Second Order Tensor Wave Equations for Electron Field

Bulikunzira Sylvestre

University of Rwanda
University Avenue, B.P 117, Butare, Rwanda
Email: S.BULIKUNZIRA [AT] ur.ac.rw

ABSTRACT--- ***In previous works, with the use of Cartan map, spinor Dirac equation for half-spin particle such as electron has been written in tensor form, in the form of non-linear Maxwell's like equations, through two complex isotropic vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$. It has been proved, that the complex vector $\vec{F} = \vec{E} + i\vec{H}$ satisfies non-linear condition $\vec{F}^2 = 0$, equivalent to two conditions for real quantities $\vec{E}^2 - \vec{H}^2 = 0$ and $\vec{E}.\vec{H} = 0$, obtained by equating to zero separately real and imaginary parts of equality $\vec{F}^2 = 0$. It has been proved, that the vectors $\vec{E}$ and $\vec{H}$ have the same properties as those of the strengths $\vec{E}$ and $\vec{H}$, components of electromagnetic tensor $F_{\mu\nu}$. Furthermore, it has been proved, that the solution of these non-linear equations for free particle as well fulfils Maxwell's equations for vacuum (with zero at right side).***

In this work, in order to investigate and to understand these non-linear tensor wave equations for electron, we derived the corresponding second order tensor wave equations.

Keywrods---- Second order, tensor, wave equations, electron field.

---

1. INTRODUCTION

In previous works, spinor Dirac equation for electron has been written in tensor form, in the form of non-linear Maxwell's like equations for two electromagnetic fields $(\vec{E}, \vec{H})$ and $(\vec{E}', \vec{H}')$. It has been proved, that these fields satisfy non-linear conditions $\vec{E}^2 - \vec{H}^2 = 0$ and $\vec{E}.\vec{H} = 0$ and each of the fields $(\vec{E}, \vec{H})$ and $(\vec{E}', \vec{H}')$ satisfies Maxwell's equations for vacuum (with zero at the right side). This enables us to interpret electron as a system of two electromagnetic waves $(\vec{E}, \vec{H})$ and $(\vec{E}', \vec{H}')$, propagating with the phase velocity equal to the velocity of light and the velocity of the particle is equal to the group velocity.

In this work, in order to understand these electron waves, we shall derive the corresponding second order tensor wave equations.

2. RESEARCH METHOD

In this work, we shall derive the second order tensor wave equations for electron. To derive these equations, we shall start with the first order tensor wave equations for electron. Here, we shall use the general mathematical method, based on general theorems of tensor analysis and usually used in derivation of the second order tensor wave equations for electromagnetic field.

3. SECOND ORDER TENSOR WAVE EQUATIONS FOR ELECTRON

In previous works, Dirac equation for electron

$$(\gamma^{\mu}\partial_{\mu} - m)\psi = 0, \tag{1}$$

has been written in tensor form as follows

$$\begin{cases} D_0\vec{F} + v_i(\vec{D}F_i) - i\vec{D} \times \vec{F} = -\frac{m}{\sqrt{2}}\frac{\vec{F}\times\vec{F}'}{(\vec{F}.\vec{F}')^{1/2}} \\ D_0\vec{F}' - v'_i(\vec{D}F'_i) + i\vec{D} \times \vec{F}' = -\frac{m}{\sqrt{2}}\frac{\vec{F}\times\vec{F}'}{(\vec{F}.\vec{F}')^{1/2}} \end{cases}. \tag{2}$$

Where

$$\vec{F} = \vec{E} + i\vec{H}, \tag{3}$$

$$\vec{F}' = \vec{E}' - i\vec{H}', \tag{4}$$

are two complex isotropic vectors, and

$$D_0 = i\frac{\partial}{\partial t}, \qquad \vec{D} = -i\vec{\nabla}, \qquad \vec{v} = \frac{\vec{E}\times\vec{H}}{\vec{E}^2}. \tag{5}$$

Here we use the natural system of units in which c=ℏ=1.

Separating real and imaginary parts in equations (2), we obtain a system of non-linear Maxwell's like equations for strengths $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$

$$\begin{cases} \mathrm{rot}\vec{E} + \frac{\partial\vec{H}}{\partial t} = v_i(\vec{\nabla}H_i) + m\vec{j}_a \\ \mathrm{rot}\vec{H} - \frac{\partial\vec{E}}{\partial t} = -v_i(\vec{\nabla}E_i) + m\vec{j}_v \\ \mathrm{rot}\vec{E}' + \frac{\partial\vec{H}'}{\partial t} = -v'_i(\vec{\nabla}H'_i) - m\vec{j}_a \\ \mathrm{rot}\vec{H}' - \frac{\partial\vec{E}'}{\partial t} = v'_i(\vec{\nabla}E'_i) + m\vec{j}_v \end{cases}. \tag{6}$$

Here

$$\vec{j}_a = \sqrt{2}\frac{(\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}')\cos\varphi/2+(\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E})\sin\varphi/2}{\left[(\vec{E}\vec{E}')^2+(\vec{H}\vec{H}')^2+2(\vec{E}\vec{E}')(\vec{H}\vec{H}')+(\vec{E}\vec{H}')^2+(\vec{E}'\vec{H})^2+2(\vec{E}\vec{H}')(\vec{E}'\vec{H})\right]^{1/4}}, \tag{7}$$

$$\vec{j}_v = -\sqrt{2}\frac{(\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}')\sin\varphi/2+(\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E})\cos\varphi/2}{\left[(\vec{E}\vec{E}')^2+(\vec{H}\vec{H}')^2+2(\vec{E}\vec{E}')(\vec{H}\vec{H}')+(\vec{E}\vec{H}')^2+(\vec{E}'\vec{H})^2+2(\vec{E}\vec{H}')(\vec{E}'\vec{H})\right]^{1/4}}, \tag{8}$$

$$\varphi = \tan^{-1}\frac{\vec{E}'\vec{H}-\vec{E}\vec{H}'}{\vec{E}\vec{E}'+\vec{H}\vec{H}'}. \tag{9}$$

In particular, when $\varphi = 0$, i.e., $\vec{E}//\vec{E}'$ and $\vec{H}//\vec{H}'$, we obtain a simple system

$$\begin{cases} \mathrm{rot}\vec{E} + \frac{\partial\vec{H}}{\partial t} = v_i(\vec{\nabla}H_i) + \sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}} \\ \mathrm{rot}\vec{H} - \frac{\partial\vec{E}}{\partial t} = -v_i(\vec{\nabla}E_i) + \sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}} \\ \mathrm{rot}\vec{E}' + \frac{\partial\vec{H}'}{\partial t} = -v'_i(\vec{\nabla}H'_i) - \sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}} \\ \mathrm{rot}\vec{H}' - \frac{\partial\vec{E}'}{\partial t} = v'_i(\vec{\nabla}E'_i) + \sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}} \end{cases}. \tag{10}$$

Let us derive the second order tensor wave equations satisfied by the fields $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$.

Acting on both sides of the system (10) by rotational operator, we obtain

$$\begin{cases} \vec{\nabla}\times\vec{\nabla}\times\vec{E} + \vec{\nabla}\times\left(\frac{\partial\vec{H}}{\partial t}\right) = \vec{\nabla}\times v_i(\vec{\nabla}H_i) + \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \\ \vec{\nabla}\times\vec{\nabla}\times\vec{H} - \vec{\nabla}\times\left(\frac{\partial\vec{E}}{\partial t}\right) = -\vec{\nabla}\times v_i(\vec{\nabla}E_i) + \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \\ \vec{\nabla}\times\vec{\nabla}\times\vec{E}' + \vec{\nabla}\times\left(\frac{\partial\vec{H}'}{\partial t}\right) = -\vec{\nabla}\times v'_i(\vec{\nabla}H_i) - \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \\ \vec{\nabla}\times\vec{\nabla}\times\vec{H}' - \vec{\nabla}\times\left(\frac{\partial\vec{E}'}{\partial t}\right) = -\vec{\nabla}\times v'_i(\vec{\nabla}E_i) + \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \end{cases}. \tag{11}$$

Using the well known formula of tensor analysis

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\Delta\vec{E} + \vec{\nabla}(\vec{\nabla}\vec{E}), \tag{12}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{H} = -\Delta\vec{H} + \vec{\nabla}(\vec{\nabla}\vec{H}), \tag{13}$$

we find

$$\begin{cases} -\Delta\vec{E} + \vec{\nabla}(\vec{\nabla}\vec{E}) + \vec{\nabla}\times\left(\frac{\partial\vec{H}}{\partial t}\right) = \vec{\nabla}\times v_i(\vec{\nabla}H_i) + \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \\ -\Delta\vec{H} + \vec{\nabla}(\vec{\nabla}\vec{H}) - \vec{\nabla}\times\left(\frac{\partial\vec{E}}{\partial t}\right) = -\vec{\nabla}\times v_i(\vec{\nabla}E_i) + \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \\ -\Delta\vec{E}' + \vec{\nabla}(\vec{\nabla}\vec{E}') + \vec{\nabla}\times\left(\frac{\partial\vec{H}'}{\partial t}\right) = -\vec{\nabla}\times v'_i(\vec{\nabla}H_i) - \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \\ -\Delta\vec{H}' + \vec{\nabla}(\vec{\nabla}\vec{H}') - \vec{\nabla}\times\left(\frac{\partial\vec{E}'}{\partial t}\right) = -\vec{\nabla}\times v'_i(\vec{\nabla}E_i) + \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \end{cases} . \quad (14)$$

Using the relation $\vec{\nabla}\times v_i(\vec{\nabla}H_i) = \vec{\nabla}.\vec{v}(\vec{\nabla}\times\vec{H})$, which gives zero for $\vec{v} = \text{constant}$, i.e., $\vec{\nabla}\times v_i(\vec{\nabla}H_i) = 0$, we obtain

$$\begin{cases} -\Delta\vec{E} + \vec{\nabla}(\vec{\nabla}\vec{E}) + \vec{\nabla}\times\left(\frac{\partial\vec{H}}{\partial t}\right) = \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \\ -\Delta\vec{H} + \vec{\nabla}(\vec{\nabla}\vec{H}) - \vec{\nabla}\times\left(\frac{\partial\vec{E}}{\partial t}\right) = \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \\ -\Delta\vec{E}' + \vec{\nabla}(\vec{\nabla}\vec{E}') + \vec{\nabla}\times\left(\frac{\partial\vec{H}'}{\partial t}\right) = -\vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \\ -\Delta\vec{H}' + \vec{\nabla}(\vec{\nabla}\vec{H}') - \vec{\nabla}\times\left(\frac{\partial\vec{E}'}{\partial t}\right) = \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) \end{cases} . \quad (15)$$

From the system (6) we have,

$$\begin{cases} \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = -\frac{\partial^2\vec{H}}{\partial t^2} + v_i\frac{\partial(\vec{\nabla}H_i)}{\partial t} + m\frac{\partial}{\partial t}(\vec{J}_a) \\ \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}) = \frac{\partial^2\vec{E}}{\partial t^2} - v_i\frac{\partial(\vec{\nabla}E_i)}{\partial t} + m\frac{\partial}{\partial t}(\vec{J}_v) \\ \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}') = -\frac{\partial^2\vec{H}'}{\partial t^2} - v'_i\frac{\partial(\vec{\nabla}H'_i)}{\partial t} - m\frac{\partial}{\partial t}(\vec{J}_v) \\ \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}') = \frac{\partial^2\vec{E}'}{\partial t^2} - v'_i\frac{\partial(\vec{\nabla}E'_i)}{\partial t} + m\frac{\partial}{\partial t}(\vec{J}_a) \end{cases} . \quad (16)$$

Finally replacing expressions (16) in the system (15), we obtain

$$\begin{cases} \Delta\vec{E} - \frac{\partial^2\vec{E}}{\partial t^2} = \vec{\nabla}(\vec{\nabla}\vec{E}) - \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) - v_i\frac{\partial(\vec{\nabla}E_i)}{\partial t} + m\frac{\partial}{\partial t}(\vec{J}_v) \\ \Delta\vec{H} - \frac{\partial^2\vec{H}}{\partial t^2} = \vec{\nabla}(\vec{\nabla}\vec{H}) - \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) - v_i\frac{\partial(\vec{\nabla}H_i)}{\partial t} - m\frac{\partial}{\partial t}(\vec{J}_a) \\ \Delta\vec{E}' - \frac{\partial^2\vec{E}'}{\partial t^2} = \vec{\nabla}(\vec{\nabla}\vec{E}') + \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{E}\times\vec{E}'+\vec{H}\times\vec{H}'}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) - v'_i\frac{\partial(\vec{\nabla}E'_i)}{\partial t} + m\frac{\partial}{\partial t}(\vec{J}_v) \\ \Delta\vec{H}' - \frac{\partial^2\vec{H}'}{\partial t^2} = \vec{\nabla}(\vec{\nabla}\vec{H}') - \vec{\nabla}\times\left(\sqrt{2}m\frac{\vec{H}\times\vec{E}'+\vec{H}'\times\vec{E}}{(\vec{E}\vec{E}'+\vec{H}\vec{H}')^{1/2}}\right) - v'_i\frac{\partial(\vec{\nabla}H'_i)}{\partial t} - m\frac{\partial}{\partial t}(\vec{J}_a) \end{cases} . \quad (17)$$

In the simple case of plane waves, the system (17) splits into four independent similar equations for strengths

$$\Delta\vec{E} - \frac{\partial^2\vec{E}}{\partial t^2} = 0, \quad (18)$$

$$\Delta\vec{H} - \frac{\partial^2\vec{H}}{\partial t^2} = 0\ , \quad (19)$$

$$\Delta\vec{E}' - \frac{\partial^2\vec{E}'}{\partial t^2} = 0, \quad (20)$$

$$\Delta\vec{H}' - \frac{\partial^2\vec{H}'}{\partial t^2} = 0. \quad (21)$$

4. DISCUSSION AND CONCLUSION

In this work, we investigated the tensor wave equations for electron field. First, spinor Dirac equation for electron has been written in tensor form, in the form of non-linear Maxwell's like equations for two electromagnetic fields $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$. From these first order non-linear tensor wave equations for electron, we derived the second order non-linear tensor wave equations for electron field. We proved that, in the simple case of plane waves, the strengths $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$ satisfy the same homogeneous wave equations as those satisfied by $(\vec{E},\vec{H})$, components of electromagnetic field in vacuum. This again proves the similarity between Dirac theory for electron field and Maxwell's theory for electromagnetic field.

5. REFERENCES

1. Bulikunzira,S., Tensor formulation of Dirac equation through divisors, Asian Journal of Fuzzy and Applied Mathematics, vol.2, no6, pp.195-197, December, 2014.
2. Bulikunzira,S., Tensor formulation of Dirac equation in standard representation, Asian Journal of Fuzzy and Applied Mathematics, vol.2, no6, pp.203-208, December, 2014.
3. Bulikunzira,S., Solution of Dirac equation in tensor formalism in standard representation, Asian Journal of Fuzzy and Applied Mathematics, vol.3, no1, pp.1-6, February, 2015.
4. Reifler, F., Vector wave equation for neutrinos, Journal of mathematical physics, vol.25, no4, pp.1088-1092, 1984.
5. Sommers, P., Space spinors, Journal of mathematical physics, vol.21, no10, pp.2567-2571, 1980.