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On Fuzzy Topological Spaces induced by a Given Function

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Abstract - Given a nonempty set X and a function $f: X \to X$, three fuzzy topological spaces are introduced. Some properties of these spaces and relation among them are studied and discussed.

Keywords - Fuzzy points, fuzzy sets and fuzzy topological spaces.

1. Introduction

Let X be a nonempty set and $f: X \to X$ be a function where $f^{n+1} = f^n o f$ and $f^{-n-1} = f^{-n} o f^{-1}$, $\forall n \in \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers). Using this function, we introduce three fuzzy topological spaces, we study and discuss some properties of these spaces like compactness, connectedness. Finally, we give necessary and sufficient conditions under which some of these spaces coincide.

2. Construction of the spaces

We introduce the first fuzzy topology as follows:

Let X be a nonempty set and $f: X \to X$ a function. For each $n \in \mathbb{N}$, define the fuzzy set $\mathcal{A}_n = \{\left(x, \mu_{\mathcal{A}_n}(x)\right) : x \in X\}$ where

$$\mu_{\mathcal{A}_n}(x) = \begin{cases} 1 & if\ \exists m \in \mathbb{N},\ x = f^m(x) \\ \frac{1}{n} & Otherwise \end{cases}$$

Then we have $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n = X$ and $\mu_{\mathcal{A}_n \cap \mathcal{A}_m}(x) = \begin{cases} 1 & if\ \exists m \in \mathbb{N},\ x = f^m(x) \\ \frac{1}{r} & Otherwise \end{cases}$

where $r = \max\{m, n\}$. Thus, the set $\{\mathcal{A}_n : n \in \mathbb{N}\}$ is a basis for a fuzzy topology on X, we denote it by τ_1.

Example 2.1. Let $f: \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n) = n + 1$, then $\mathcal{A}_n = \{\left(x, \frac{1}{n}\right) : x \in \mathbb{N}\}$ and hence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ is a base for the fuzzy topology τ_1 on $\mathbb{N}$.

Proposition 2.2. In the fuzzy topological space (X, τ_1), if $\nexists m \in \mathbb{N}$, such that $x = f^m(x)$, $\forall x \in X$, then the set $\{{\mathcal{A}_n}^c : n \in \mathbb{N}\}$ is a base for a fuzzy topology on X.

Proof. We have $\mu_{\mathcal{A}_n{}^c}(x) = 1 - \mu_{\mathcal{A}_n}(x) = 1 - \frac{1}{n} = \frac{n-1}{n}$, thus, $\mu_{\mathcal{A}_n{}^c \cap \mathcal{A}_m{}^c}(x) = \frac{k-1}{k}$, where $k = \min\{m, n\}$. Also, $\mu_{\bigcup_{n=1}^{\infty} \mathcal{A}_n{}^c}(x) = \sup\left\{\frac{n-1}{n} : n \in \mathbb{N}\right\} = 1$, thus $\bigcup_{n=1}^{\infty} \mathcal{A}_n{}^c = X$ and this implies that the set $\{\mathcal{A}_n{}^c : n \in \mathbb{N}\}$ is a base for a fuzzy topology on X.

Now, we introduce the second fuzzy topological space.

Let X be a nonempty set and $f: X \to X$ be a function. Define the sets $J_0 = \bigcap_{n\in\mathbb{N}} f^n(X)$, $J_n = f^{n-1}(X) - f^n(X)$, $\forall n \in \mathbb{N}$. Now, for each natural number m, define a fuzzy subset $K_m = \left\{\left(x, \mu_{K_m}(x)\right) : x \in X\right\}$ of X, where $\mu_{K_m}(x) = \begin{cases} p & if\ x \in J_n,\ n > 0 \\ 1 & Otherwise \end{cases}$ and $p = min\left\{1, \frac{n}{m}\right\}$. Since $\bigcup_{m=1}^{\infty} K_m = X$ and $K_m \cap K_l = K_q$ where $q = max\{m, l\}$, the set $\mathbb{B} = \{K_m : m \in \mathbb{N}\}$ is a basis for a fuzzy topology on X denoted by τ_2.

Example 2.3. Let $f: \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n) = \begin{cases} 1 & if\ n = 1 \\ n+2 & Otherwise \end{cases}$. Then $K_1 = \{(1,1), (2,1), (3,1), \dots\} = \mathbb{N}$, $K_2 = \{(1,1), (2, 1/2), (3, 1/2), (4,1), \dots\}$, $K_3 = \{(1,1), (2, 1/3), (3, 1/3), (4, 2/3), (5, 2/3), (6,1), (7,1), \dots\}$ and so on. Since $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$, and $K_1 = \mathbb{N}$, the set $\mathbb{B} = \{K_m : m \in \mathbb{N}\}$ is a base for the topology τ_2 on $\mathbb{N}$.

Lemma 2.4. In the fuzzy topological space (X, τ_2), if f is onto, then τ_2 is the indiscrete fuzzy topology.

Proof. The proof is clear.

Finally, we introduce the third fuzzy topological space as follows:

Let X be a set containing at least one element and $f: X \to X$ be a one to one function, we define a fuzzy topology on X as follows:

Suppose x_0 is a fixed point in X and k is a fixed natural number, let $A(x_0) = \{y \in X : y = f^n(x_0), n \in \mathbb{N}\}$, $N_0 = \{n \in \mathbb{Z} : f^n(x_0) \in A(x_0)\}$, where $\mathbb{Z}$ is the set of integers. Then we define the following fuzzy sets for each n in N_0 and for each x in X:

$$C = \{(x, 1) : x \in X - A(x_0)\}$$

$$\mu_{C_n}(x) = \begin{cases} 1 & if\ x = f^n(x_0) \\ 1/k & if\ x \in A(x_0), x \neq f^n(x_0) \\ 0 & Otherwise \end{cases}$$

Now, we have $C \cap C_n = \emptyset$, $\forall n \in N_0$, $C_n \cap C_m = \left\{\left(x, \frac{1}{k}\right) : x \in A(x_0)\right\} \subseteq C_n$ and $\left(\bigcup_{n\in N_0} C_n\right) \cup \{C\} = X$. Then the collection $\mathbb{B} = \{C\} \cup \{C_n : n \in N_0\}$ is a base for a fuzzy topology on X, we denote it by τ_3.

Example 2.5. Consider the function $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 1$ and take $x_0 = 1$, $k = 2$. Then $C = \emptyset$ and $C_n = \{(n+1, 1)\} \cup \{(m, 1/2) : m \in \mathbb{N} - \{n+1\}\}$ for all $n \in N_0 = \{0, 1, 2, \dots\}$.

Lemma 2.6. Let X be a nonempty set endowed by the fuzzy topology τ_3. Then the following are true:

1. $C = \emptyset$ iff $A(x_0) = X$.
2. The fuzzy topological space (X, τ_3) is a topological space iff $k = 1$.

Proof. The proof is clear.

Theorem 2.7. $f: (X, \tau_1) \to (X, \tau_1)$ is onto iff f is open.

Proof. Let f be onto and $x \in X$. If $x = f^m(x)$, for some $m \in \mathbb{N}$, then $\mu_{A_n}(x) = 1$ and since f is onto, so $f^{-1}(x)$ is exists and one of its value is $f^{m-1}(x)$, thus $\mu_{f(A_n)}(x) = sup\{\mu_{A_n}(y): y \in X, y = f^{-1}(x)\} = \mu_{A_n}(f^{m-1}(x)) = 1$. Also, if $x \neq f^m(x)$, $\forall m \in \mathbb{N}$, then there is not exists $m \in \mathbb{N}$ such that $f^m\big(f^{-1}(x)\big) = f^{-1}(x)$. Otherwise, $f^m(x) = x$ and this is a contradiction. So, $\mu_{A_n}(f^{-1}(x)) = \frac{1}{n}$ and $\mu_{f(A_n)}(x) = sup\{\mu_{A_n}(y): y \in X, y = f^{-1}(x)\} = \frac{1}{n}$. Hence for any x in X, we have $\mu_{A_n}(x) = \mu_{f^{-1}(A_n)}(x)$ and $f^{-1}(A_n) = A_n$, $\forall n \in \mathbb{N}$. Hence f is open.

Now, let f be open, then the image of each basic open fuzzy set is open fuzzy, hence $f(A_n)$ is open fuzzy. Suppose contrary that f is not onto, so there exists y in X such that there is no x in X with $f(x) = y$ and this implies that $\mu_{f(A_n)}(y) = 0$ which means that $f(A_n)$ is not open fuzzy and this contradict with being f is open.

Theorem 2.8. $f: (X, \tau_1) \to (X, \tau_1)$ is continuous if f is one to one.

Proof. Let $x \in X$, if there exists a natural number m such that $x = f^m(x)$, then $\mu_{A_n}(x) = 1$ and $f(x) = f^{m+1}(x)$, so $\mu_{f^{-1}(A_n)}(x) = \mu_{A_n}(f(x)) = 1$. Also, if $x \neq f^m(x)$, $\forall m \in \mathbb{N}$, then $\mu_{A_n}(x) = \frac{1}{n}$ and since f is one to one, $f(x) \neq f\big(f^m(x)\big) = f^m\big(f(x)\big)$, $\mu_{A_n}(f(x)) = \frac{1}{n}$ and so $\mu_{f^{-1}(A_n)}(x) = \mu_{A_n}(f(x)) = \frac{1}{n}$. Hence $\mu_{f^{-1}(A_n)}(x) = \mu_{A_n}(f(x))$, $\forall x \in X$, this means that $f^{-1}(A_n) = A_n$ and f is continuous.

The converse of above theorem need not be true as we see it in the following example.

Example 2.9. Let $f: (\mathbb{N}, \tau_1) \longrightarrow (\mathbb{N}, \tau_1)$ be a function defined by $f(n) = 1$, then f is not one to one and since $A_n = \{1_1, 2_{\frac{1}{n}}, 3_{\frac{1}{n}}, \dots\}$, $\forall n \in \mathbb{N}$, $f^{-1}(A_n) = \{1_1, 2_1, \dots\} = \mathbb{N}$ is open and f is continuous.

Theorem 2.10. $f: (X, \tau_2) \to (X, \tau_2)$ is open iff f is onto.

Proof. Let f be onto, then by Lemma 2.4. we have τ_2 is the indiscrete fuzzy topology and hence f is open.

Now, suppose that f is open. Since $K_1 = X$, $f(K_1) = f(X)$ is open. But $f(X)$ is a nonempty classical set, so $f(K_1)$ must be a classical open set and the only nonempty classical open set is X, hence $f(K_1) = X$ that is $f(X) = X$ and this means that f is onto.

Theorem 2.11. $f: (X, \tau_2) \to (X, \tau_2)$ is continuous if it is onto.

Proof. Let f be onto, then by Lemma 2.4. we have τ_2 is the indiscrete fuzzy topology and hence f is continuous.

The following example shows that the converse of above theorem is not true in general.

Example 2.12. Suppose $f: (\mathbb{N}, \tau_2) \longrightarrow (\mathbb{N}, \tau_2)$ is a function defined by $f(n) = 1$, then f is not onto. But, since $K_m = \{1_1, 2_{\frac{1}{m}}, 3_{\frac{1}{m}}, \dots\}$ and $f^{-1}(K_m) = \{1_1, 2_1, \dots\} = \mathbb{N}$, $\forall m \in \mathbb{N}$, so f is continuous.

Theorem 2.13. $f: (X, \tau_3) \to (X, \tau_3)$ is open iff f is onto.

Proof. Suppose f is onto, we have to prove f is open. We claim that $f(C_n) = C_{n+1}$ and $f(C) = C$. Let $x \in X$, if $x \in A(x_0)$, then there exists $r \in N_0$ such that $x = f^r(x_0)$. Thus,

$\mu_{C_n}(x) = \begin{cases} 1 & if\ n = r \\ \frac{1}{k} & Otherwise \end{cases}$, $\mu_{C_{n+1}}(x) = \begin{cases} 1 & if\ n + 1 = r \\ \frac{1}{k} & Otherwise \end{cases}$ and so

$\mu_{f(C_n)}(x) = \mu_{C_n}(f^{-1}(x)) = \begin{cases} 1 & if\ \ n = r-1 \\ \frac{1}{k} & Otherwise \end{cases} = \begin{cases} 1 & if\ n+1 = r \\ \frac{1}{k} & Otherwise \end{cases} = \mu_{C_{n+1}}(x)$. In another side, since $x \in A(x_0)$ and f is onto, $f^{-1}(x_0) \in A(x_0)$, so $\mu_C(x) = 0$, $\mu_C(f^{-1}(x)) = 0$ and $\mu_{f(C)}(x) = \mu_C(f^{-1}(x)) = 0$. Thus, $\mu_{f(C)}(x) = \mu_C(x) = 0$. Hence $f(C) = C$, $f(C_n) = C_{n+1}$, $\forall x \in X$ and this implies that f is open. Also, if $x \notin A(x_0)$, then $f^{-1}(x) \notin A(x_0)$ because f is onto, so $\mu_C(x) = 1$, $\mu_C(f^{-1}(x)) = 1$ and $\mu_{f(C)}(x) = \mu_C(f^{-1}(x)) =1$. Also $\mu_{C_n}(x) = 0$ and $\mu_{f(C_n)}(x) = \mu_{C_n}(f^{-1}(x)) = 0$. Thus $\mu_{C_n}(x) = \mu_{f(C_n)}(x) = 0$ and we obtain that $f\ is\ open$.

Now, Suppose that f is open, we have to prove f is onto. Let f be a non onto function, so there exists an element y in X with $y \neq f(x)$, $\forall x \in X$. If $y \in A(x_0)$, then there exists $n \in \mathbb{Z}$ such that $y = f^n(x_0)$. So $\mu_{C_n}(y) = 1$, $\mu_{C_{n+1}}(y) = \frac{1}{k}$ and $\mu_{f(C_n)}(y) = \mu_{C_n}(f^{-1}(y)) = 0$. Thus $f(C_n)$ is not open which is contradiction, since f is open. But, if $y \notin A(x_0)$, then $\mu_C(y) = 1$, $\mu_{f(C)}(y) = \mu_C(f^{-1}(y)) = 0$, so $f(C) \neq C$ which means that $f(C)$ is not open and hence f is not open which is contradiction. Thus f is onto.

Lemma 2.14. In (X, τ_3), $f^{-1}(C) = C$.

Proof. For $x \in X$, if $x \notin A(x_0)$, then $x \neq f^n(x_0)$, $\forall n \in \mathbb{Z}$ and so $f(x) \neq f^{n+1}(x_0)$ because f is one to one, so $f(x) \notin A(x_0)$ and $\mu_{f^{-1}(C)}(x) = \mu_C(f(x)) = 1$. If $x \in A(x_0)$, then $f(x) \in A(x_0)$ and $\mu_{f^{-1}(C)}(x) = \mu_C(f(x)) = 0$, hence for $x \in X$, $\mu_{f^{-1}(C)}(x) = \begin{cases} 1 & x \notin A(x_0) \\ 0 & Otherwise \end{cases} = \mu_C(x)$ and $f^{-1}(C) = C$.

Theorem 2.15. $f: (X, \tau_3) \longrightarrow (X, \tau_3)$ is continuous if k=1.

Proof. Let $k = 1$, by Lemma 2.14. $f^{-1}(C) = C$, so $f^{-1}(C)$ is open. Now, we have to show that $f^{-1}(C_n)$ is open, $\forall n \in N_0$., where $\mu_{C_n}(x) = \begin{cases} 1 & x \in A(x_0) \\ 0 & Otherwise \end{cases}$

Let $x \in X$, if $x \in A(x_0)$, then $\mu_{f^{-1}(C_n)}(x) = \mu_{C_n}(f(x)) = 1$. If $x \notin A(x_0)$, then $\forall n \in \mathbb{Z}$, $x \neq f^n(x_0)$ and since f is one to one, $f(x) \neq f^{n+1}(x_0)$, thus $f(x) \notin A(x_0)$ and $\mu_{f^{-1}(C_n)}(x) = \mu_{C_n}(f(x)) = 0$. Therefore, $\mu_{f^{-1}(C_n)}(x) = \mu_{C_n}(x)$, $\forall x \in X$ and $f^{-1}(C_n)$ is open. Hence f is continuous.

The converse of above theorem is not true, for example, consider the function $f: \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = 5$ and put $k = 2$ and $x_0 = 5$, then $\mu_{f^{-1}(C_0)}(n) = \mu_{C_0}(f(n)) = \mu_{C_0}(5) = 1$, so $f^{-1}(C_0) = \mathbb{N}$ and from Lemma 2.14. we have $f^{-1}(C) = C$. Hence f is continuous.

3. Some properties of the spaces

In this section we discuss some properties for the spaces that were introduced in section two and we give necessary and sufficient conditions for the spaces to satisfy some of these properties like compactness and connectedness.

Proposition 3.1. (X, τ_1) is a compact space.

Proof. Let $\{G_\lambda: \lambda \in \Lambda\}$ be an open fuzzy cover for X, then $\exists \lambda_0 \in \Lambda$ such that $G_{\lambda_0} = A_1 = X$. Thus every finite open fuzzy subset of $\{G_\lambda: \lambda \in \Lambda\}$ that containing G_{λ_0} can be consider as a finite fuzzy subcover for X. Hence (X, τ_1) is a compact space.

Proposition 3.2. (X, τ_1) is a connected space.

Proof. Since for every two basic open fuzzy sets A_n, A_m we have either $A_n \subseteq A_m$ or $A_m \subseteq A_n$ depending on the values of m and n that either $m \leq n$ or $n \leq m$ respectively. Hence there are no disjoint open fuzzy sets whose union is X. Therefore, (X, τ_1) is connected.

Proposition 3.3. (X, τ_1) is not $T_0 - space$.

Proof. Since any two distinct fuzzy points (x, p) and (y, p) with equal memberships p are belong to the same basic open fuzzy set A_n where $p \leq \frac{1}{n}$. Hence there is no open set that contain one of the elements not the other and this implies that (X, τ_1) is not a $T_0 - space$.

Proposition 3.4. (X, τ_1) is regular iff there is no an element x in X such that $f^m(x) = x$, for some $m \in \mathbb{N}$.

Proof. Suppose there is no an element x in X such that $f^m(x) = x$, for some $m \in \mathbb{N}$, thus $\nexists x \in X$ with $\mu_{A_n}(x) = 1$, $\forall n \in \mathbb{N}$, so $A_n = \{(x, \frac{1}{n}) : x \in X\}$ and ${A_n}^c = \{(x, \frac{n-1}{n}) : x \in X\}$. Thus there is no element $x \in X$ and closed fuzzy set F with $x \notin F$ and this means that (X, τ_1) is regular.

Now, suppose (X, τ_1) is a regular space. If there is an element $y \in X$ with $\mu_{A_n}(y) = 1$, then y does not belong to any closed fuzzy set, let F be one of such closed fuzzy set. But (X, τ_1) is regular, so there are disjoint open fuzzy sets one containing y and the other containing F, but this contradict the definition of τ_1 that there are no disjoint open fuzzy sets. Hence $\nexists y \in X$ with $\mu_{A_n}(y) = 1$ and this implies that $\nexists x \in X$ such that $f^m(x) = x$, for some $m \in \mathbb{N}$.

Remark 3.5. Since every two open fuzzy sets in (X, τ_1) have nonempty intersection, so there are no disjoint closed fuzzy sets and this implies that (X, τ_1) is a normal space.

Proposition 3.6. The fuzzy topological space (X, τ_i) is a Lindelof space for $(i = 1,2,3)$.

Proof. Since the base of the space (X, τ_i), $i = 1,2,3$, is countable, hence every open cover has countable subcover.

Proposition 3.7. (X, τ_2) is a connected space.

Proof. Since $K_1 = X$ and $K_m \subseteq K_1$, $\forall m \in \mathbb{N}$. So there are no disjoint open fuzzy sets whose union be X. Hence (X, τ_2) is connected.

Proposition 3.8. (X, τ_2) is a compact space.

Proof. Let $A = \{G_\lambda : \lambda \in \Lambda\}$ be a open fuzzy cover for X. Since $K_1 = X$, there exists an element $G_{\lambda_i} \in A$ such that $K_1 \subseteq G_{\lambda_i}$ and this implies that $\{G_{\lambda_i}\}$ is a finite open subcover for X.

Proposition 3.9. (X, τ_2) is not $T_0 - space$.

Proof. Since each basic open fuzzy set contain every element of X with nonzero membership, (X, τ_2) is not $T_0 - space$.

Proposition 3.10. (X, τ_2) is regular iff f is onto.

Proof. Suppose f is onto, then by Lemma 2.4. we have τ_2 is the indiscrete fuzzy topology and hence X is regular.

Now, let X be a regular space and f be a non-onto function, then $A_1 \neq \emptyset$. Take $a \in A_1$, then $\mu_{K_2}(a) = \frac{1}{2}$, so $\{(a, \frac{3}{4})\}$ is a fuzzy point in X and ${K_2}^c$ is a closed fuzzy set not containing the fuzzy point $\{(a, \frac{3}{4})\}$, then there exist two disjoint open fuzzy sets G and H such that ${K_2}^c \subseteq G$ and H containing the fuzzy point $\{(a, \frac{3}{4})\}$. But from the definition of τ_2 we have no disjoint open sets, thus we have a contradiction. Hence f must be onto.

Proposition 3.11. (X, τ_2) is a normal space.

Proof. Since the basic open fuzzy sets have the property that $X = K_1 \supseteq K_2 \supseteq \cdots$, so the closed fuzzy sets have the property that $\emptyset = {K_1}^c \subseteq {K_2}^c \subseteq \cdots$, hence there are no disjoint closed sets and consequently X is normal.

Proposition 3.12. (X, τ_3) is connected iff $C = \emptyset$.

Proof. Suppose X is connected, we have to prove $C = \emptyset$. If $C \neq \emptyset$, then take $A = \bigcup_{n \in N_0} C_n$ and hence C and A are two disjoint nonempty open fuzzy sets whose union is X and this is a contradiction. Thus, $C = \emptyset$.

Now, let $C = \emptyset$, since for any basic open fuzzy sets K_m and K_n we have $K_m \cap K_n = \{(x, \frac{1}{k}) : x \in X\}$ which is nonempty. Hence X is connected.

Proposition 3.13. (X, τ_3) is a compact iff N_0 is finite.

Proof. Suppose N_0 is a finite set say $N_0 = \{n_1, n_2, \dots, n_m\}$ and let $\{G_\lambda : \lambda \in \Lambda\}$ be an open fuzzy cover for X, then there exist G_{λ_1}, G_{λ_2}, ..., G_{λ_m}, G_{λ_0} in $\{G_\lambda : \lambda \in \Lambda\}$ such that $C_{n_i} \subseteq G_{\lambda_i}$ for $i = 1,2,..,m$ and $C \subseteq G_{\lambda_0}$. Hence $\{G_{\lambda_0}, G_{\lambda_1}, G_{\lambda_2}, \dots, G_{\lambda_m}\}$ is a finite open fuzzy subcover for X and this means that X is compact.

Now, let X be a compact space and N_0 infinite set, then $\{C_n : n \in N_0\} \cup \{C\}$ is an open fuzzy cover for X that have no finite subcover, so X is not compact and this contradict with our assumpsion. Thus N_0 must be finite.

Proposition 3.14. (X, τ_3) is $T_0 - space$ iff C has atmost one element and $f(x_0) = x_0$.

Proof. Suppose C has atmost one element and $f(x_0) = x_0$. If $C = \emptyset$, then $X = \{x_0\}$ and X is $T_0 - space$. If $C \neq \emptyset$, then there exists $y \neq x_0$ in X such that $C = \{y\}$, that is $X = \{x_0, y\}$ and there is an open fuzzy set C contain y but not x_0. Hence X is $T_0 - space$.

Now, let (X, τ_3) be a $T_0 - space$. If $f(x_0) \neq x_0$, then there exists an element y in X such that $f(x_0) = y$ and this means that $x_0 \neq y$ and there is no open fuzzy set contains only one of them, so X is not $T_0 - space$ which is contradiction. Also, if C contains two elements, then it makes X to be non $T_0 - space$. Hence must $f(x_0) = x_0$ and C contains atmost one element.

Proposition 3.15. (X, τ_3) is regular iff $f(x_0) = x_0$.

Proof. Suppose $f(x_0) = x_0$ and $x \in X$. Let F be a nonempty closed fuzzy set not containing x. We have two cases; first, if $= x_0$, then $F = {C_0}^c$, so there exist two disjoint open fuzzy sets C and C_0 such that $F \subseteq C$ and $x \in C_0$. Second, if $\neq x_0$, then $x \in C$ and $F = C^c$, so there exist two disjoint open fuzzy sets C and C_0 such that $F \subseteq C_0$ and $x \in C$. In both cases we conclude that X is regular.

Now, let X be a regular space and $f(x_0) \neq x_0$. Put $x = x_0$ and $y = f(x_0)$, then for the closed fuzzy set ${C_0}^c$ not containing x, there exist two disjoint open fuzzy sets G and H such that $x \in G$ and ${C_0}^c \subseteq H$. Since $x \notin H$, $H = C$, but $\mu_{{C_0}^c}(y) = \frac{k-1}{k}$ and $\mu_C(y) = 0$, so ${C_0}^c \nsubseteq C$ which is a contradiction. Hence $x = y$, that is $f(x_0) = x_0$.

Proposition 3.16. (X, τ_3) is a normal space.

Proof. To prove X is normal, we have two cases;

1. If $f(x_0) = x_0$ and $C \neq \emptyset$, then there are only two nonempty disjoint closed fuzzy sets C^c and ${C_0}^c$, so there are two disjoint open fuzzy sets C and C_0 such that ${C_0}^c \subseteq C$, $C^c \subseteq C_0$ and hence X is normal.
2. If either $f(x_0) \neq x_0$ or $C = \emptyset$, then we have no disjoint nonempty closed fuzzy sets, thus X is normal.

4. Relation among the spaces

In this section we study the necessary conditions for some of the spaces to be coincide. For this purpose we have the following theorems.

Theorem 4.1. The two fuzzy topologies τ_1 and τ_2 are equal iff for every $x \in X$, there exists $m \in \mathbb{N}$ such that $f^m(x) = x$.

Proof. Assume that for every $x \in X$, there exists $m \in \mathbb{N}$ such that $f^m(x) = x$, then $K_m = X$, $\forall m \in \mathbb{N}$ and τ_2 is the indiscrete fuzzy topology. Furthermore, we have from the assumption that for every $x \in X$, $\mu_{A_n}(x) = 1$, so τ_1 is the indiscrete fuzzy topology. Therefore, $\tau_1 = \tau_2$.

Now, suppose that $\tau_1 = \tau_2$ and according to the definitions of τ_1 and τ_2, we have for every $x \in X$, $\mu_{\mathcal{A}_n}(x) = \begin{cases} 1 & if\ \exists m \in \mathbb{N},\ x = f^m(x) \\ \frac{1}{n} & Otherwise \end{cases}$ and $\mu_{K_m}(x) = \begin{cases} min\{1, \frac{i}{m}\} & if\ x \in J_i,\ i > 0 \\ 1 & Otherwise \end{cases}$. By contrary that if there exists x in X such that for every natural number m, $f^m(x) \neq x$, then $\mu_{\mathcal{A}_n}(x) = \frac{1}{n}$, $\mu_{K_m}(x) = min\{1, \frac{i}{m}\}$ which they are not equal. Therefore, for every $x \in X$, there exists $m \in \mathbb{N}$ such that $f^m(x) = x$ and this completes the proof.

Theorem 4.2. The two fuzzy topologies τ_1 and τ_3 are never be equal.

Proof. From the fuzzy topology τ_3, $\mu_C(x_0) = 0$, but there is no open fuzzy set G in τ_1 and x in X such that $\mu_G(x) = 0$. Thus $\tau_1 \neq \tau_3$.

Theorem 4.3. The two fuzzy topologies τ_2 and $\tau_3$are equal iff f is onto, $A(x_0) = X$ and $k = 1$.

Proof. Suppose f is onto, $A(x_0) = X$ and $k = 1$, then $A_n = \emptyset$, $\forall n > 0$ and $A_0 = X$, so $\mu_{K_m}(x) = 1$, $\forall x \in X$, $\forall m \in \mathbb{N}$ and hence τ_2 is the indiscrete fuzzy topology. Also, since $A(x_0) = X$, $C = \emptyset$ and $\mu_{C_n}(x) = 1$. Thus $C_n = X$, $\forall n \in N_0$ and τ_3 is the indiscrete fuzzy topology. Hence $\tau_2 = \tau_3$.

Now, let $\tau_2 = \tau_3$, since for each x in X, we have $\mu_{K_m}(x) = \begin{cases} min\{1, \frac{i}{m}\} & if\ x \in J_i,\ i > 0 \\ 1 & Otherwise \end{cases}$

and $\mu_{C_n}(x) = \begin{cases} 1 & if\ x = f^n(x_0) \\ 1/k & if\ x \in A(x_0), x \neq f^n(x_0) \\ 0 & Otherwise \end{cases}$

So they are equal if $\frac{i}{m} \geq 1$, $C = \emptyset$ and $\frac{1}{k} = 1$ that is $i \geq m$, $A(x_0) = X$ and $k = 1$. But for K_2, we have $i \geq 2$ and this implies that $J_1 = \emptyset$. Hence f is onto, $A(x_0) = X$ and $k = 1$.

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