Redefined submultiset-based multiset ordering via grid

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ABSTRACT—***We present a grid form of the Jouannaud-Lescanne submultiset-based multiset ordering. A flexible definition of the multiset ordering is hereby presented. The grid approach has been used in this paper to prove some assertions. One such result is the submultiset-based pair-wise equality theorem for multisets.***

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1. Introduction

The articles [1] and [4] are some of the earliest references known to introduce the Dershowitz-Manna ordering and using it to prove the termination of programs and term rewriting systems. The Dershowitz-Manna ordering was also used in [7] to define *tree structure* for membrane computing. Huet and Oppen gave a definition of multiset ordering in [2] – a refinement of the Dershowitz-Manna ordering. Jouannaud and Lescanne introduced some multiset orderings based on multiset partition, which were proved to be stronger than the Dershowitz-Manna ordering [3].

The idea of the *grid* of a partially ordered multiset was first introduced in [5], where the *references* were based on the subsets of the multiset in question. We introduce here *submultiset-based grid* of a multiset, *submultiset-based difference grid* of two multisets and the *submultiset-based references* in the grids in question. The approach is an extension of the method used by Joannaud and Lescanne. This paper verifies that the approach is flexible. Such flexibility potential is demonstrated in the proofs of some important results, the *submultiset-based pair-wise equality theorem* in particular.

2. Preliminaries

Definition 1. *Partial ordering*

An ordering $\prec$ on a set E is a *partial* or *strict total ordering* on E, i.e., an irreflexive and transitive relation on E (or equivalently, a transitive but not an equivalence relation). Similar to the notation used in [3], we write $x \# y$ to mean the following:

$$\neg(x \prec y \text{ or } x = y \text{ or } y \prec x) \tag{1}$$

read as "x and y are *incomparable* under '$\prec$' or '$=$'." Interestingly, $x \neq y$ can be defined as:

$$\neg(x \prec y \text{ or } x \# y \text{ or } y \prec x).$$

Definition 2. *Comparably unequal*

Two elements x and y are said to be *comparably unequal* if and only if either $x \prec y$ or $x \succ y$. In other words x and y are comparably unequal if they satisfy the property:

$$\neg\left(x = y \text{ or } x \# y\right).$$

Definition 3. *Inordering*

If $x = y$ or $x \# y$ for all the elements x and y of a multiset M with respect to a given partial ordering $\prec$, then M is said to be *unordered* with respect to $\prec$. That is, a multiset is said to be unordered if it contains no comparably unequal objects. Given that $a \# b$, $b \# c$ and $a \# c$, a typical unordered multiset is $M = \{a, a, a, b, b, c\}$.

Definition 4. *Multiset equality*

Let E be a base set of the multisets M and N, then $M = N$ if and only if $M(x) = N(x)\ \forall x \in E$ where $M(x)$ and $N(x)$ are the multiplicities of x in M and N, respecively.

Definition 5. *The Dershowitz-Manna definition of multiset ordering*

Let M and N be multisets in $M(S)$ a class of multisets whose base set is S, then $M \lll N$ if there exist two multisets X and Y in $M(S)$ satisfying

(i) $\varnothing \neq X \subseteq N$,
(ii) $M = (N \backslash X) + Y$, and
(iii) $(\forall y \in Y)(\exists x \in X)[y \prec x]$.

In other words, $M \lll N$ if M is obtained from N by removing none or at least one element (those in X) from N, and replacing each such element x by zero or any finite number of elements (those in Y), each of which is strictly less than (in the ordering $\prec$) one of the elements x that have been removed.

Informally, we say that M is smaller than N in this case. Similarly, $\ggcurly$ on $M(S)$ with $(S, \succ)$ can be defined. For example, let $S = (\{0, 1, 2, \dots\} = \aleph)$, then under the corresponding multiset ordering $\lll$ over $\aleph$, each of the following multisets [3, 4], [3, 2, 2, 1, 1, 1, 4, 0] and [3, 3, 3, 3, 2, 2] is less than the multiset [3, 3, 4, 0]. The empty set $\varnothing$ is smaller than any multiset. It is also easy to see that $\forall y \in N \implies [\exists x \in M \wedge x \succ y] \implies M \ggcurly N$. See [3] for more on the Dershowitz-Manna ordering. Where necessary, this definition will be denoted by $\lll_{DM}$.

Definition 6. *The Huet-Oppen definition of multiset ordering*

Let M and N be multisets, then

$M \lll N$ if and only if $M \neq N$ and

$$[M(x) > N(x) \Longrightarrow (\exists y \in E)[x \prec y] \; \& \; M(y) < N(y)]$$

([2] and [3] for details).

We shall denote this ordering by $\lll_{HO}$. Next, we prove the following Lemma, which will be used in the concluding part of Theorem 3.

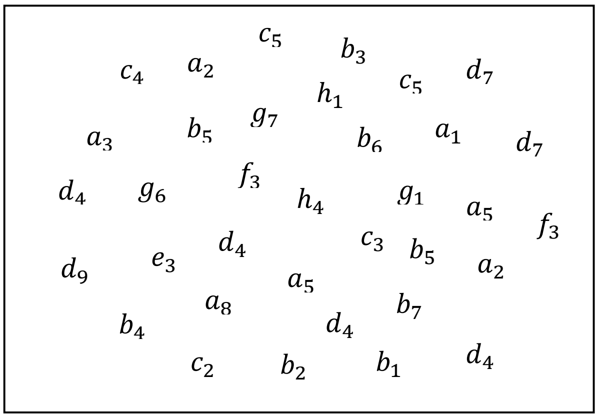
Lemma 1. *Let E be a base set of multisets M and N. If $M \# N$ holds, then there exist incomparable elements x and y of E such that $M(x) > N(x)$ and $N(y) > M(y)$.*

Proof. Suppose $M \# N$ holds, we have the following properties by the definition of incomparability of multisets: $M \neg\lll N$, $N \neg\lll M$ and $M \neq N$. Since $M \neg\lll N$, by the Huet-Open definition of multiset ordering, for all x such that $M(x) > N(x)$, $\nexists y \succcurlyeq x$ such that $N(y) > M(y)$. Either $y \prec x$ or $y \# x$ for such x and y. If $y \prec x$ for all of such x and y, then $M \ggcurly N$. This is a contradiction of $M \# N$. Therefore, there exists incomparable objects x and y such that $M(x) > N(x)$ and $N(y) > M(y)$.

In the section that follows, we construct a submultiset-based grid for multiset ordering. First, we define a grid, then a difference grid. We thereafter define multiset ordering based on the difference grid.

3. The concept of submultiset-based grid of a partially ordered multiset

Consider the following (diagrammatic representation of a) multiset:

$M =$ 

Assume the elements of M to satisfy the following properties:

(i) $\omega_i \succ \omega_{i+1}$, where ω represents each of the alphabets in M.

(ii) Elements with different letterings are incomparable.

(iii) Elements with same lettering and index are equal.

Consider now the following monotonic non-increasing sequences of elements of M.

$a_1 \succ a_2 = a_2 \succ a_3 \succ a_5 = a_5 \succ a_8,$
$b_1 \succ b_2 \succ b_3 \succ b_4 \succ b_5 = b_5 \succ b_6 \succ b_7,$
$c_2 \succ c_3 \succ c_4 \succ c_5 = c_5,$
$d_4 = d_4 = d_4 = d_4 \succ d_7 = d_7 \succ d_9,$
$e_3,$
$f_3 = f_3,$
$g_1 \succ g_6 \succ g_7,$
$h_1 \succ h_4.$

Let F_i be the set containing all occurrences of the i^{th} object from each monotonic non-increasing sequence of elements of M as follows:

$F_1 = [a_1, b_1, c_2, d_4, d_4, d_4, d_4, e_3, f_3, f_3, g_1, h_1],$
$F_2 = [a_2, a_2, b_2, c_3, d_7, d_7, g_6, h_4],$
$F_3 = [a_3, b_3, c_4, d_9, g_7],$
$F_4 = [a_5, a_5, b_4, c_5, c_5],$
$F_5 = [a_8, b_5, b_5],$
$F_6 = [b_6],$
$F_7 = [b_7].$

The elements of F_i for each i are unordered (see Definition 3). Consider now the following basic definition of ordering on $[F_i]$, a class of multisets with unordered elements.

Definition 7. *The Multiset ordering $\gg_{\mathcal{R}}$ defined on the submultiset-based grid of a multiset.*

Let E be a base set of unordered multisets G and H. The relation $G \gg_{\mathcal{R}} H$

holds if and only if $G \neq H$ and

(i) $G(x) \nless H(x)$ for any x in E.

(ii) $x \nless y$ for any $x \in G$ and for any $y \in H$ such that $x \neg\# y$.

The following result immediately comes to mind.

Lemma 2. *The collection $[F_i]$ is a decreasing sequence of submultisets of M with respect to $\ggcurly_{\mathcal{R}}$.*

Proof. In fact, only the second property of Definition 4 suffices. This is obvious since the elements of F_i are strictly less than the elements of F_{i-1} with respect to $\ggcurly_{\mathcal{R}}$ for all $i > 1$. $\square$

Lemma 3. *The ordering $\llcurly_{HO}$ is stronger than the ordering $\llcurly_{\mathcal{R}}$.*

Proof. Suppose $G \ggcurly_{\mathcal{R}} H$ holds where G and H are multisets whose elements are from a base set E. Consider the properties of $\ggcurly_{\mathcal{R}}$ on G and H. Property (i) implies $G(x) \geqslant H(x)\ \forall x \in E$. Since $G \neq H$, the relation $G(x) = H(x)\ \forall x \in E$ does not hold, which implies $G \ggcurly_{HO} H$. Property (ii) implies $x \succcurlyeq y$ for any $x \in G$ and for any $y \in H$ such that $x \neg\# y$. Again since $G \neq H$ holds, then $x \neq y\ \forall x \in G$ and $\forall y \in H$ does not hold. This also implies $G \ggcurly_{HO} H$. Next, we show that there exist multisets which are ordered by $\ggcurly_{HO}$ but not by $\ggcurly_{\mathcal{R}}$. Consider the simple multisets $[5]$ and $[4, 4]$. The relation $[5] \ggcurly_{HO} [4, 4]$ is obvious, whereas the two multisets are incomparable under $\ggcurly_{\mathcal{R}}$. $\square$

The above theorem makes it clear that $\llcurly_{HO}$ is stronger than $\llcurly_{R}$. However, $\llcurly_{\mathcal{R}}$ sufficiently orders $[F_i]$ and will be used as the reference ordering for a given multiset. The ordering $\llcurly_{HO}$ will be used when the references involves at least two multisets. Note that $[F_i]$ is a permutation (an ordered sequence of elements with repetition allowed). The number of submultisets of M in $[F_i]$ equals the highest number of comparably unequal elements of M. In this illustration, the highest number of comparably unequal elements of M is the number of elements of $[b_i]$. This sequence may not be unique. We denote this number by α_M. The collection $[F_i]$ is called *the submultiset-based grid* of M and each member of $[F_i]$ is called a *submultiset-based grid reference of* M (or simply, a *reference* of M or M-*reference*). Below is a formal definition of the concept.

Definition 8

Let $\preccurlyeq$ be a partial order defined on a set E and let M be a multiset of cardinality n over E. The permutation $[M_i]$ of submultisets $M_1,\ M_2,\ \ldots M_m$ of M is called the *submultiset-based grid* of M if the following properties are satisfied:

(i) $M(x) = M_i(x)\ \ \forall x \in E, \forall i$ (*Whole-submultiset* property. See [6] for the definition of a 'whole submultiset' of a multiset). From (1) it follows that M_i's are sets.

(ii) $\forall (x, y, i) [x \in M_i, \ y \in M_i] \implies [(x = y) \ \text{or} \ (x \# y)]$ (*Inordering* property). In other words, the elements of M_i are unordered.

(iii) $\forall(i < j)[x \in M_j] \implies \exists y(y \in M_i) \ [x \prec y]$ (*Strict order* property).

If such properties exist, we say that the submultiset-based reference M_j of M precedes the submultiset-based reference M_i of M, and we denote this by $M_j \lll_{\mathcal{R}} M_i$. Also, the *set* M_i is non-empty for all i. If no two elements of M are one strictly less than the other, then all the elements are members of the only reference available, which is M. We say that the relation $\lll_{\mathcal{R}}$ is the *submultiset-based reference ordering* on the grid of the multiset M over E. Thus, the submultiset-based grid of M is the monotonic decreasing sequences of submultisets of elements of M, in which the i^{th} submultiset contains the i^{th} element from each of all the longest possible monotonic non-increasing sequences of comparably unequal objects of M. See submultiset-based partition and multiset ordering in [3] for further details on the original variant of the construct.

4. Submultiset-based difference grid

Definition 9

Let $[M_i]$ and $[N_j]$ be the submultiset-based grids of the multisets M and N for $i = 1, \ 2, \ldots, m$ and $j = 1, \ 2, \ldots, n$, respectively. Let $p = \max\{m, n\}$ and let $k = 1, \ 2, \ldots, p$. We construct the *submultiset-based difference grid* $[M_k, \ N_k]$ of M and N as follows:

(i) $M_p \neq \emptyset$ or $N_p \neq \emptyset$,

(ii) If $M_k \neq \emptyset$ and $N_k \neq \emptyset$ for a given k then $M_{k-1} \neq \emptyset$ and $N_{k-1} \neq \emptyset$.

The submultisets M_k and N_k are the submultiset-based references of M and N, respectively, in the grid $[M_k, \ N_k]$ of M and N.

If $m < n$ then $p = n$ and M_k is empty for $k = m + 1, \ m + 2, \ldots, \ n$, and if $n < m$ then $p = m$ and N_k is empty for $k = n + 1, \ n + 2, \ldots, \ m$. Unlike the references in the grid of a multiset, the references in a difference grid of two multisets are empty up to the number of references with which the grid with more references exceeds the grid with fewer references. Thus, the difference grid of two multisets is a collection of all the references from the individual grids of the multisets, where some references are empty in the difference grid up to the number of references with which the grid with more references exceeds the grid with fewer references.

Definition 10

Let $[M_k, \ N_k]$ be the difference grid of two multisets M and N. A property p of references is said to be *pair-wise* if and only if p is *attributed to* or *connects* any two references M_i and N_j for $i = j$. For instance, if p stands for non-empty, and M_i and N_j are non-empty for $i = j$ then M_i and N_j are pair-wise non-empty; if p stands for disjoint, and M_i and N_j are disjoint for $i = j$ then M_i and N_j are pair-wise disjoint. We now prove the theorem that follows.

Theorem 3 (Submultiset-based pair-wise equality theorem for multisets.) *Let M and N be multisets. $M = N$ if and only if $M_i = N_j$ for all $i = j$ where M_i and N_j are the respective submultiset-based references of M and N in the difference grid $[M_k,\ N_k]$ of M and N.*

Proof. Let M and N be multisets over a domain set E and suppose $M_i = N_j$ for all $i = j$ where M_i and N_j are the submultiset-based references of M and N, respectively. By Definition 4, $M_i\,(x) = N_j\,(x)$ for all $x \in E$. By Property (i) of Definition 8 (whole-submultiset property), M_i is the only reference of M containing all occurrences of x in M. Similarly, N_j is the only reference of N containing all occurrences of x in N. Thus, $M_i\,(x) = M(x)$ and $N_j\,(x) = N\,(x)$. It follows that $M(x) = N(x)\ \forall x \in E$. Therefore, $M = N$.

Conversely, let $M = N$ where M and N are multisets over the domain set E. Let M_i and N_j be the submultiset-based references of M and N, respectively. By Definition 4, $M\,(x) = N(x)$ for all $x \in E$. We claim $M_i = N_j\ \forall i = j$. Suppose the contrary, and let l_0 be the smallest integer for which the claim is not true. Either $M_{l_0} \lll_{HO} N_{l_0}$ or $M_{l_0} \ggg_{HO} N_{l_0}$ or $M_{l_0} \# N_{l_0}$. If $M_{l_0} \lll_{HO} N_{l_0}$, by Definition 6, $M_{l_0} \neq N_{l_0}$ holds. Thus, $\exists a \in E$ such that $M_{l_0}(a) \neq N_{l_0}(a)$. By the whole-submultiset property, $M(a) \neq N(a)$. This contradicts $M = N$. Similarly, if $M_{l_0} \lll_{HO} N_{l_0}$, we get a contradiction of $M = N$. If $M_{l_0} \# N_{l_0}$, then by Lemma 1 there exist incomparable elements u and v of E such that $M_{l_0}(u) > N_{l_0}(u)$ and $N_{l_0}(v) > M_{l_0}(v)$. It follows by the whole-submultiset property that $M(u) > N(u)$ and $N(v) > M(v)$. Again, this is a contradiction of $M = N$. Therefore, $M_i = N_j\ \forall i = j$.

5. Grid approach to the Jouannaud-Lescanne submultiset-based multiset ordering

Definition 11

Let M and N be multisets. $M \lll N$ if and only if the following property is satisfied:

If $M_i = N_i$ for all i such that $M_i \neq \varnothing$ and $N_i \neq \varnothing$ then $M_j = \varnothing\ \forall j > i$; otherwise if $\exists i$ such that $M_i \neg\lll_{HO} N_i$ then $\exists j$ with $j < i$ such that $M_j \lll_{HO} N_j$, for all non-empty references M_i, N_i and N_j in the submultiset-based difference grid $[M_k,\ N_k]$ of M and N.

Theorem 4 *Definition 11 is equivalent to the Jouannaud-Lescanne submultiset-based multiset ordering.*

Proof. Let $\lll_{JL}$ be the Jouannaud-Lescanne submultiset-based multiset ordering (defined using *submultiset-based partition*) and let $\lll_{PS}$ be the multiset ordering in Definition 11. Suppose M and N are multisets such that $M \lll_{PS} N$.

We first show that $M \neq N$. Consider the difference grid $[M_k,\ N_k]$ of M and N. By Property (i) of Definition 9, we have $M_p \neq \varnothing$ or $N_p \neq \varnothing$. If either M_p or N_p is non-empty then $M \neq N$. If both M_p and N_p are non-empty then by the 'if' part of Definition 11 or by the contrapositive of the 'if otherwise' part of Definition 11, $M_k \neq N_k$ for some k. Therefore, by Theorem 3, $M \neq N$.

The monotonic non-increasing sequence of all the non-empty references in the grid of M from the difference grid $[M_k,\ N_k]$ is a submultiset-based partition $\overline{M}$ of M in lexicographic order, each submultiset containing only unordered elements. Conditions (i), (ii) and (iii) of the submultiset-based multiset ordering (Definition 11) coincide with those of $\lll_{JL}$.

Let $M_i = N_i$ for all i such that $M_i \neq \varnothing$ and $N_i \neq \varnothing$ imply $M_i = \varnothing$. Then the number of sets in the partition of N is greater than the number of sets in the partition of M. Thus N is greater than M by the lexicographic ordering of $\lll_{JL}$.

If $N_i \lll_{HO} M_i$ implies that $\exists j$ with $j < i$ such that $M_j \lll_{HO} N_j$, then again the partition of N is greater than the partition of M by the lexicographic ordering of $\lll_{JL}$.

Conversely, suppose $M \lll_{JL} N$. Let p and q be the number of sets in the partitions of M and N respectively. Lexicographical extension of the ordering entails the following:

(i) for all $i = j$ implies $p < q$.

(ii) there exists i such that $N_i \lll_{HO} M_i$ implies there exists j where $j < i$ such that $M_j \lll_{HO} N_j$.

In both cases, if we introduce difference grid then M_i and N_i are the references of M and N. Consider the repeated empty set $\varnothing, \varnothing, \ldots, \varnothing$ ($j - i$ many times). Then $[M_i, \varnothing, \varnothing, \ldots, \varnothing\ ; N_j]$ is the difference grid of M and N. The above theorem reveals that the Jouannaud-Lescanne set-based multiset ordering and the grid-based multiset ordering are not one stronger than the other. The two definitions are, in fact, the same. We emphasize here what we mean by the strength of a multiset over another. Generally, an ordering $\prec_1$ is *stronger than* or is *more powerful than* an ordering $\prec_2$ if and only if $\prec_2$ implies $\prec_1$ but not $\prec_1$ implying $\prec_2$. Though, the grid-based ordering (Definition 11) and the set-based multiset ordering are equivalent, the choice of the former over the latter lies in its flexibility.

Theorem 5 *Definition 11 is stronger than the Huet-Oppen multiset ordering*

Proof. Let $M \lll_{HO} N$ hold. Let i_0 be such that $[M_{i_0}; N_{i_0}]$ is the first pairwise unequal references in the difference grid of M and N. We claim $N_{i_0} \ggg_{HO} M_{i_0}$. Suppose the contrary, and let $M_{i_0}(x) > N_{i_0}(x)$ hold for a given x such that $\nexists y > x$ for which $N_{i_0}(y) > M_{i_0}(y)$. By the whole submultiset property, $M(x) > N(x)$ holds and $\nexists y > x$ such that $N(y) > M(y)$ This contradicts $M \lll_{HO} N$. Therefore the Definition 11 is stronger than the Huet-Oppen multiset ordering.

Next, we show that the converse is not always true. Consider the multisets $M = \{a, 1, 2\}$, $N = \{a, a, 2\}$. It is easy to see that $M \lll_{PS} N$ since the submultiset-based reference $\{a, a, 2\}$ in the grid of N is greater than either of the two references $\{a, 2\}$ and $\{1\}$ in the grid of M. However, M and N are incomparable using $\lll_{HO}$ since no element in N greater than all the elements in M whose multiplicity in N is greater than its multiplicity in M and vice versa.

Theorem 6 *Definition 11 is equivalent to the Jouannaud-Lescanne submultiset-based multiset ordering.*

Proof. The article [3] contains a proof of the Dershowitz-Manna definition being equivalent to the Huet-Oppen definition. Since by Theorem 5, Definition 11 is stronger than the Huet-Oppen definition, it follows by transitivity that Definition 11 is stronger than the Dershowitz-Manna definition.

Theorem 7 Let $\lll_{PS}$ be th ordering in Definition 11. *Let $\mathfrak{M}(S)$ be the set of all finite multisets on S. Then $\lll_{\mathcal{PS}}$ is well-founded on $\mathfrak{M}(S)$ if and only if $\prec$ is well-founded on S.*

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Proof. The article [1] contains a proof of the well-foundedness of $\lll_{\mathcal{JL}}$, since $\lll_{\mathcal{PS}}$ is equivalent to $\lll_{\mathcal{JL}}$ then a proof of the well-foundedness of $\lll_{\mathcal{JL}}$ is also a proof of the well-foundedness of $\lll_{\mathcal{PS}}$.

6. A direction for future research

The redefined (using the grid approach) set-based multiset ordering appeared in [5]. The construct has proved to be elegant being useful in proving several results. The submultiset counterparts of some of the results have been presented in this paper. It is easy to see (see [3]) that the two definitions are incomparable. The criteria for which the set-based multiset ordering implies the submultiset ordering and vice versa immediately comes to mind.

7. Conclusion

Similar to their definition of set-based multiset ordering, the Jouannaud-Lescanne definition and the grid-based definition of submultiset-based multiset ordering are equivalent. Moreover, in view of Theorems 3 and 5, the grid-based definition seems to have a greater potential in the implementation of the ordering.

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