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The Modified Galerkin Method for Solving the Helmholtz Equation for Low Frequencies on Planet Mars

Yajni Warnapala[1*], Hy Dinh[2] and Jill Resh[3]

[1]Roger Williams University
Bristol, RI 02809

[2]Roger Williams University
Bristol, RI 02809

[3]Roger Williams University
Bristol, RI 02809

[*]*Corresponding author's email: ywarnapala [AT] rwu.edu*

ABSTRACT—*The objective of this paper is to investigate numerical solutions of several boundary value problems for the Helmholtz equation for two smooth surfaces. The superellipsoid is a shapethat is controlled by two parameters. There are some numerical issues in this type of an analysis;any integration method is affected by the wave number k, because of the oscillatory behavior ofthe fundamental solution. The Biconcave Disk is a closed, simply connected bounded shapemodified from a sphere where the two sides concave toward the center, mapped by a sine curve. This project was funded by NASA RI Space Grant and the NASA EPSCoR Grant for testingof boundary conditions for these shapes. One practical value of all these computations can be a shape for the part of the space shuttle that might one day land on planet Mars. Theatmospheric condition on Mars is conducive for small atmospheric wave numbers or lowfrequencies. We significantly reduced the number of terms in the infinite series needed tomodify the original integral equation and used the Green's theorem to solve the integralequation on the boundary of the surface.*

Keywords—Helmholtz Equation, Galerkin Method, Biconcave Disk

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1. INTRODUCTION

The objective of this manuscript is to solve boundary value problems for the Helmholtz equation given by

$$\Delta u+k^2u=0,\ A=Im k\geq 0, \qquad (1)$$

where *k* is the wave number. To overcome the non-uniqueness problem arising in integral equations for the exterior boundary-value problems for the Helmholtz's equation, Jones [5] suggested adding a series of outgoing waves to the free-space fundamental solution. Here we use Jones' modified integral equation approach, where we solve exterior boundary value problems for the modified integral equation. In this paper we looked specifically at the Super ellipsoid and the Biconcave Disk (blood cell) regions. To date there are no numerical results for the Biconcave Disk, but there are results available for the Helmholtz equation,Superellipsoid with the Dirichlet and Robin boundary conditions [13].The Helmholtz Equation, also known as the wave equation, emerges when the topics of electromagnetism and radiation are discussed. It consists of a combination of partial differential equations that investigates how an object reacts towards incoming waves from all directions. In reality, most objects are exposed to various types of waves. However, the reactions are neglected for being insignificant and also not observable by the naked eye. In this paper, we focus on the acoustic aspects of the Helmholtz Equation when theobject is in outer space. The surfaces should be smooth and simply connected. The HelmholtzEquation was solved analytically via separation of variables for the circular membrane by AlfredClebach[3] and for the elliptical membrane by Emile Mathieu [10] under the condition that the wave particles move in predictable straight lines. There is no exact solution for the waveequation if the movement is more complicated. Therefore, the solutions are often approximatedthrough numerical methods. The first step for any numerical method is to discretize the problem; the common techniques are the finite difference method -- for simple shapes and the finite elementmethod -- for more complicated geometries. They share the general idea of dividing the objectinto smaller sub-regions. We utilized the Modified Galerkin Method -- finite element method for multi-dimensional space with non-congruent grids. However, since the fundamental solution, has a weak singularity at points near the boundary, we had to enhance the method by adding an infinite series to reduce the discontinuity effects.

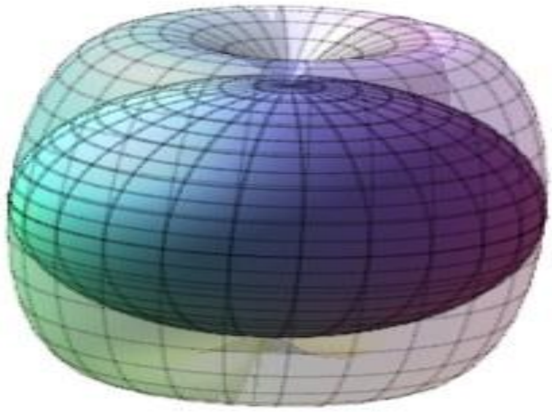

Figure 1: Biconcave Disk (Blood Cell) with λ=0.4 and A=1, B=1, C=1.

2. DEFINITIONS

The Superellipsoid is defined by:

$$x = A\sin^{n}(\varphi)\cos(\theta)$$
$$y = B\sin^{n}(\varphi)\sin(\theta)$$
$$z = C\cos(\varphi)$$

n ranges from [0.5 to 1.8]. The cross sectional superellipsoidal shape varies from a pinched-in diamond to a diamond and expands towards a square-like shape. All the deviations could be obtained by changing the *n*-value (a sphere has *n*=1 and the larger *n* becomes, the closer to a square the cross sectional shape gets). We limited the range of the superellipsoid so that it does not have any sharp edges and is not a perfect sphere.

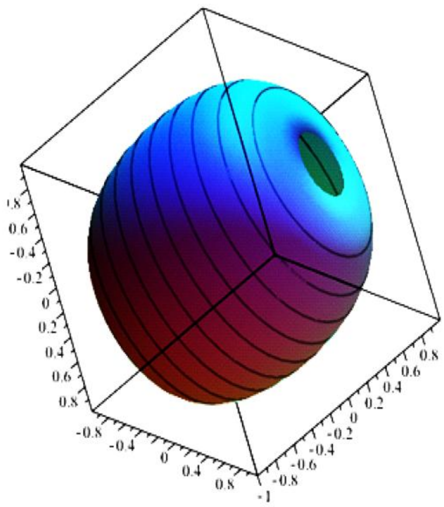

Figure 2: The Superellipsoid for *n*= 1.4.

The Biconcave Disk is defined by

$$x = A\sin(\varphi)\cos(\theta)$$
$$y = B\sin(\varphi)\sin(\theta)$$
$$z = C\left(\left(1-\frac{\lambda}{2}\right)+\sin(\varphi)\right)\cos(\varphi)$$

where $0<\lambda<1$, and *A, B,* and *C* are constants, with $0\leq\theta<2\pi$ and $0\leq\phi<\pi$. Depending on the coefficients *A* and *B,* the symmetry of the shape will vary, which will affect the drag coefficient, which in turn will have an impact on the orientation of the space craft while landing and taking off. The formula for the Biconcave Disc was obtained by running tests of a few different *z*-values that map the concavity using different sine and cosine curves. The $z = C\left(\left(1-\frac{\lambda}{2}\right)+\sin(\varphi)\right)\cos(\varphi)$ gave the best convergence.

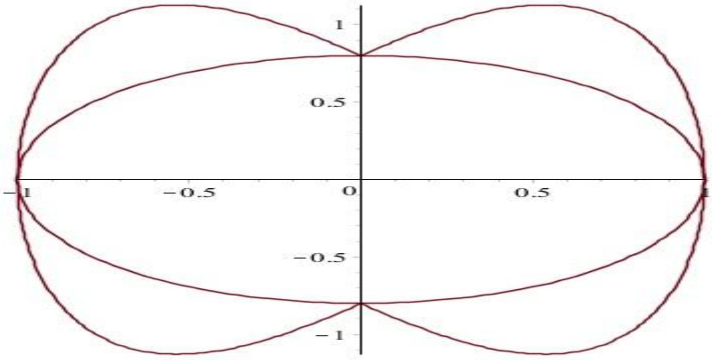

Figure 3: Approximating the radius of the Biconcave Disk by the ellipsoid.

It is necessary to develop a method which is uniquely solvable for all frequencies k which is a challenge. In the case of planet Mars small$k$ values are desirable because of atmospheric conditions on the planet. Let S be a closed bounded surface in $\Re^3$ and assume it belongs to the class of C^2. Let D_-, D_+, denote the interior and exterior of the Superellipsoid or Biconcave Disk (boundary) respectively. We use the Green's theorem as the background for the problem. The exterior Neumann problem for the Helmholtz's equation is given by

$$\Delta u(A)+k^2u(A)=0,\ A=(x,y,z)\in D_+,\ Imk\geq 0$$
$$\frac{\partial u(p)}{\partial \nu_p} = f(p),\ p \in S \ (2) \tag{2}$$

with f a given function and $u(p)$ satisfying the Sommerfeld radiation condition.

2.1 Framework of the Neumann Boundary Value Problem

The exterior Neumann problem will be written as an integral equation. We represented the solution as a modified single layer potential, based on the modified fundamental solution [1, 2, 4].

$$u(A) = \int_S u(q)\left(\frac{e^{ikr}}{4\pi r} + \chi(A,q)\right)d\sigma_q \text{ with} A\in D_+ \text{where } r=|A-q| \tag{3}$$

The series of radiating waves is given by

$$\chi(A,q) = ik\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}h_n^{(1)}(k|A|)Y_n^m\left(\frac{A}{|A|}\right)h_n^{(1)}(k|q|)\overline{Y_n^m}\left(\frac{q}{|q|}\right) \tag{4}$$

This addition of the infinite series to the fundamental solution is in order to remove the singularity that occurs when $A=q$. As in [4], here we assume that D_-(Superellipsoid or the Biconcave Disk) to be a connected domain containing the origin and we choose a ball B or an ellipsoid E of approximate radius $R$and center at the origin such that $\overline{B} \subset D_-$ or $\overline{E} \subset D_-$. On the coefficients a_{nm}[7,9]we imposed the condition that the series $\chi(p,q)$ is uniformly convergent in p and in q in any region $|p|,|q|\geq R+\varepsilon, \varepsilon>0$, and that the series can be two times differentiated term by term with respect to any of the variables with the resulting series being uniformly convergent. We also assumed that the series χ is a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition for $|p|,|q|>R$. By letting A tend to a point $p\in S$, we obtain the following integral equation based on the Fredholm equations of the second kind.

$$-2\pi\mu(p) + \int_S \mu(q)\frac{\partial\left(\frac{-e^{ikr}qp}{r} - 4\pi\chi(p,q)\right)}{\partial\nu_q}d\sigma_q = -4\pi f(p),\ p \in S \tag{5}$$

By the assumptions on the series $\chi(p,q)$ the kernel $\dfrac{\partial\chi(p,q)}{\partial\nu_q}$ is continuous on $S \times S$, and hence K is compact from $C(S)$ to $C(S)$and $L^2(S)$ to $L^2(S)$.

Kleinman and Roach [9] gave an explicit form of the coefficient a_{nm} that minimizes the upper bound on the spectral radius where they are restricted to surfaces $|x|=R+\varepsilon F(\theta,\varphi)$ with θ ranging from zero to π and φ ranging from zero to 2π. If

B is the exterior of a sphere radius R with center at the origin then the optimal coefficient for the boundary problem was given by

$$a_{nm} = -\frac{1}{2}\left(\frac{j_n(kR)}{h_n^{(1)}(kR)} + \frac{j_n'(kR)}{h_n^{(1)'}(kR)}\right) + O(\varepsilon) \text{ for } n = 0, 1, 2 \ldots \text{ and } m = -n, \ldots n. \quad (6)$$

This choice of the coefficient minimizes the condition number. We used the same coefficient choice for the ellipsoid with approximate radius R.

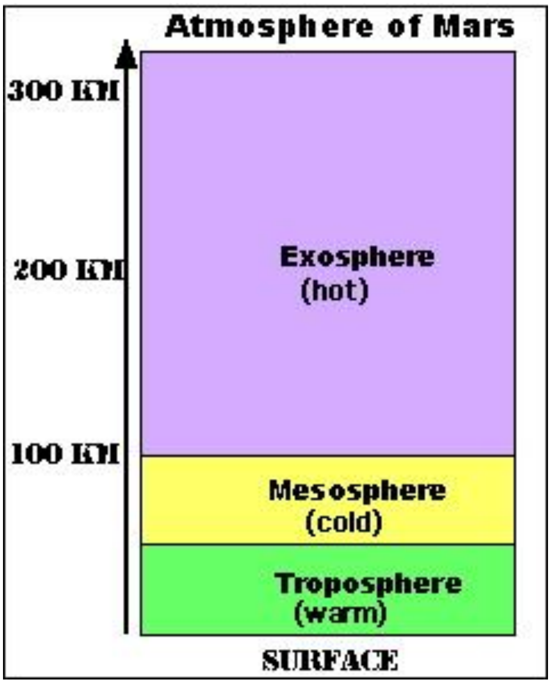

Figure 4: The three atmospheric layers on planet Mars, 50km Troposphere, 100km Mesosphere and 300km Exosphere.

3. PROPERTIES OF THE INTEGRAL OPERATOR *K*

The series χ is a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition for $|x|,|y|>R$, when $B=\{x:|x|\leq R\}$ or $E\approx\{x: |x| \leq R\}$ is contained in D. As the series $\chi(p,q)$ is infinitely differentiable with respect to any of the variables p,q. Furthermore it is easy to see that if μ is bounded and integrable and $S\in C^1$ then $\int_S \frac{\partial\chi(p,q)}{\partial\nu_q}\mu(q)d\sigma_q \in C^1(S)$.

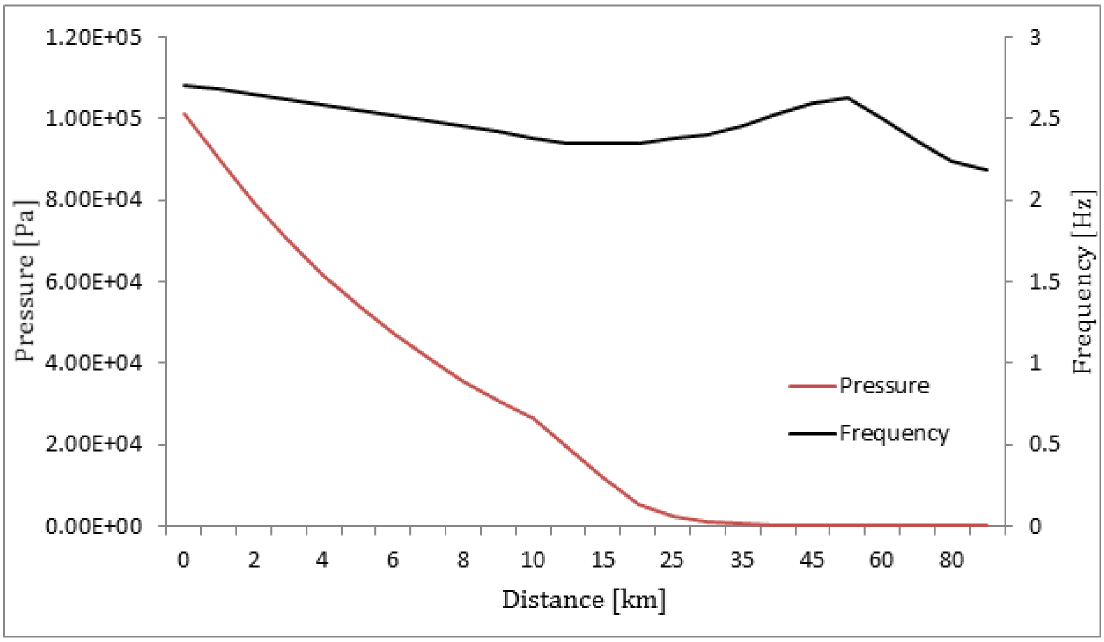

Figure 5: As the pressure varies the frequencies stay somewhat constant on planet Mars.

4. THE THEORETICAL FRAMEWORK

By converting to a new integral equation defined on the unit sphere we could apply theModified Galerkin Method to this new equation, using spherical polynomials to define the approximating subspaces. The new equation over U,

$$-2\pi\hat{\mu}+\hat{K}\hat{\mu} = -4\pi\hat{f}, ,\hat{f} \in C(U) \tag{7}$$

The notation "^" denotes the change of variable from S to U. Galerkin's method for solving (7) for the Neumann boundary condition is given by

$$(-2\pi+P_N\ \hat{K})\hat{\mu}_{\hat{N}} = -4\pi\ P_N\ \hat{f} \tag{8}$$

The solution given by $\hat{\mu}_{\hat{N}} = \sum_{j=1}^{d}\alpha_j h_j$

$$-2\pi\alpha(h_i,h_i)+\sum_{j=1}^{d}\alpha_j(\hat{K}h_j,h_i) = -4\pi(\hat{f},h_i), i = 1\ldots d \tag{9}$$

The convergence of μ_N to μin $L^2(S)$ is straightforward. We know from previous literature that $P_N\hat{\mu} \to \hat{\mu}$ for all $\hat{\mu} \in L^2(U)$. Combining the fact that $\|f - p_N\| \le \frac{c_l H_{l,\lambda}(f)}{N^{l+\lambda}}$, $N \ge 0$ where $H_{l,\lambda}(f)$ is uniform over l^{th}order derivativesand p_n is the sequence of spherical polynomials, and the fact that $\left\|\hat{K} - P_n\hat{K}\right\| = \sup_{\psi\in F}\|(I - P_n)\psi\|_\infty$ with $F = \left\{\hat{K}\psi / \|\psi\|_\infty \le 1, \psi \in C(U)\right\}$ we obtained $\left\|\hat{K} - P_n\hat{K}\right\| \le \frac{C}{N^{\lambda^1 - 1/2}}$ where $\lambda' > \frac{1}{2}$ and $\|P_n\| = O(\sqrt{N})$. Given μ_N an approximate solution, we defined the approximate solution μ_N by

$$\mu_N(A) = \int_S \mu_N(q)\frac{\partial}{\partial\upsilon_q}\left(\frac{e^{ikr_{qA}}}{4\pi r_{qA}} + \chi(A,q)\right)d\sigma_q, A \in D_+ \tag{10}$$

The coefficients $(\hat{K}h_j, h_i)$ are fourfold integrals with a singular integrand. Because the Galerkin coefficients $(\hat{K}h_j, h_i)$ depends only on the surface S, we calculated them separately for $N \le N_{\max}$. To decrease the effect of the singularity in computing $\hat{K}h_j(\hat{p})$ we did the following manipulation $\left(\frac{e^{ikr_{qp-1}}}{r_{qp}} + \chi(q,p)\right) + \frac{e^{ikr_{qp}}}{r_{qp}}$.

5. NUMERICAL RESULTS FOR THE SUPERELLIPSOID AND THE BICONCAVE DISK

For the true solution we used the fundamental solution of the Helmholtz equation, $u(x, y, z) = \frac{e^{ikr}}{r}$.

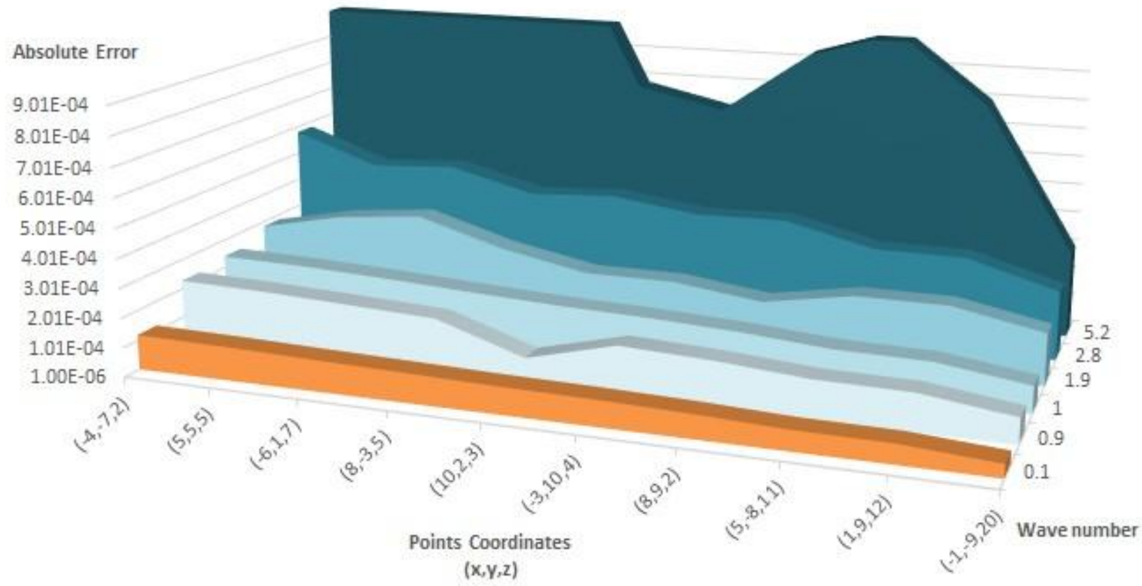

Figure 6: The convergence results are for 8 exterior nodes, 16 interior nodes and 5 terms from the infinite series.

From the above graph, we see that for the points away from the boundarythere is much greater accuracy than for points near the boundary. This is because the integrand is more singular at points near the boundary. Also the convergence results are much better for small values of *k*.

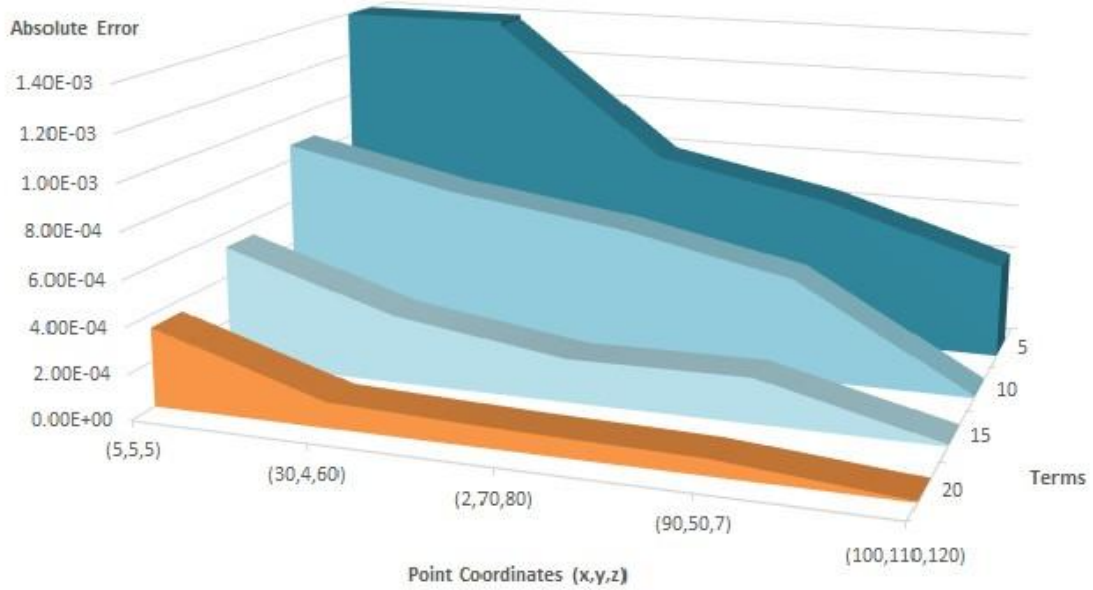

Figure 7: The absolute error decreases when 20 terms are added from the infinite series.

From the above graph we see that increasing the number of terms from the infinite seriesgives a better result. This is due to the following fact: the kernel function involves sin *kr* and cos *kr*. These trigonometric functions are somewhat oscillatory. Therefore in this case we must increase the impact of the series in order to minimize the singularity occurring at the points near the boundary to achieve the same accuracy. We chose more interior nodes because the integrand of $(h_i, \hat{K}h_j)$ is smoother than the integrand of $\hat{K}h_j$.

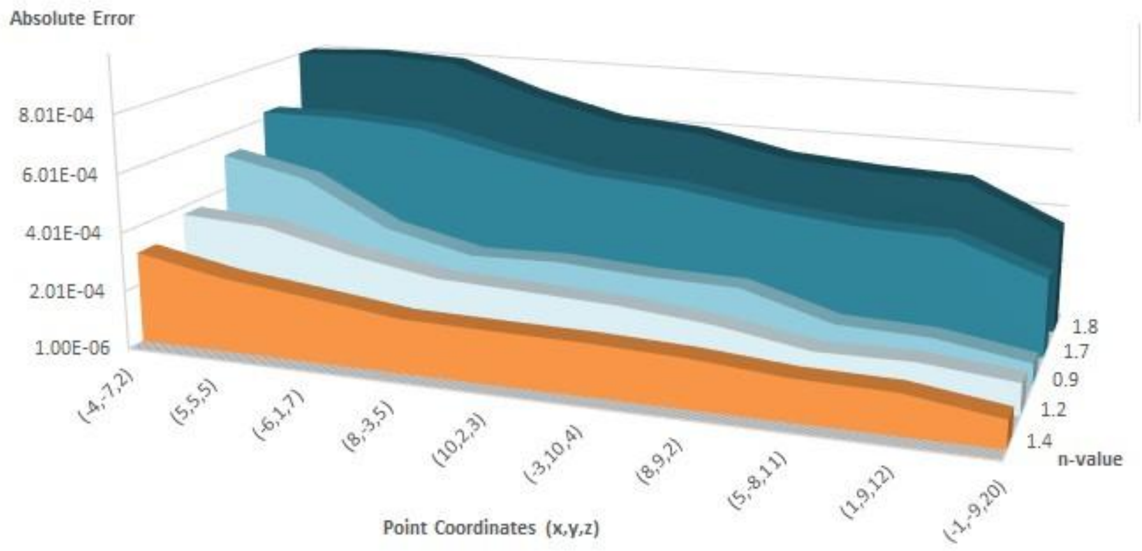

Figure 8: The best convergence results were obtained when *n* was 1.4.

The above results were obtained for the Superellipsoid.

Table 1: Numerical results obtained for the Biconcave Disk

z-coordinate	Absolute Error
$((1-\frac{\lambda}{2})-\frac{\lambda}{2})\cos(\varphi)$	1.910D-05
$((1-\frac{\lambda}{2})-\cos(\varphi))\cos(\varphi)$	2.764D-04
$((1-\frac{\lambda}{2})-\sin(\varphi))\cos(\varphi)$	2.049D-04
$((1-\frac{\lambda}{2})-\sin(2\varphi))\cos(\varphi)$	3.134D-04
$((1-\frac{\lambda}{2})-\cos(2\varphi))\cos(\varphi)$	8.494D-05
$((1-\frac{\lambda}{2})+\cos(\varphi))\cos(\varphi)$	2.033D-04
$((1-\frac{\lambda}{2})+\sin(\varphi))\cos(\varphi)$	9.295D-07
$((1-\frac{\lambda}{2})+\sin(2\varphi))\cos(\varphi)$	2.197D-04
$((1-\frac{\lambda}{2})+\cos(2\varphi))\cos(\varphi)$	4.588D-05
$((1-\frac{\lambda}{2})+\frac{\lambda}{2}\sin(\varphi))\cos(\varphi)$	1.808D-06

In table 1 various z-values were tested that mapped the Biconcave Disc, and the one with the best convergence result was chosen. Using *radius*=1, *N*=0, 5; *k*=1; λ=0.4; number of exterior nodes is 16; number of interior nodes is 8; point (1, 2, 3000). The boundary function is $f(p) = \frac{e^{ikr}}{r}$.

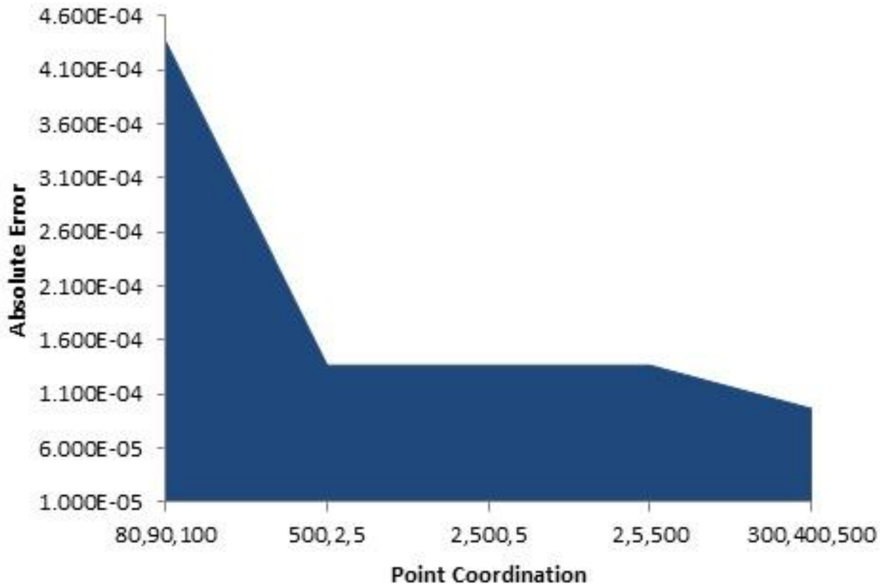

Figure 9: As we go further from the boundary the error decreases.

6. CONCLUSION

The Biconcave Disk and the Superellipsoid both gave similar convergence results. TheBiconcave Disk is a better shape in terms of practicality as its drag coefficient is much less compared to the superellipsoidal shape. In each case the radiuswas assumed to be a constant R=1 which might be a slight drawback. Also we allowedonly a finite number of the coefficients a_{nm} to be differentfrom zero. According to Jones and Ursell [5,11], this is sufficient to ensure uniqueness for themodified integral equations in a finite range of wave numbers k. In our case we were onlyconcerned with a finite range of k. From the above examples, we see that the absolute error isaffected by the boundary S, interior nodes, exterior nodes, k,C, and n.

The role of k is more significant for ill-behaved boundary shapes such as the Biconcave Disk. In order to obtain more accuracy, one must increase the number of integration nodes for calculating the Galerkin coefficients $(\hat{K}h_j, h_i)$. Some of the increased cost comesfrom the complex number calculations, which is an intrinsic property of theHelmholtz equation. Furthermore any integration method is affected by k, due to the oscillatory behavior of the fundamental solution $\frac{e^{ikr}}{r}$. Therefore by increasing the Gaussian quadrature weights we should be able to obtain better results for the Biconcave Disk. This is the task that we will be looking at in our future work.

7. ACKNOWLEDGEMENTS

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