1 = \alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \dots\dots \geq \alpha_n$ *and* $0 = \beta_0 \leq \beta_1 \leq \beta_2 \leq \dots\dots \leq \beta_n$ *;* $\alpha_i, \beta_i \in [0,1]$ *such that* $\alpha_i + \beta_i \leq 1$, $\forall\ i = 0, 1, 2, ..., n$.

The converse of above theorem is also true i.e., any intuitionistic fuzzy G-module A of G-module M can be expressed in the above form.

Definition 2.15. ([19]) Let $A \in G^M$ (where G^M denotes the intuitionistic fuzzy power set of G-module M). Then A is called an intuitionistic fuzzy submodule (IFSM) of G-module M, if it satisfies the following:
(i) $\mu_A(0) = 1$ and $\nu_A(0) = 0$;
(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, $\forall\ g \in G, m \in M$;
(iii) $\mu_A(m_1 + m_2) \geq \mu_A(m_1) \wedge \mu_A(m_2)$ and $\nu_A(m_1 + m_2) \leq \nu_A(m_1) \vee \nu_A(m_1)$, $\forall m_1, m_2 \in M$.
We denote the set of all intuitionistic fuzzy submodules of G-module M by $G(M)$.

Definition 2.16. ([4],[14]) We define two IFSs Ω and $\Omega(M)$ of G-module M by

$$\mu_\Omega(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}; \quad \nu_\Omega(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases}$$

and $\mu_{\Omega(M)}(x) = 1$, $\mu_{\Omega(M)}(x) = 0$, $\forall x \in M$.
Then it can be easily verified that both Ω and $\Omega(M) \in G(M)$. These are called trivial IFSMs of G-module M. Any IFSM of G-module M other than these is called proper IFSM of M.

Lemma 2.17. *Let M be a G-module and let A be any IFSM of M, then* $A^* = \{0\}$ *if and only if* $A = \Omega$.

Proof. If $A = \Omega$, then clearly $A^* = \{0\}$.
Conversely, let $A^* = 0$. Then $\mu_A(0) > 0$ and $\nu_A(0) < 1$.
Also, $\mu_A(x) = 0$ and $\nu_A(x) = 1$ for all $x (\neq 0) \in M$. But $\mu_A(0) = 1$ and $\nu_A(0) = 0$. Therefore,

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases}, \forall x \in M.$$

This shows that $A = \Omega$. □

3. Direct sum of intuitionistic fuzzy submodules of G-modules

Definition 3.1. If A and B are two IFSMs of G-module M, then the sum $A + B$ is called the direct sum of A and B if $A \cap B = \Omega$ and we write it as $A \oplus B$.

Definition 3.2. Let $A_i = (\mu_{A_i}, \nu_{A_i})$ be IFSMs of a G-module M, then we say that $A = (\mu_A, \nu_A)$ is the direct sum of $\{A_i : i \in J\}$ denoted by $\oplus_{i\in J} A_i$ if
(i) $A = \Sigma_{i\in J} A_i$
(ii) $A_j \cap \sum_{i\in J\setminus\{j\}} A_i = \Omega, \forall j \in J$.

Example 3.3. Let $G = \{1, -1\}, M = R^2 = \{(p,q) : p, q \in R\}$ is a vector space over the field R. Then M is a G-module. Define IFSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B), C = (\mu_C, \nu_C)$ of M by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = (0,0) \\ 0.25, & \text{if } x = (p,0), p \neq 0 \\ 0.25, & \text{if } x = (p,q), q \neq 0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = (0,0) \\ 0.5, & \text{if } x = (p,0), p \neq 0 \\ 0.5, & \text{if } x = (p,q), q \neq 0; \end{cases}$$

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = (0,0) \\ 0.25, & \text{if } x = (p,0), p \neq 0 \\ 0, & \text{if } x = (p,q), q \neq 0 \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = (0,0) \\ 0.5, & \text{if } x = (p,0), p \neq 0 \\ 1, & \text{if } x = (p,q), q \neq 0; \end{cases}$$

$$\mu_C(x) = \begin{cases} 1, & \text{if } x = (0,0) \\ 0.25, & \text{if } x = (0,q), q \neq 0 \\ 0, & \text{if } x = (p,q), p \neq 0 \end{cases}; \quad \nu_C(x) = \begin{cases} 0, & \text{if } x = (0,0) \\ 0.5, & \text{if } x = (0,q), q \neq 0 \\ 1, & \text{if } x = (p,q), p \neq 0. \end{cases}$$

It can be easily verified that A, B and C are IFSMs of M such that $A = B + C$ and $B \cap C = \Omega$. Hence $A = B \oplus C$.

Theorem 3.4. *Let $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B), C = (\mu_C, \nu_C)$ be IFSMs of G-module M such that $A = B \oplus C$, then $A^* = B^* \oplus C^*$.*

Proof. Let $x \in A^*$, then $x \in (B \oplus C)^* \Rightarrow \mu_{B\oplus C}(x) > 0$ and $\nu_{B\oplus C}(x) < 1$
$\Rightarrow \mu_{B+C}(x) > 0$ and $\nu_{B+C}(x) < 1$ $[\because A = B \oplus C \Rightarrow A = B + C, B \cap C = \Omega]$
Now, $\mu_{B+C}(x) = \vee_{x=y+z}\{\mu_B(y) \wedge \mu_C(z)\} > 0$ implies $\exists^{'}s$ $y, z \in M$ with $y + z = x$ such that $\mu_B(y) > 0$ and $\mu_C(z) > 0$ and thus $\nu_B(y) < 1$ and $\nu_C(z) < 1$,
i.e., there exists $y, z \in M$ such that $y \in B^*$ and $z \in C^*$, so that $x = y + z \in B^* + C^*$.
Therefore, $A^* \subseteq B^* + C^*$..(1)
Again, let $x \in B^* + C^*$ be any element, then $\exists^{'}s$ $y \in B^*, z \in C^*$ such that $x = y + z$
i.e., $\mu_B(y) > 0, \mu_C(z) > 0$ and $\nu_B(y) < 1, \nu_C(z) < 1$ such that $x = y + z$
i.e., $\mu_B(y) \wedge \mu_C(z) > 0$ and $\nu_B(y) \vee \nu_C(z) < 1$ is true for all $y, z \in M$ such that $x = y + z$
i.e., $\vee_{x=y+z}\{\mu_B(y) \wedge \mu_C(z)\} > 0$ and $\wedge_{x=y+z}\{\nu_B(y) \vee \nu_C(z)\} < 1$
$\Rightarrow \mu_{B+C}(x) > 0$ and $\nu_{B+C}(x) < 1$ i.e., $\mu_A(x) > 0$ and $\nu_A(x) < 1$. Thus $x \in A^*$.
Therefore, $B^* + C^* \subseteq A^*$..(2)
From (1) and (2), we get $A^* = B^* + C^*$.
Now, $x \in B^* \cap C^* \Rightarrow x \in B^*$ and $x \in C^*$
$\Rightarrow \mu_B(x) > 0, \nu_B(x) < 1$ and $\mu_C(x) > 0, \nu_C(x) < 1$
$\Rightarrow \mu_B(x) \wedge \mu_C(x) > 0$ and $\nu_B(x) \vee \nu_C(x) < 1$
$\Rightarrow \mu_{B\cap C}(x) > 0$ and $\nu_{B\cap C}(x) < 1$
$\Rightarrow \mu_{B\cap C}(x) = 1$ and $\nu_{B\cap C}(x) = 0$ [As $A = B \oplus C$, so $B \cap C = \Omega$] $\Rightarrow x = 0$.
Thus, $B^* \cap C^* = 0$. Hence $A^* = B^* \oplus C^*$. □

Remark 3.5. The converse of the theorem (3.4) need not be true.

Example 3.6. Let $G = \{1, -1\}, M = R^2 = \{(p, q) : p, q \in R\}$ as in example (3.3).Then M is a G-module. Define IFSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B), C = (\mu_C, \nu_C)$ of M by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = (0,0) \\ 0.25, & \text{if } x = (p,0), p \neq 0 \\ 0.5, & \text{if } x = (p,q), q \neq 0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = (0,0) \\ 0.75, & \text{if } x = (p,0), p \neq 0 \\ 0.25, & \text{if } x = (p,q), q \neq 0 \end{cases}; \forall x \in M$$

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = (0,0) \\ 0.25, & \text{if } x = (p,0), p \neq 0 \\ 0, & \text{if } x = (p,q), q \neq 0 \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = (0,0) \\ 0.5, & \text{if } x = (p,0), p \neq 0 \\ 1, & \text{if } x = (p,q), q \neq 0 \end{cases}; \forall x \in M$$

$$\mu_C(x) = \begin{cases} 1, & \text{if } x = (0,0) \\ 0.25, & \text{if } x = (0,q), q \neq 0 \\ 0, & \text{if } x = (p,q), p \neq 0 \end{cases}; \quad \nu_C(x) = \begin{cases} 0, & \text{if } x = (0,0) \\ 0.5, & \text{if } x = (0,q), q \neq 0 \\ 1, & \text{if } x = (p,q), p \neq 0 \end{cases}; \forall x \in M.$$

It is easy to verify that A, B and C are IFSMs of M.
Now, $A^* = \{x \in R^2 : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\} = R^2$.
Similarly, $B^* = (R, 0)$ and $C^* = (0, R)$. Therefore, we have $A^* = B^* \oplus C^*$.

$$\mu_{B+C}(x) = \begin{cases} 1, & \text{if } x = (0,0) \\ 0.25, & \text{if } x = (p,0), p \neq 0 \\ 0.25, & \text{if } x = (p,q), q \neq 0 \end{cases}; \quad \nu_{B+C}(x) = \begin{cases} 0, & \text{if } x = (0,0) \\ 0.5, & \text{if } x = (p,0), p \neq 0 \\ 0.5, & \text{if } x = (p,q), q \neq 0. \end{cases}$$

Clearly, $B + C \neq A$ and so $A \neq B \oplus C$.

Definition 3.7. Let M and N be G-modules. Let $A = (\mu_A, \nu_A)$ be an IFSM of M and $B = (\mu_B, \nu_B)$ be an IFSM of N. Consider the direct sum $M \oplus N$. We extend the definition A and B to $M \oplus N$ to get $A' = (\mu_{A'}, \nu_{A'})$ and $B' = (\mu_{B'}, \nu_{B'})$, the IFSs in $M \oplus N$ as follows

$$\mu_{A'}(m,n) = \begin{cases} \mu_A(m), & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}; \quad \nu_{A'}(m,n) = \begin{cases} \nu_A(m), & \text{if } n = 0 \\ 1, & \text{if } n \neq 0 \end{cases}; \forall (m,n) \in M \oplus N.$$

$$\mu_{B'}(m,n) = \begin{cases} \mu_B(n), & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases}; \quad \nu_{B'}(m,n) = \begin{cases} \nu_B(n), & \text{if } m = 0 \\ 1, & \text{if } m \neq 0 \end{cases}; \forall (m,n) \in M \oplus N.$$

Theorem 3.8. *The IFSs $A' = (\mu_{A'}, \nu_{A'})$ and $B' = (\mu_{B'}, \nu_{B'})$ defined above are IFSMs of $M \oplus N$.*

Proof. Let $x = (m_1, n_1), y = (m_2, n_2) \in M \oplus N, a, b \in K, g \in G$ be any elements, then

$$\mu_{A'}(ax + by) = \mu_{A'}(am_1 + bm_2, an_1 + bn_2) = \begin{cases} \mu_A(am_1 + bm_2); & \text{if } an_1 + bn_2 = 0 \\ 0, & \text{if } an_1 + bn_2 \neq 0. \end{cases}$$

Case(i) if $an_1 + bn_2 = 0$, then
Subcase(i) when both $n_1 = n_2 = 0$, then
$\mu_{A'}(ax + by) = \mu_{A'}(am_1 + bm_2) \geq \mu_A(m_1) \wedge \mu_A(m_2) = \mu_{A'}(x) \wedge \mu_{A'}(y)$.
Subcase(ii) when $n_2 \neq 0$ and $n_1 = 0$ then $bn_2 = -an_1 = 0$ implies $b = 0$, so

$\mu_{A'}(ax+by) = \mu_A(am_1+bm_2) = \mu_A(am_1) \geq \mu_A(m_1) \geq \mu_A(m_1) \wedge 0 = \mu_{A'}(x) \wedge \mu_{A'}(y)$. The subcase when $n_1 \neq 0$ and $n_2 = 0$ can be dealt similarly.
subcase (iii) when $n_1 \neq 0$ and $n_2 \neq 0$ then $bn_2 = -an_1 \neq 0$, so
$\mu_{A'}(ax+by) = \mu_A(am_1+bm_2) \geq \mu_A(m_1) \wedge \mu_A(m_2) \geq 0 \wedge 0 = \mu_{A'}(x) \wedge \mu_{A'}(y)$.

Case(ii) if $an_1 + bn_2 \neq 0$, then
Subcase(i) when $n_2 \neq 0$ and $n_1 = 0$ then $an_1 + bn_2 = bn_2 \neq 0$ implies $b \neq 0$, so
$\mu_{A'}(ax+by) = 0 = \mu_A(m_1) \wedge 0 = \mu_{A'}(x) \wedge \mu_{A'}(y)$.
The subcase when $n_1 \neq 0$ and $n_2 = 0$ can be dealt similarly.
Subcase (ii) when $n_1 \neq 0$ and $n_2 \neq 0$, then
$\mu_{A'}(ax+by) = 0 = 0 \wedge 0 = \mu_{A'}(x) \wedge \mu_{A'}(y)$.

Thus in all the cases, we see that $\mu_{A'}(ax+by) \geq \mu_{A'}(x) \wedge \mu_{A'}(y)$.
Similarly, we can show that $\nu_{A'}(ax+by) \leq \nu_{A'}(x) \vee \nu_{A'}(y)$.
Also,

$$\mu_{A'}(gx) = \mu_{A'}(gm_1, gn_1) = \begin{cases} \mu_A(gm_1), & \text{if } gn_1 = 0 \\ 0, & \text{if } gn_1 \neq 0. \end{cases}$$

Case(i) when $gn_1 = 0 \Rightarrow n_1 = 0$, then
$\mu_{A'}(gx) = \mu_{A'}(gm_1, gn_1) = \mu_A(gm_1) \geq \mu_A(m_1) = \mu_{A'}(x)$.
Case(ii) when $gn_1 \neq 0 \Rightarrow n_1 \neq 0$, then
$\mu_{A'}(gx) = \mu_{A'}(gm_1, gn_1) = 0 = \mu_{A'}(x)$.
Thus in all cases we see that $\mu_{A'}(gx) \geq \mu_{A'}(x)$.
Similarly, we can show that $\nu_{A'}(gx) \leq \nu_{A'}(x)$.
Thus, $A^{'}$ is an IFSM of $M \oplus N$.
In a similar way we can prove that $B^{'}$ is also an IFSM of $M \oplus N$. □

Remark 3.9. Since $A^{'} = (\mu_{A'}, \nu_{A'})$, $B^{'} = (\mu_{B'}, \nu_{B'})$ are IFSMs of $M \oplus N$, their sum $A^{'} + B^{'}$ is also IFSM of $M \oplus N$. Now, $A^{'} \cap B^{'} = (\mu_{A' \cap B'}, \nu_{A' \cap B'})$, where

$$\mu_{A' \cap B'}(m,n) = \mu_{A'}(m,n) \wedge \mu_{B'}(m,n) = \begin{cases} 1, & \text{if } (m,n) = 0 \\ 0, & \text{if } (m,n) \neq 0 \end{cases}$$

and

$$\nu_{A' \cap B'}(m,n) = \nu_{A'}(m,n) \vee \nu_{B'}(m,n) = \begin{cases} 0, & \text{if } (m,n) = 0 \\ 1, & \text{if } (m,n) \neq 0. \end{cases} \; i.e., A^{'} \cap B^{'} = \Omega.$$

i.e., $A^{'} + B^{'}$ is infact a direct sum and is denoted by $A \oplus B$ which is an IFSM of $M \oplus N$.

Remark 3.10. we have $A \oplus B = (\mu_{A \oplus B}, \nu_{A \oplus B}) = A^{'} + B^{'} = (\mu_{A'+B'}, \nu_{A'+B'})$, where

$$\begin{aligned} \mu_{A \oplus B}(m,n) &= \mu_{A'+B'}(m,n) \\ &= \vee_{(m,n)=(m_1,n_1)+(m_2,n_2)} \{\mu_{A'}(m_1,n_1) \wedge \mu_{B'}(m_2,n_2)\}, \forall (m,n) \in M \oplus N \\ &= \mu_{A'}(m,0) \wedge \mu_{B'}(0,n) \\ &= \mu_A(m) \wedge \mu_B(n). \end{aligned}$$

Similarly, $\mu_{A\oplus B}(m,n) = \nu_A(m) \vee \nu_B(n)$, $\forall\ (m,n) \in M \oplus N$.

Theorem 3.11. *Let A and B be intuitionistic fuzzy submodules of G-module M and G-module N respectively, then $(A \oplus B)^* = A^* \oplus B^*$.*

Proof. from remark (3.10) we have $A \oplus B = A' + B'$ is the intuitionistic fuzzy submodule of G- module $M \oplus N$

$$\begin{aligned}
\text{Let } (m,n) \in (A \oplus B)^* &\Rightarrow \mu_{A\oplus B}(m,n) > 0 \text{ and } \nu_{A\oplus B}(m,n) < 1 \\
&\Rightarrow \mu_A(m) \wedge \mu_B(n) > 0 \text{ and } \nu_A(m) \vee \nu_B(n) < 1 \\
&\Rightarrow \mu_A(m) > 0, \mu_B(n) > 0 \text{ and } \nu_A(m) < 1, \nu_B(n) < 1 \\
&\Rightarrow m \in A^* \text{ and } n \in B^* \\
&\Rightarrow m + n \in A^* + B^*.
\end{aligned}$$

$$\begin{aligned}
\text{Now, } x \in A^* \cap B^* &\Rightarrow x \in A^* \text{ and } x \in B^* \\
&\Rightarrow \mu_A(x) > 0, \nu_A(x) < 1 \text{ and } \mu_B(x) > 0, \nu_B(x) < 1 \\
&\Rightarrow \mu_A(x) \wedge \mu_B(x) > 0 \text{ and } \nu_A(x) \vee \nu_B(x) < 1 \\
&\Rightarrow \mu_{A\cap B}(x) > 0 \text{ and } \nu_{A\cap B}(x) < 1 \\
&\Rightarrow \mu_{A\cap B}(x) = 1 \text{ and } \nu_{A\cap B}(x) = 0 [For A \oplus B, A \cap B = \Omega] \\
&\Rightarrow x = 0.
\end{aligned}$$

$\therefore A^* + B^*$ is the direct sum denoted as $A^* \oplus B^*$, which is a submodule of G-module $M \oplus N$. Thus $(A \oplus B)^* \subseteq A^* \oplus B^*$............................(1)

$$\begin{aligned}
\text{Further, let } (m,n) \in A^* + B^* &\Rightarrow m \in A^* \text{ and } n \in B^* \\
&\Rightarrow \mu_A(m) > 0, \nu_A(m) < 1 \text{ and } \mu_B(n) > 0, \nu_B(n) < 1 \\
&\Rightarrow \mu_A(m) \wedge \mu_B(n) > 0 \text{ and } \nu_A(m) \vee \nu_B(n) < 1 \\
&\Rightarrow \mu_{A\oplus B}(m,n) > 0 \text{ and } \nu_{A\oplus B}(m,n) < 1 \\
&\Rightarrow (m,n) \in (A \oplus B)^*.
\end{aligned}$$

Hence $A^* \oplus B^* \subseteq (A \oplus B)^*$.......................................(2)
from (1) and (2) we get $(A \oplus B)^* = A^* \oplus B^*$. ☐

Corollary 3.12. *If A and B are two IFSMs of a G-module M, then $(A \oplus B)^* = A^* \oplus B^*$.*

Theorem 3.13. *If $A_i, i \in J$ and B are IFSM of G-module M, where $\Sigma_{i\in J} A_i$ is the direct sum $\oplus_{i\in J} A_i$ and if $B \cap \Sigma_{i\in J} A_i = \Omega$, then $B + \Sigma_{i\in J} A_i$ is the direct sum $B \oplus (\oplus_{i\in J} A_i)$.*

Proof. Given that $\Sigma_{i\in J} A_i$ is a direct sum, hence $A_j \cap (\Sigma_{i\in J\setminus\{j\}} A_i) = \Omega\ \forall j \in J$.
Also given that $B \cap \Sigma_{i\in j} A_i) = \Omega$. For any $x \in M, j \in J$
$(A_j \cap (B + \Sigma_{i\in J\setminus\{j\}} A_i))(x) = (\mu_{A_j \cap (B+\Sigma_{i\in J\setminus\{j\}} A_i)}(x), \nu_{A_j \cap (B+\Sigma_{i\in J\setminus\{j\}} A_i)}(x))$

Now, $\mu_{A_j \cap (B+\Sigma_{i\in J\setminus\{j\}} A_i)}(x)$
$= \mu_{A_j}(x) \wedge \mu_{B+\Sigma_{i\in J\setminus\{j\}} A_i}(x)$
$= \mu_{A_j}(x) \wedge [\vee_{x=y+\Sigma_{i\in J\setminus\{j\}} A_i} \{\mu_B(y) \wedge \{\wedge_{i\in J\setminus\{j\}} \mu_{A_i}(x_i)\}\}]$, where $x_i, y \in M, i \in J\setminus\{i\}$

$= \mu_{A_j}(y + \Sigma_{i\in J\setminus\{j\}}x_i) \wedge [\vee_{x=y+\Sigma_{i\in J\setminus\{j\}}x_i}\{\mu_B(y) \wedge \{\wedge_{i\in J\setminus\{i\}}\mu_{A_i}(x_i)\}\}]$
$= (\vee_{x=y+\Sigma_{i\in J\setminus\{j\}}x_i}[\mu_{A_j}(y)\wedge\{\wedge_{i\in J\setminus\{j\}}\mu_{A_i}(x_i)\}])\wedge(\vee_{x=y+\Sigma_{i\in J\setminus\{j\}}x_i}[\mu_B(y)\wedge\{\wedge_{i\in J\setminus\{j\}}\mu_{A_i}(x_i)\}])$
$= \vee_{x=y+\Sigma_{i\in J\setminus\{j\}}x_i}[\mu_{A_j}(y) \wedge \{\wedge_{i\in J\setminus\{j\}}\mu_{A_i}(x_i)\} \wedge \mu_B(y) \wedge_{i\in J\setminus\{j\}} \mu_{A_i}(x_i)\}]$
$= \vee_{x=y+\Sigma_{i\in J\setminus\{j\}}x_i}[(\mu_{A_j}(y) \wedge \mu_B(y)) \wedge \{\wedge_{i\in J\setminus\{i\}}\mu_{A_i}(x_i) \wedge \mu_{A_i}(x_i)\}]$
$= \vee_{x=y+\Sigma_{i\in J\setminus\{j\}}x_i}[\mu_{A_j\cap B}(y) \wedge \{\wedge_{i\in J\setminus\{j\}}\mu_{A_j\cap A_i}(x_i)\}]$

$$= \begin{cases} 1, & \text{if } y = 0, x_i = 0 \forall i \in J\setminus\{i\} \\ 0, & \text{if } y \neq 0 \text{ or } x_i \neq 0 \text{ for some } i \in J\setminus\{i\} \end{cases} (\because A_j\cap B = \Omega \forall j \in J \text{ and } A_j\cap A_i = \Omega \forall i \in j\setminus\{i\}).$$

$$= \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases} [\therefore x = y + \Sigma_{i\in J\setminus\{j\}}x_i]$$

$= \mu_\Omega(x)$

Similarly, we can show that

$$\nu_{A_j\cap(B\cap\Sigma_{i\in J\setminus\{j\}}A_i)}(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases} = \nu_\Omega(x)$$

Therefore $B + \Sigma_{i\in J}A_i$ is the direct sum $B \oplus (\oplus_{i\in J}A_i)$.
Thus the theorem is proved.

□

4. Decomposability and indecomposability of intuitionistic fuzzy G-modules

Definition 4.1. An IFSM $A(\neq \Omega)$ of a G-module M is said to be an indecomposable intuitionistic fuzzy G-module if there donot exists IFSM B and $C(\neq \Omega, A)$ of M such that $A = B\oplus C$, otherwise A is called a decomposable intuitionistic fuzzy G-module.

Example 4.2. Let $M = K = GF(p)$ be the Galois field, p is a prime, and let $G = M - \{0\}$. Then M is a G-module over the field K. Then the IFS A on M defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ r, & \text{if } x \neq 0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ s, & \text{if } x \neq 0 \end{cases}, \forall x \in M,$$

where $r, s \in [0, 1]$ such that $r + s \leq 1$ is an fuzzy indecomposable intuitionistic G-module.

Proposition 4.3. *$\Omega(M)$ is an indecomposable intuitionistic fuzzy G-module if and only if M is indecomposable G-module.*

Proof. First let $\Omega(M)$ be an indecomposable intuitionistic fuzzy G-module.
Let $M = P \oplus Q$, where P and Q are non-zero proper submodules of a G-module M.
Let χ_P and χ_Q be the charactersistic intuitionistic fuzzy sets on P and Q respectively.
Then χ_P and χ_Q are IFSMs of M such that $(\chi_P)^* = P$ and $(\chi_Q)^* = Q$.
So $M = (\chi_P)^* \oplus (\chi_Q)^* = (\chi_P \oplus \chi_Q)^* \Rightarrow \Omega(M) = \chi_P \oplus \chi_Q$.
Since P and Q are non-zero proper submodules of M so $\chi_P, \chi_Q \neq \Omega, \Omega(M)$.
This contradict that $\Omega(M)$ is an indecomposable.
Hence M is indecomposable G-module.
Conversely, let M is indecomposable G-module. Let A and B be two IFSMs of

M such that $A, B \neq \Omega, \Omega(M)$ and $\Omega(M) = A \oplus B$. Then $M = A^* \oplus B^*$. Since $A, B \neq \Omega(M)$ so $A^*, B^* \neq M$. Also, if $A^* = \{0\}$ then $B^* = M$ and so $B = \Omega(M)$, which is not true. This contradicts the fact that M is indecomposable G-module. Hence $\Omega(M)$ is an indecomposable intuitionistic fuzzy G-module. □

Theorem 4.4. *If a G-module M is decomposable, then there exist a decomposable intuitionistic fuzzy G-module .*

Proof. The result follows from proposition (4.3). □

Remark 4.5. Let M and N be G-modules and $f : M \to N$ be a G-module homomorphism. If $A = B \oplus C$ be a decomposable intuitionistic fuzzy submodule of G-module M. Then $f(A)$ need not be equal to $f(B) \oplus f(C)$.

Example 4.6. Let $G = \{1, -1\}, M = R^2 = \{(p, q) : p, q \in R\}$ as in example (3.3) and let $N = R$, the set of real numbers, then N is a G-module. Define IFSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ as in example (3.3).
Then we have $A = B \oplus C$.
Now consider the mapping $f : M \to N$ defined by $f(p, q) = p + q \ \forall (p, q) \in M$.
It is easy to check that M is a G-module homomorphism.

$$\text{Now, } f(A)(y) = \begin{cases} (\vee\{\mu_A(x) : f(x) = y\}, \wedge\{\mu_A(x) : f(x) = y\}), & \text{if } f^{-1}(y) \neq \phi \\ (0, 1), & \text{otherwise.} \end{cases}$$

Hence we have $\mu_{f(A)}(0) = \mu_A(0, 0) \vee \mu_A(p, -p) = 1 \vee 0.25 = 0.25$ and
$\nu_{f(A)}(0) = \nu_A(0, 0) \wedge \nu_A(p, -p) = 0 \wedge 0.5 = 0$.

$$\begin{aligned} \text{Also, } \mu_{f(A)}(p + q) &= \mu_A(p + q, 0) \vee \mu_A(0, p + q) \vee \mu_A(p, q) \vee \mu_A(q, p) \\ &= 0.25 \vee 0.25 \vee 0.25 \vee 0.25 \\ &= 0.25 \end{aligned}$$

$$\begin{aligned} \text{and } \nu_{f(A)}(p + q) &= \nu_A(p + q, 0) \wedge \nu_A(0, p + q) \wedge \nu_A(p, q) \wedge \nu_A(q, p) \\ &= 0.5 \wedge 0.5 \wedge 0.5 \wedge 0.5 \\ &= 0.5. \end{aligned}$$

Thus,

$$f(A)(y) = \begin{cases} (1, 0), & \text{if } y = 0 \\ (0.25, 0.5), & \text{if } y \neq 0 \end{cases}, \forall y \in R.$$

It can be easily checked that $f(A) = f(B) + f(C)$, but $f(B) \cap f(C) \neq \Omega$ implies that $f(A) \neq f(B) \oplus f(C)$.

5. Conclusions

In this paper, we introduce the notion of direct sum of intuitionistic fuzzy submodules of G-module. This notion is very useful in studying some properties of intuitionistic fuzzy G-module like projectivity, injectivity and semi-simplicity of intuitionistic fuzzy submodules in term of exact sequences. This work is under progress [22].

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