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Formulation of Conservation Law of Energy for Electron Field in Tensor Formalism

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ABSTRACT--- ***In previous works, Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell´s like equations through two isotropic complex vectors $\vec{F}=\vec{E}+i\vec{H}$ and $\vec{F}'=\vec{E}'-i\vec{H}'$. The complex vectors $\vec{F}=\vec{E}+i\vec{H}$ and $\vec{F}'=\vec{E}'-i\vec{H}'$ satisfy non-linear condition $\vec{F}^2=0$, equivalent to two conditions for real quantities $\vec{E}^2-\vec{H}^2=0$ and $\vec{E}.\vec{H}=0$, obtained by equating to zero separately real and imaginary parts of equality $\vec{F}^2=0$. It has been proved, that the vectors $\vec{E}$ and $\vec{H}$ have the same properties as those of $\vec{E},\vec{H}$, components of electromagnetic field.***

In this work, in order to understand these non-linear equations, we investigated the deducted from them law of conservation of energy for electron field.

Keywords--- Dirac equation, tensor, conservation law, energy

1. INTRODUCTION

In previous works, using Cartan map, Dirac equation for half-spin particle such as electron has been written in tensor form, in the form of non-linear Maxwell's like equations for two "electromagnetic fields" $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$. In these works, it has been proved, that the vectors $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$ have the same properties as those of $(\vec{E},\vec{H})$, components of the antisymmetric tensor $F_{\mu\nu}$ of electromagnetic field. For example, the solution of Dirac equation in tensor form for free particle as well fulfils Maxwell´s equations for vacuum (with zero at the right side).

In this work, in order to understand better these non-linear tensor equations for electron, we shall derive from them the law of conservation of energy.

2. RESEARCH METHOD

In this work, in order to derive the law of conservation of energy for electron field in tensor formalism, we shall use the general mathematical method for derivation of conservation laws, used in classical field theory. In particular, we shall use the method often used in the derivation of the conservation law of energy for electromagnetic field. We shall write non-linear Maxwell's like equations for two "electromagnetic fields" $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$, corresponding to Dirac equation in tensor formalism. Using the above mentioned method and applying the well known theorems and identities of vector analysis, we shall derive the law of conservation of energy for electron field in tensor formalism.

3. CONSERVATION LAW OF ENERGY FOR ELECTRON FIELD IN TENSOR FORMALISM

In previous works, using Cartan map, Dirac equation for electron

$$(\gamma_\mu \partial_\mu - m)\,\psi = 0, \qquad (1)$$

has been written in tensor form, through two isotropic complex vectors $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{F}' = \vec{E}' - i\vec{H}'$ as follows

$$\begin{cases} D_0\vec{F} + v_i\left(\vec{D}\,F_i\right) - i\vec{D}\times\vec{F} = -\frac{m}{\sqrt{2}}\frac{\vec{F}\times\vec{F}'}{\left(\vec{F}\vec{F}'\right)^{1/2}} \\ D_0\vec{F}' - v'_i\left(\vec{D}F_i{}'\right) + i\vec{D}\times\vec{F}' = -\frac{m}{\sqrt{2}}\frac{\vec{F}\times\vec{F}'}{\left(\vec{F}\vec{F}'\right)^{1/2}} \end{cases}. \qquad (2)$$

Where

$$D_0 = \frac{i}{2}\frac{\partial}{\partial t},$$

$$\vec{D} = -\frac{i}{2}\vec{\nabla}, \qquad (3)$$

$$\vec{v} = \frac{\vec{E}\times\vec{H}}{\vec{E}^2}.$$

Here we use the natural system of units in which c=ħ=1.

Separating real and imaginary parts in equations (2), we obtain a system of non-linear Maxwell´s like equations for two electromagnetic fields ($\vec{E}$,$\vec{H}$) and ($\vec{E}'$,$\vec{H}'$)

$$\begin{cases} \operatorname{rot}\vec{E} + \frac{\partial\vec{H}}{\partial t} = v_i\left(\vec{\nabla}H_i\right) + m\vec{j}_a \\ \operatorname{rot}\vec{H} - \frac{\partial\vec{E}}{\partial t} = -v_i\left(\vec{\nabla}E_i\right) + m\vec{j}_v \\ \operatorname{rot}\vec{E}' + \frac{\partial\vec{H}'}{\partial t} = -v'_i\left(\vec{\nabla}H'_i\right) - m\vec{j}_a \\ \operatorname{rot}\vec{H}' - \frac{\partial\vec{E}'}{\partial t} = v'_i\left(\vec{\nabla}E'_i\right) + m\vec{j}_v \end{cases}. \qquad (4)$$

Here

$$\vec{j}_a = \sqrt{2}\,\frac{\left(\vec{E}\times\vec{E}' + \vec{H}\times\vec{H}'\right)\cos\varphi/2 + \left(\vec{H}\times\vec{E}' + \vec{H}'\times\vec{E}\right)\sin\varphi/2}{\left[\left(\vec{E}\vec{E}'\right)^2 + \left(\vec{H}\vec{H}'\right)^2 + 2\left(\vec{E}\vec{E}'\right)\left(\vec{H}\vec{H}'\right) + \left(\vec{E}\vec{H}'\right)^2 + \left(\vec{E}'\vec{H}\right)^2 + 2\left(\vec{E}\vec{H}'\right)\left(\vec{E}'\vec{H}\right)\right]^{1/4}}, \qquad (5)$$

$$\vec{j}_v = -\sqrt{2}\,\frac{\left(\vec{E}\times\vec{E}' + \vec{H}\times\vec{H}'\right)\sin\varphi/2 + \left(\vec{H}\times\vec{E}' + \vec{H}'\times\vec{E}\right)\cos\varphi/2}{\left[\left(\vec{E}\vec{E}'\right)^2 + \left(\vec{H}\vec{H}'\right)^2 + 2\left(\vec{E}\vec{E}'\right)\left(\vec{H}\vec{H}'\right) + \left(\vec{E}\vec{H}'\right)^2 + \left(\vec{E}'\vec{H}\right)^2 + 2\left(\vec{E}\vec{H}'\right)\left(\vec{E}'\vec{H}\right)\right]^{1/4}}. \qquad (6)$$

$$\varphi = \operatorname{arctag}\frac{\vec{E}'\vec{H} - \vec{E}\vec{H}'}{\vec{E}\vec{E}' + \vec{H}\vec{H}'}. \qquad (7)$$

In particular, if φ=0, i.e., $\vec{E}//\vec{E}'$ and $\vec{H}//\vec{H}'$, we find a simple system

$$\begin{cases} \mathrm{rot}\,\vec{E} + \frac{\partial \vec{H}}{\partial t} = v_i(\vec{\nabla} H_i) + \sqrt{2}m \frac{\vec{E}\times\vec{E}' + \vec{H}\times\vec{H}'}{(\vec{E}\vec{E}' + \vec{H}\,\vec{H}')^{1/2}} \\ \mathrm{rot}\,\vec{H} - \frac{\partial \vec{E}}{\partial t} = -v_i(\vec{\nabla} E_i) + \sqrt{2}m \frac{\vec{H}\times\vec{E}' + \vec{H}'\times\vec{E}}{(\vec{E}\vec{E}' + \vec{H}\,\vec{H}')^{1/2}} \\ \mathrm{rot}\,\vec{E}' + \frac{\partial \vec{H}'}{\partial t} = -v'_i(\vec{\nabla} H'_i) - \sqrt{2}m \frac{\vec{E}\times\vec{E}' + \vec{H}\times\vec{H}'}{(\vec{E}\vec{E}' + \vec{H}\,\vec{H}')^{1/2}} \\ \mathrm{rot}\,\vec{H}' - \frac{\partial \vec{E}'}{\partial t} = v'_i(\vec{\nabla} E'_i) + \sqrt{2}m \frac{\vec{H}\times\vec{E}' + \vec{H}'\times\vec{E}}{(\vec{E}\vec{E}' + \vec{H}\,\vec{H}')^{1/2}} \end{cases} \qquad (8)$$

Now, we shall derive the law of conservation of energy for electron field.

Multiplying the second equation of the system (4) by $\vec{E}$, we find

$$\vec{E}\,\mathrm{rot}\,\vec{H} - \vec{E}\frac{\partial \vec{E}}{\partial t} = -\vec{E}v_i(\vec{\nabla} E_i) + m\vec{E}\vec{j}_v. \qquad (9)$$

Using a well known formula in vector analysis

$$\vec{E}\,\mathrm{rot}\,\vec{H} - \vec{H}\,\mathrm{rot}\,\vec{E} = \mathrm{div}\,\vec{E}\times\vec{H}, \qquad (10)$$

we obtain

$$\vec{H}\,\mathrm{rot}\,\vec{E} - \vec{E}\frac{\partial \vec{E}}{\partial t} - \mathrm{div}\,\vec{E}\times\vec{H} = -\vec{E}v_i(\vec{\nabla} E_i) + m\vec{E}\vec{j}_v. \qquad (11)$$

Using the first equation of the system (4) we find after some rearrangements

$$-\vec{H}\frac{\partial \vec{H}}{\partial t} - \vec{E}\frac{\partial \vec{E}}{\partial t} - \mathrm{div}\,\vec{E}\times\vec{H} = -\vec{E}v_i(\vec{\nabla} E_i) - \vec{H}v_i(\vec{\nabla} H_i) + m(\vec{E}\vec{j}_v - \vec{H}\vec{j}_a). \qquad (12)$$

Similarly, the third and fourth equations of system (4) give

$$-\vec{H}'\frac{\partial \vec{H}'}{\partial t} - \vec{E}'\frac{\partial \vec{E}'}{\partial t} - \mathrm{div}\,\vec{E}'\times\vec{H}' = \vec{E}'v'_i(\vec{\nabla} E'_i) + \vec{H}'v'_i(\vec{\nabla} H'_i) + m(\vec{E}'\vec{j}_v + \vec{H}'\vec{j}_a). \qquad (13)$$

Equations (12) and (13) can be written in the form

$$-\frac{\partial w}{\partial t} - \mathrm{div}\vec{s} = -\vec{E}v_i(\vec{\nabla} E_i) - \vec{H}v_i(\vec{\nabla} H_i) + m(\vec{E}\vec{j}_v - \vec{H}\vec{j}_a), \qquad (14)$$

$$-\frac{\partial w'}{\partial t} - \mathrm{div}\vec{s}' = \vec{E}'v'_i(\vec{\nabla} E'_i) + \vec{H}'v'_i(\vec{\nabla} H'_i) + m(\vec{E}'\vec{j}_v + \vec{H}'\vec{j}_a). \qquad (15)$$

Where

$$w = \frac{1}{2}(\vec{E}^2 + \vec{H}^2), \quad \vec{s} = \vec{E}\times\vec{H}, \qquad (16)$$

$$w' = \frac{1}{2}\left(\vec{E}'^2 + \vec{H}'^2\right), \quad \vec{s}' = \vec{E}'\times\vec{H}'. \qquad (17)$$

Combining formulas (14) and (15), we find

$$-\frac{\partial W}{\partial t} - \mathrm{div}\vec{S} = -\vec{E}v_i(\vec{\nabla} E_i) - \vec{H}v_i(\vec{\nabla} H_i) + \vec{E}'v'_i(\vec{\nabla} E'_i) + \vec{H}'v'_i(\vec{\nabla} H'_i) + m((\vec{E}+\vec{E}')\,\vec{j}_v + (\vec{H}' - \vec{H})\vec{j}_a). \qquad (18)$$

Here

$$W = w + w', \tag{19}$$

$$\vec{S} = \vec{s} + \vec{s}'. \tag{20}$$

If the right side of formula (18) vanishes (for example, in the case of a plane wave) we obtain the continuity equation expressing the law of conservation of energy

$$\frac{\partial W}{\partial t} + \mathrm{div}\vec{S} = 0. \tag{21}$$

4. DISCUSSION AND CONCLUSION

In this work, we investigated the law of conservation of energy for electron field in tensor formalism. Starting with the non-linear tensor equations, equivalent to spinor Dirac equation for electron and using the general method for derivation of conservation laws, we found the general formula (18), expressing the conservation law of energy for electron field in terms of vectors ($\vec{E}$,$\vec{H}$) and ($\vec{E}'$,$\vec{H}'$). We see that, apart the different from zero right side in formula (18), this expression is exactly analogous to that obtained for electromagnetic field. In particular, if the right side of formula (18) vanishes (For example, in the case of plane wave) we obtain the well known equation of continuity (21). Once again, this result shows the similarity between electron field and electromagnetic field.

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