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Sufficient Conditions for the Stability of Trivial Solutions of a Certain Class of Nonlinear Delay Differential Equations

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ABSTRACT---- ***In This paper we study the stability of a trivial solution of certain nonlinear delay differential equations of the form*** $x(t) = F(t_1, x_t) \rightarrow (1.1)$ ***where*** $x_{t_0} = \Phi_0 \rightarrow (1.2)$ ***for us to improves on the existence literature, equation (1.1) was re-written as a perturbation of the linear homogeneous system of the form*** $y'(t) = \sum_j^M = 0 \; A_j(t) y\left(t - \tau_j(t)\right) + \int_{-G}^{0} B(t,\theta)\, y(t+\theta) d\theta \rightarrow (1.3)$ ***and*** $x'(t) = \sum_j^M = 0 \; A_j(t) x\left(t - \tau_j(t)\right) +$ $-G0Bt,\theta\ xt+\theta d\theta\ +ht,xt \rightarrow 1.4$ ***and we use existences an uniqueness theorem of a linear system to establish sufficient conditions that guarantee the stability of the trivial solutions of a certain class of nonlinear delay differential equations. The goal of this paper is to give a simple criterion for the stability of (1.1) when re-written as a perturbation of a linear homogeneous system of the form (1.3) and (1.4).***

Keywords---- Stability, Trivial solution, nonlinear delay differential equations,(DDES) Perturbation, linear and homogeneous system

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1. INTRODUCTION

Delay differential equations (DDE) have been studied extensively for the past 60 years. They have Applications in domains as diverse as engineering, biology and medicine where information transmission and/or response in control system is not instantaneous. For a good introduction to the subject, see [8] works that has been done treats DDES with one or a few discrete delays. A number of Realistic physiological models however include distributed delays and a problem of particular interest is to determine the stability of the steady state solutions. For applications in physiological system see [10,11,11 3,1,2].

The study of machine tool dynamics has led to many problems involving delay differential Equations (DDES) Bakhandran [4]. For example in turning operations, a cutting tool passes over a work Piece many times successively. The forces on the tool depends on the chip thickness which dependents on the tool's current position and its position on previous revolution of the work piece, thus introducing a delay effect. The delay effect of the irregularities can introduce self-sustained oscillations of the tool against the work piece, called regenerative chatter. This phenomenon has been studied by Tlusty and placek[16] and Tobias[117] as early as the 1960's. Therefore, being able to determine the Nature of the bounded delay solutions to differential equations, it is crucial to determining the quality Of the work piece surface finish. Reader refer to obtain literature[3,5,6,7,9,12,14,15].

In this paper will consider the DDE initial value problem

$$x'(t) = F(t_1, x_t) \rightarrow 1.1$$

$$x_{t_0} = \Phi_0 \rightarrow 1.2$$

Where $C_D = C([-r, 0], D), F: (\alpha, \infty) \times C_D \rightarrow R^n, t_0 > \alpha, \Phi_0 \epsilon C_D \; and \; D$ is an open subset of R^n, we will Assume that $\mathrm{F}(t, \varphi)$ is continuous, locally Lipschiz with repect to φ and quasi-bounded on its domain, So that existence of a unique non continual solution to the initial value problem is guaranteed.

In order to study the stability of the trivial solution of the nonlinear DDE of the form (1.1), we write it as

Perturbation of the linear, homogeneous system of the form,

$y'(t) = \sum_{j=0}^{M} Aj(t)y\big(t - \tau j(t)\big) + \int_{-\sigma}^{0} B(t,\theta)\, y(t+\theta)d\theta \to 1.3$

Defining

$h(t,\varphi) = F(t,\varphi) - \sum_{j=0}^{M} Aj(t)\varphi\big(-\tau j(t)\big) - \int_{-\sigma}^{0} B(t,\theta)\, \varphi(\theta)d\theta$, and equation (1.1) can then be

Rewritten as $x'(t) = \sum_{j=0}^{M} Aj(t)x\big(t - \tau j(t)\big) + \int_{-\sigma}^{0} B(t,\theta)\, x(t+\theta)d\theta + h(t_1, x_t) \to 1.4$

2. NOTATIONS AND DEFINITIONS

Definition 2.1; The DDE (1.1) is called autonomous if F is independent of t, i.e. $F = F(x_t)$, otherwise it is called nonautonomous

Definition 2.2. An equilibrium solution of (1.1) is a constant solution $x(t) = x^*\ t \geq \propto -r$ The trivial solution (or zero solution) of (1.1) is the equilibrium solution $x(t) = 0t \geq \propto -r$

Note that equation (1.1) admits the trivial solution only if $0 \epsilon D\ and\ F(t,0) = 0$ for all $t > \propto$ i.e. if $\varphi(\theta) = 0, -r \leq \theta \leq 0 \Rightarrow F(t,\varphi) = 0, t > \propto$

Definition 2.3. The trivial solution of (1.1) is said to be stable at $t_0(t_0 > \alpha)$. If for each $\epsilon > 0$ there is $\delta = \delta(\epsilon, t_0) > 0$ such that whenever $\|\Phi_0\|r < \delta$, the solution (1.1) (1.2) exist On $[t_0 - r, \infty)$ and $\|x(t)\| < \epsilon$ for all $t \geq t_0$. Otherwise the trivial solution is said to be Unstable at t_0

Definition 2.4; The trivial solution of (1.1) is said to uniformly stable on (α, ∞) if it is Stable at each $t_0 > \alpha$ and the number δ is independent of t_0

Definition 2.5; The trivial solution of (1.1) is said to be asymptotically stable at t_0 if It is stable at t_0 and there exists $\delta_1(t_0) > 0$ such that whenever $\|\Phi_0\|r < \delta_1$ the solution Of (1.1) and (1.2) satisfies $\lim_{t\to\infty}\|x(t)\| = 0$

Note that this is sometimes referred to as local asymptotic stability as it only applies to Solutions with initial functions "close enough" to the trivial

3. PRELIMINARY

We begin with a few preliminary results about (1.3) which will enable us to establish the main results for (1.4).

Lemma 3.1; the trivial solution of (1.3) is uniformly stable if and only if there exists a Constant $M_1(\geq 1)$ such that for any $t_0 > \alpha$ and any $\Phi_0 \epsilon\ c$, the solution of (1.3) and (1.2) Satisfies $\|y(t)\| \leq M_1\|\Phi_0\|r$ for all $t \geq t_0 - r.$ (1.5)

Proof; suppose that the trivial solution of (1.3) is uniformly stable and let $\varepsilon = 1$ and $\delta = min(\sigma(1), 1)$. then $\|y(t)\| < 1$ whenever $\|\Phi_0\|r < \delta$ and $t \geq t_0 > \alpha$. Define $M_1 = \frac{1}{5}$clearly (1.5) holds if $\Phi_0 = 0$. Consider an arbitrary $\Phi_0 \neq 0$ then for each $\delta^* \epsilon (0,\delta)$, the solution $\bar{y}$ Of (1.3) with the initial condition $\bar{y}_{t_0} = \delta^*\Phi_0/\|\Phi_0\|r$ satisfies$\|\bar{y}\| < 1$. By linearity the

Solution of (1.3) with the initial condition (1.2) is $y(t) = \|\Phi_0\|r/\delta^*\bar{y}$ and hence satisfies $\|y(t)\| \leq \frac{\|\Phi_0\|r}{\delta^*}$ for all $t \geq t_0$ Now for each fixed $t_0 > \alpha$ and $t \geq t_0$ let $\delta^* \to \delta$, to find $\|y(t)\| \leq \frac{\|\Phi_0\|r}{\delta} = M_1\|\Phi_0\|r$ which implies $M_1 \geq 1$ so this inequality also holds for $t_0 - r \leq t \leq t_0$

Lemma 3.2. (Halanay [1966]).

Let K and P be constants with$0 < p < k$. Let v be a continous nonnegative function on $[\,t_0 - r, \beta]$ satisfying

$v'(t) \leq -kv(t) + p\|v_t\|r\ \ for\ t_0 \leq t < \beta$

then $\|v(t)\| \leq \left\|v_{t_0}\right\|r^{e^{\lambda(t-t_0)}}\ for\ t_0 \leq t \leq \beta.$

Where λ is the (unique) positive solution of λ$= k - pe^{\lambda r}$

Proof; The proof that $\lambda = k - pe^{\lambda r}$ has a unique positive solution requires only simple argumenst from calculus, which is trivial.

Define $w(t) = \|v_{t_0}\| r^{e^{\lambda(t-t_0)}}$ $for\ t_0 - r \le t < \beta$ and $k > 1$ be arbitrary. Note that w

Satisfies $w'(t) = -kw(t) + pw(t-r)\ and\ v(t) < kw(t) for\ t_0 - r \le t \le t_0$. now suppose (for Contradiction) that $v(t) = kw(t)$ for some $t\epsilon(t_0, \beta)$. then, since v and w are continous,

There must exist $t_1 \epsilon (t_0, \beta)\ such\ that\ v(t) < kw(t)\ for\ t_0 - r \le t \le t_1\ and\ v(t_1) = kw(t_1)$.

This could not occur if $v'(t_1) < kw'(t_1)$. But on the other hand,

$v'(t_1) - kv(t_1) + \mathrm{p}\|\mathrm{v}_{t_1}\|\mathrm{r}$

$< -kkw(t_1) + pkw(t-r)$

$= kw'(t_1)$.

This is a contradiction. Thus we conclude that $v(t) < kw(t)\ for\ t_0 \le t < \beta$. Since $k > 1$ is arbitrary, it follows that $and\ v(t) \le w(t)$. This completes the proof.

4. THE MAIN RESULTS

Theorem 4.1; the trivial solution of (1.3) is uniformly asymptotically stable if and only If there exist constants $M \ge 1$ and $k > 0$ such that for every $(t_0, \Phi_0)\epsilon(\alpha, \infty) \times c$, the Solution of (1.3) and (1.2) satisfies.

$\|y(t)\| \le M\|\Phi_0\| r^{e^{-K(t-t_0)}}\ for\ all\ t \ge t_0$ (4.1)

Or, equivalently, $\|y_t\| r \le Me^{kr}\|\Phi_0\| r^{e^{-K(t-t_0)}}\ for\ all\ t \ge t_0$ (4.2)

Proof; assume that the trivial solution of (1.3) is uniformly asymptotically stable. Then It is uniformly stable, and by Lemma 3.1 given $t_0 > \mp\alpha\ and\ \Phi_0 \epsilon\ c$. there exist a constant $M_1 \ge 1$ Such that the solution of (1.3)-(1.2) satisfies $\|y(t)\| \le M_1\|\Phi_0\| r, for\ all\ t \ge t_0 - r$

Now let $\delta_1 > 0$ be as in the definition of uniformly asymptotic stability. Then if we take

$\eta = \frac{\delta_1}{2}$, there exists T> 0. Such that for every $t_0 > \alpha\ and\ \Phi_0 \epsilon\ c\ \ with\ \|\Phi_0\| r < \delta_1 \|y(t)\| < \frac{\delta_1}{2}\ for\ all\ t \ge t_0 + T$. Consider any $\Phi_0 \ne 0$. Then it can be shown using linearity (as in the Proof of lemma 3.1) that for each $\delta_1{}^* \epsilon\ (0, \delta_1)$ the solution of (1.3) and (1.2) satisfies

$\|y(t)\| \le \frac{\delta_1}{2\,\delta_1{}^*}\|\Phi_0\| r\ \ for\ all\ t \ge t_0 + T$. Now for fixed $t_0 > \alpha$ and $t \ge t_0 + T$, let $\delta_1{}^* \to \delta_1$. To

Find $\|y(t)\| \le \frac{1}{2}\|\Phi_0\| r\ \ for\ all\ t \ge t_0 + T$. This implies that

$\|y_t\| r \le \frac{1}{2}\|\Phi_0\| r = \frac{1}{2}\|y_{t_0}\| r\ \ for\ all\ t \ge t_0 + T + r$. Repeated application of this inequality

Gives, for $t_0 + k(T+r) \le t \le t_0 + (k+1)(T+r)$,

$\|y_t\| r \le 2^{-k}\|\Phi_0\| r = \|\Phi_0\| re^{-(In\ 2)k}, k = 1,2, \ldots$

It follows that $\|y_t\| r \le \|\Phi_0\| re^{-(In\ 2)(t-t_0-T-r)/(T+r)} = 2\|\Phi_0\| re^{-(In\ 2)(t-t_0)/(T+r)}$. For

$t \ge t_0 + T + r$. This, combined with the inequality (1.5), gives for all $t \ge t_0$,

$\|y_t\| r \le 2M_1\|\Phi_0\| re^{-(In\ 2)(t-t_0)/(T+r)}$. Taking $M = 2M_1 e^{-Kr}$ and $k = (In2)/(T+r)$ yield the Result. This completes the proof.

Corollary 4.1; if the trivial solution of (1.3) is uniformly asymptotically stable, $\tau j(t)$ is Continuous and $\tau' j(t) \neq 1$, then for each $t > 0 > \alpha$ and each $\xi \in R^n$, The solution (1.3)

And the initial condition.

$$y(t) = \begin{cases} 0 & t_0 - r, \leq t < t_0 \\ \xi & t = t_0 \end{cases} \quad 4.3$$

Satisfies $\|y(t)\| \leq M\|\xi\|e^{-K(t-t_0)}$ For all $t \geq t_0$ Where M and k are the same as in Theorem 4.1

Proof; Let $u(\theta)$ be the unit step function, i.e.

$$u(\theta) = \begin{cases} 0 & -r \leq \theta \leq 0 \\ 1 & \theta = 0 \end{cases}$$

Then the initial condition (4.3) can be written $y_{t_0} = \xi u$. Approximate u by the Continuous functions

$$u(l) = \begin{cases} 0 & -r \leq \theta < -\frac{r}{l} \\ \frac{l}{r}\theta + 1 & -\frac{r}{l} \leq \theta \leq 0 \end{cases}$$

Now let $\hat{y}(t)$ be the solution to (1.3), (4.1), (4.2) and (4.3) and let $y(l)(t)$ be the solution Of (1.3) and the initial condition $y_{t_0} = \xi u(l)$. Consider, for $t \geq t_0$;

$$\hat{y}(t) - y(l)(t) = \int_{t_0}^{t} \left\{ \sum_{j=0}^{M} Aj(s)\left[\hat{y}\left(s - \tau j(s)\right) - y(l)\left(s - \tau j(s)\right)\right] + \int_{\sigma(s)}^{0} B(S,\theta)[\hat{y}(s+\theta) - y(l)(s+\theta)]d\theta \right\}$$

Defining $Kj(t) = \max_{t_0 \leq s \leq t} \left\| \frac{Aj\,(s)}{1 - \tau j\,(s)} \right\|, \qquad j = 0,1, \dots m,$

$K_{m+1}(t) = r \max_{t_0 \leq s \leq t \, -r \leq \theta \leq 0} sup\|B(s,\theta)\|$ and $k(t) = \sum_{j=0}^{m+1} kj(t)$, we find for $t \geq t_0$

$$\|\hat{y}(t) - y(l)(t)\| \leq \sum_{j=0}^{m+1} kj(t) \int_{t_0 - \frac{r}{l}}^{t} \|\hat{y}(s) - y(l)\|ds \leq k(t)\|\xi\| \frac{r}{2l} + k(t) \int_{t_0}^{t} \|\hat{y}(s) - y(l)(s)\|ds.$$

By Generalized Gromwell's Lemma,

$$\|\hat{y}(t) - y_L(t)\| \leq \frac{r}{2l} k(t)\|\xi\|e^{k(t)(t-t_0)} \text{ for } t \geq t_0$$

But, since $\xi u(l) \in C$, we can apply Theorem 4.1 to obtain

$\|\hat{y}(t)\| \leq \|\hat{y}(t) - y_L(t)\| + \|y_L(t)\| \leq \frac{r}{2l} k(t)\|\xi\|e^{k(t)(t-t_0)} + M\|\xi\|e^{-k(t-t_0)}$ for $t \geq t_0$. since this Holds for each r, we can let $r \to \infty$ to obtain the result. This completes the proof.

Theorem 4.2; Let the trivial solution of (1.3) be uniformly asymptotically stable and let M and K be The constants in theorem 4.1. If for some constant $N \epsilon \left(0, \frac{K}{M}\right)$

$\|h(t,\varphi)\| \leq N\|\varphi\|r, for\ all\ (t,\varphi)\epsilon(\alpha,\infty) \times C_D$, Then the trivial solution of (1.4) is also uniformly Asymptotically stable.

Proof; Let $0 < N < \frac{K}{M}$ and let h satisfy the condition given. Let $(t_0, \Phi_0)\epsilon(\alpha,\infty) \times C_D$ be given, and let $x(t)$ be the unique, noncontinuable solution of (1.4)-(1.2) as $x(t) = x_\emptyset + x_h, for\ t_0 - r \leq t < \beta_1$

Where is the solution of (1.4)-(1.2) and

$$x_h(t) = \int_{t_0}^{t} C(t;s)h(s,x_s)ds = \int_{t_0}^{t} z(t;s)ds.$$

Then $\|x(t)\| \leq \|x_\emptyset(t)\| + \int_{t_0}^{t} \|z(t;s)\|ds.$

Since the trivial solution of (1.3) is uniformly asymptotically stable, there exist constants M, $k > 0$ Such that for all $\Phi_0 \epsilon C_D$. $\|x_\Phi(t)\| \leq M\|\Phi_0\| r e^{-k(t-t_0)}$ for $t \geq t_0$

Whish implies that $Z(t;s) = C(t;s)h(s, x_s)$ satifies the initial problem.

$\frac{\partial z}{\partial t} = \sum_{j=1}^{M} Aj(t)\, z(t - \tau j(t); s) + \int_{-\sigma(t)}^{0} B(t,\theta) z(t+\theta; s) d\theta, s \leq t \leq \beta_1$

$Z(t;s) = \begin{pmatrix} 0, & s - r \leq t < s \\ h(s, x_s), & t = s \end{pmatrix}$

It follows from corollary 4.1 and the hypothesis on h that

$\|z(t,s)\| \leq M\|h(s;x_s)\| e^{-k(t-s)} \leq MN\|x_s\| r e^{-k(t-s)}$,

Thus $\|x(t)\| \leq M\|\Phi_0\| r e^{-k(t-t_0)} + \int_{t_0}^{t} MN\|x_s\| r e^{-k(t-s)}\, ds\ for\ t_0 \leq t \leq \beta_1$.Now define

$$v(t) = \begin{pmatrix} M\|\Phi_0\| r & t_0 - r \leq t \leq t_0 \\ M\|\Phi_0\| r e^{-k(t-t_0)} + e^{-kt} \int_{t_0}^{t} MN\|x_s\| r e^{-k(t-s)}\, ds\ for\ t_0 < t < \beta_1 \end{pmatrix}.$$

Then V is continuous and nonnegative, and $\|x(t)\| \leq v(t)$ for $t_0 - r \leq t < \beta_1$. moreover, For $t_0 \leq t \leq \beta_1$ $v'(t) = -kv(t) + MN\|x_t\| r \leq -kv(t) + MN\|v_t\| r$. Since $MN < k$, it follows

From lemma 3.2 that $\|x(t)\| \leq v(t) \leq M\|\Phi_0\| r e^{-\lambda(t-t_0)} for\ t_0 \leq t < \beta_1$, where λ is the

unique positive solution of λ= $k - MNe^{\lambda r}$. Recall that D= $\{\xi \| \xi\| < H\}, for\ some\ H > 0$.

Then provided $M\|\Phi_0\| r < H, \rightarrow$ 4.4. Guarantees $x(t)\epsilon D\ for\ t_0 \leq t \leq \beta_1$.

So $\|x(t)\| \leq M\|\Phi_0\| r e^{-\lambda(t-t_0)} for\ t \geq t_0$. This completes the proof.

Below are useful applications of this theorem.

Example 4.1 (Linear DDE'S as Perturbations of linear ODE'S)

Consider the linear system.

$x'(t) = A_0(t)x(t) + \sum_{j=1}^{M} Aj(t)x\big(t - \tau j(t)\big), \rightarrow$ 4.5

Where each aj is a continuous matrix-valued function, and each τj is continuous with $0 \leq \tau j \leq r\ for\ all\ t > \alpha$. Suppose that the trivial solution of $y'(t) = A_0(t)y(t) \rightarrow$ 4.6 isUniformly asymptotically stable, so that given $t_0 > \alpha$ and $y_0 \epsilon R^n$ there are constants

$M \geq 1\ and\ k > 0$, Such that the solution (4.6) with initial condition $y(t_0) = y_0$ satisfies $\|y(t)\| \leq M\|y_0\| e^{-k(t-t_0)}\ for\ all\ t \geq t_0$. Regading (4.6) as a DDE, this gives

$\|y(t)\| \leq M\|\Phi_0\| r e^{-k(t-t_0)} for\ t \geq t_0$.

$where\ \Phi_0 \epsilon C$ with $\Phi_0(0) = y_0$. it now follows from Theorem 4.2 that the trivial solution of (4.5) is uniformly asymptotically stable provided that there is a number N such that

$\sum_{j=1}^{M} \|Aj(t)\| \leq N < \frac{K}{M}\ \ for\ all\ t > \alpha$.

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