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Tensor Formulation of Dirac Equation in Standard Representation

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ABSTRACT--- ***In previous works, via Cartan map, Dirac equation for half-spin particle has been written in tensor form by using the spinor representation of Dirac matrices. However, it is well known, that there exist different representations of Dirac matrices. The most used representations are the standard and the spinor representations. It has been proved that, Dirac equation written in tensor form by using the spinor representation takes the form of non-linear Maxwell´s like equations for two electromagnetic fields ($\vec{E},\vec{H}$) and ($\vec{E}',\vec{H}'$).To compare the results of both the spinor and the standard representations, in this work, we wrote Dirac equation in tensor form, using the standard representation.***

Keywords--- Dirac equation, Cartan map, tensor form, standard representation.

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1. INTRODUCTION

In previous works, Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell´s like equations for two electromagnetic fields ($\vec{E},\vec{H}$) and ($\vec{E}',\vec{H}'$). In these works, to write Dirac equation in tensor form, Cartan map has been used. First, Dirac equation has been written in components by using the spinor representation of Dirac matrices. However, it is known that there exist different representations of Dirac matrices, the very commonly used being the spinor and the standard representations. In this work, we shall write Dirac equation in tensor form by using the standard representation of Dirac matrices. We shall prove that the form of Dirac equation in tensor form varies with the used representation of Dirac matrices.

2. RESEARCH METHOD

In this work we shall use Cartan map elaborated at the beginning of the twentieth century. This transformation will help us to pass from the two dimensional space of spinors C^2 into the four dimensional space of four vectors C^4. Thus, via Cartan map, we shall derive tensor equations, which are exactly equivalent to spinor Dirac equation for half-spin particle, like electron.

3. SPINOR FORMULATION OF DIRAC EQUATION

Relativistic particle with spin $1/2$ and different from zero rest mass is described by the wave equation, proposed by Dirac in 1928. This equation, written in symmetric form is

$$(\gamma^{\mu}\,\partial_{\mu} - m)\psi=0. \qquad (1)$$

Here γ^{μ} are square matrices of 4^{th}rank, satisfying the relations (Klifford-Dirac algebra)

$$\gamma_{\mu}\gamma_{\nu}+\gamma_{\nu}\gamma_{\mu}=2\delta_{\mu\nu}, \qquad (2)$$

Where μ,ν=0,1,2,3.

It is natural to emphasize that, in general, Dirac matrices γ^{μ} are defined with accuracy to correspondence transformation. Thus, the representation of these matrices can be chosen in different forms. Ordinary, it is commonly used the representation of Dirac matrices in which γ^0 is diagonal:

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}. \qquad (3)$$

Here $\vec{\sigma}$ are second rank Pauli spin matrices, having the form

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (4)$$

This representation is often called the standard representation.

In this representation, Dirac bispinor ψ is written as

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \qquad (5)$$

Here φ, χ are tridimensional (but two components) Pauli spinors.

Using formulas (3) and (5), equation (1) can be written in the form of a system of two equations:

$$\begin{cases} p_0\varphi - (\vec{p}\vec{\sigma})\chi = -m\,\varphi \\ p_0\chi - (\vec{p}\vec{\sigma})\varphi = m\chi \end{cases}. \qquad (6)$$

Another representation of Dirac matrices is the spinor representation. In this representation γ^μ-matrices and Dirac bispinor ψ are written in the form

$$\gamma^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \ \vec{\gamma} = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \qquad (7)$$

$$\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \qquad (8)$$

With the help of formulas (7) and (8), Dirac equation (1) can be written in the form of a system of two equations

$$\begin{cases} \left(p_0 + (\vec{p}\vec{\sigma})\right)\eta = m\xi \\ \left(p_0 - (\vec{p}\vec{\sigma})\right)\xi = m\eta. \end{cases} \qquad (9)$$

It follows from equation (1), that each component of the wave function ψ satisfies the Klein-Gordon equation

$$(\square - m^2)\psi_i = 0. \qquad (10)$$

Where i=0,1,2,3,4; $\square = \frac{\partial}{\partial t^2} - \vec{\nabla}^2 -$ DÁlembert operator .

4. CARTAN MAP DEFINITION AND ALGEBRAIC PROPERTIES

We shall denote by C^n, the complex vector space of dimension "n". We shall consider only C^2, C^3 and C^4.

Elements of C^2 will be denoted by Geek syllables

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \qquad (11)$$

and will be called spinors.

Elements of C^3 will be denoted by Latin syllables

$$\vec{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}, \qquad (12)$$

and will be called vectors.

Finally, elements of C^4 will be denoted by Latin syllables

$$j_\mu = \begin{bmatrix} j_0 \\ j_x \\ j_y \\ j_z \end{bmatrix}, \qquad (13)$$

and will be called four vectors.

Definition1: Cartan map is a bilinear transformation b from space $C^2 \times C^2$ into space C^4, defined as follows:

$$b^0(\xi,\tau) = -(\xi_1\tau_2 - \xi_2\tau_1) \qquad (14)$$

$$\vec{b}(\xi,\tau) = \begin{bmatrix} \xi_1\tau_1 - \xi_2\tau_2 \\ i(\xi_1\tau_1 + \xi_2\tau_2) \\ -(\xi_1\tau_2 + \xi_2\tau_1) \end{bmatrix}. \qquad (15)$$

From the definitions (14) and (15) follows that b^0 is antisymmetric and $\vec{b}$ is symmetric relative to the change ξ by τ, i.e.,

$$b^0(\xi,\tau) = -b^0(\tau,\xi) \qquad (16)$$

$$\vec{b}(\xi,\tau) = \vec{b}(\tau,\xi). \qquad (17)$$

In particular, for any spinor ξ

$$b^0(\xi,\xi) = 0 \qquad (18)$$

Using the definitions (14)-(15), one can prove the following properties of Cartan map:

Lemma1: For any arbitrary spinors ρ, ξ, τ of space C^2, the following identities are verified

$$\vec{b}(\rho,\xi)\,\vec{b}(\tau,\tau) = -2\, b^0(\rho,\tau)\, b^0(\xi,\xi) \qquad (19)$$

$$\vec{b}(\rho,\xi)\,\vec{b}(\xi,\tau) = -2\, b^0(\rho,\xi)\, b^0(\xi,\tau) \qquad (20)$$

$$\vec{b}(\rho,\tau)\,\vec{b}(\xi,\tau) = \ b^0(\rho,\tau)\, b^0(\xi,\tau) \qquad (21)$$

$$\vec{b}(\xi,\xi)\,\vec{b}(\tau,\tau) = -2 b^0(\xi,\tau)^2 \qquad (22)$$

$$\vec{b}(\xi,\tau)\,\vec{b}(\tau,\xi) = \ b^0(\xi,\tau)^2 \qquad (23)$$

$$\vec{b}(\xi,\xi)\,\vec{b}(\tau,\xi) = 0 \qquad (24)$$

Lemma2: For any two spinors ξ and τ of space C^2, the following identity is verified

$$\vec{b}(\xi,\xi) \times \vec{b}(\tau,\tau) = 2i\, b^0(\xi,\tau)\,\vec{b}(\xi,\tau) \qquad (25)$$

Definition2: If

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in C^2 \qquad (26)$$

is a spinor, then the conjugate spinor ξ^* of the spinor ξ is defined as follows

$$\xi^* = \begin{bmatrix} -\overline{\xi_2} \\ \overline{\xi_1} \end{bmatrix} \in C^2, \qquad (27)$$

Where $\overline{\xi_1}$, $\overline{\xi_2}$ are complex conjugates of spinor components ξ_1 and ξ_2.

Lemma3: For any two spinors ξ and τ of space C^2, the following identities are verified

$$b^0(\xi,\tau^*) = \overline{b^0(\tau,\xi^*)}, \qquad (28)$$

$$\vec{b}(\xi,\tau^*) = \overline{\vec{b}(\tau,\xi^*)}\,, \qquad (29)$$

$$b^0(\xi^*,\tau^*) = \overline{b^0(\xi,\tau)}, \qquad (30)$$

$$\vec{b}(\xi^*,\tau^*) = -\overline{\vec{b}(\xi,\tau)}. \qquad (31)$$

Let us introduce vectors $\vec{F} \in C^3$ and $j_\mu \in C^4$ as follows:

$$\vec{F} = \vec{E} + i\vec{H} = i\vec{b}(\xi,\xi), \qquad (32)$$

$$j_\mu = b_\mu(\xi,\xi^*)\,. \qquad (33)$$

Here $\vec{E}$ and $\vec{H}$ are real vectors.

From formula (31) follows that,

$$\vec{F^*} = \vec{E} - i\vec{H} = \overline{i\vec{b}(\xi,\xi)} = i\,\vec{b}(\xi^*,\xi^*). \qquad (34)$$

Lemma4: From formulas (15) and (32) follows identity

$$\vec{F}^2 = \vec{F}.\vec{F} = 0. \qquad (35)$$

i.e., $\vec{F}$ is isotropic vector.

Formula (35) is equivalent to two conditions, obtained by equating to zero separately real and imaginary parts of equality $\vec{F}^2=0$:

$$\vec{E}^2 = \vec{H}^2, \qquad (36)$$

$$\vec{E}.\vec{H} = 0. \qquad (37)$$

One can also prove, that

$$j_0 = \left[\frac{\vec{F}.\vec{F}^*}{2}\right]^{1/2} = |\vec{E}|,$$

$$\vec{j} = i\,\frac{\vec{F}\times\vec{F}^*}{2j_0} = \frac{\vec{E}\times\vec{H}}{|\vec{E}|}\,. \qquad (38)$$

Lemma5: For any spinor $\xi \in C^2$, the following identities are verified

$$j_0 = |\vec{E}| = |\xi|^2, \qquad (39)$$

$$\vec{j} = \frac{\vec{E}\times\vec{H}}{|\vec{E}|} = \bar{\xi}^T\vec{\sigma}\xi. \qquad (40)$$

Where $\bar{\xi}^T$ is the transposed conjugate of the spinor ξ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli spin matrices.

From formulas (39)-(40) follows that, under Lorentz relativistic transformations j_μ transforms as a four vector. Vectors $\vec{E}$ and $\vec{H}$ transform as components of electromagnetic field, i.e., form a second rank tensor $F_{\mu\nu}$.

Lemma6: For any pair of spinors ξ and τ of space C^2 and any vector $\vec{v}$ the following identities are verified

$$b^0(\vec{v}.\vec{\sigma}\xi,\tau) = \vec{v}.\vec{b}(\xi,\tau), \qquad (41)$$

$$\vec{b}(\vec{v}.\vec{\sigma}\xi,\tau) = \vec{v}b^0(\xi,\tau) + i\,\vec{v}\times\vec{b}(\xi,\tau), \qquad (42)$$

$$\vec{b}(\vec{v}.\vec{\sigma}\xi,\xi) = (\vec{v}.\vec{s})\,\vec{b}(\xi,\xi). \qquad (43)$$

Here $\vec{s} = (s_x, s_y, s_z)$ are Proka spin matrices, with $S_i = i\varepsilon_{ijk}$, where ε_{ijk} is the tridimensional antisymmetric tensor Levi-Cevita.

From formula (43) follows that, if ξ is eigenvector of operator $(\vec{v}.\vec{\sigma})$ with eigenvalue λ, then $\vec{b}(\xi,\xi)$ is eigenvector of operator $(\vec{v}.\vec{s})$ with the same eigenvalue λ.

Definition3: Let ξ be a spinor field and $\widetilde{A}$, an operator acting on ξ. Let $\vec{b}$ maps spinor ξ on isotropic vector $\vec{F} = i\vec{b}(\xi,\xi)$. We shall say that, the operator $\widetilde{A}$ commutes with Cartan map and becomes $\widehat{A}$, acting on $\vec{F}$, if:

$$\widehat{A}\,\vec{F} = i\widehat{A}\,\vec{b}(\xi,\xi) = i\vec{b}(\widetilde{A}\xi,\xi). \qquad (44)$$

From formula (44) follows that, if ξ is eigenvector of operator $\widetilde{A}$ with eigenvalue λ, then $\vec{F}$ is eigenvector of operator $\widehat{A}$ with the same eigenvalue λ; i.e., Cartan map conserves eigenvectors and eigenvalues.

Lemma7: For any spinor ξ of space C^2, the following identities are verified

$$b^0(\vec{p}\xi,\xi) = -i\{\vec{D}\vec{b}(\xi,\xi)\}.\vec{v}, \qquad (45)$$

$$\vec{b}(\vec{p}\xi,\xi) = \vec{D}\vec{b}(\xi,\xi), \qquad (46)$$

Where $\quad \vec{v} = \frac{\vec{j}}{j_0} = \frac{\vec{E}\times\vec{H}}{\vec{E}^2}$

$$\vec{D} = -i\frac{\hbar}{2}\vec{\nabla}.$$

5. DIRAC EQUATION IN TENSOR FORM

Let us consider Dirac equation in standard representation

$$\begin{cases} p_0\varphi - (\vec{p}\vec{\sigma})\chi = m\,\varphi \\ p_0\chi - (\vec{p}\vec{\sigma})\varphi = -m\chi \end{cases} \qquad (47)$$

We shall transform this system of equations by using Cartan map.

Let us begin by transforming the first equation

$$p_0\varphi - (\vec{p}\vec{\sigma})\chi = m\,\varphi. \qquad (48)$$

We have,

$$\vec{b}(p_0\varphi,\varphi) - \vec{b}[(\vec{p}\vec{\sigma})\chi,\varphi] = m\,\vec{b}(\varphi,\varphi). \qquad (49)$$

Using formula (42), the second term of the left side of equation (49) gives

$$\vec{b}[(\vec{p}\vec{\sigma})\chi,\varphi] = \vec{p}\,'b^0(\chi,\varphi) + i\,\vec{p}\,' \times \vec{b}(\chi,\varphi). \qquad (50)$$

Where $\vec{p}\,'$ is the momentum operator, acting only on the first argument of the quantity $\vec{b}(\chi,\varphi)$.

With the help of formula (22), the first term of the right side of equation (50) can be written in the following form

$$\vec{p}\,'b^0(\chi,\varphi) = \vec{p}\,'\left[\frac{\vec{b}(\chi,\chi).\vec{b}(\varphi,\varphi)}{-2}\right]^{1/2}, \qquad (51)$$

Or

$$\vec{p}\,'b^0(\chi,\varphi) = \vec{D}\left[\frac{\vec{b}(\chi,\chi).\vec{b}(\varphi,\varphi)}{-2}\right]^{1/2}. \qquad (52)$$

Using formula (25) the second term of the right side of equation (50) becomes

$$\vec{b}(\chi, \varphi) = \frac{\vec{b}(\chi,\chi)\times\vec{b}(\varphi,\varphi)}{2ib^0(\chi,\varphi)} . \qquad (53)$$

Where, by formula (22), we have

$$b^0(\chi, \varphi) = \left[\frac{\vec{b}(\chi,\chi).\vec{b}(\varphi,\varphi)}{-2}\right]^{1/2} . \qquad (54)$$

Hence, equation (53) takes the form

$$\vec{b}(\chi, \varphi) = \frac{-i}{\sqrt{2}} \frac{[\vec{b}(\chi,\chi)\times\vec{b}(\varphi,\varphi)]}{[-\vec{b}(\chi,\chi).\vec{b}(\varphi,\varphi)]^{1/2}} . \qquad (55)$$

Thus

$$i\vec{p}' \times \vec{b}(\chi, \varphi) = \frac{1}{\sqrt{2}} \vec{p}' \times \frac{[\vec{b}(\chi,\chi)\times\vec{b}(\varphi,\varphi)]}{[-\vec{b}(\chi,\chi).\vec{b}(\varphi,\varphi)]^{1/2}} . \qquad (56)$$

Combining equations (50), (52) and (56), formula (49) can be written as follows

$$D_0\, i\vec{b}(\varphi, \varphi) - \frac{1}{\sqrt{2}} i\vec{D} \left[i\vec{b}(\chi, \chi). i\vec{b}(\varphi, \varphi)\right]^{1/2} + \frac{i}{\sqrt{2}} \vec{D} \times \frac{[i\vec{b}(\chi,\chi)\times i\vec{b}(\varphi,\varphi)]}{[i\vec{b}(\chi,\chi).i\vec{b}(\varphi,\varphi)]^{1/2}} = m\, i\vec{b}(\varphi, \varphi). \qquad (57)$$

Introducing complex isotropic vectors

$$\vec{F} = \vec{E} + i\vec{H} = i\vec{b}(\varphi, \varphi), \qquad (58)$$

$$\vec{F}' = \vec{E}' - i\vec{H}' = i\vec{b}(\chi, \chi), \qquad (59)$$

equation (57) becomes

$$D_0\, \vec{F} - \frac{i}{\sqrt{2}} \vec{D}\left(\vec{F}\vec{F}'\right)^{1/2} + \frac{i}{\sqrt{2}} \vec{D} \times \left[\frac{\vec{F} \times \vec{F}'}{\left(\vec{F}\vec{F}'\right)^{1/2}}\right] = m\, \vec{F}. \qquad (60)$$

In the same way, the second equation of the system (47) can be transformed.

Finally, we obtain Dirac equation in tensor form

$$\begin{cases} D_0\, \vec{F} - \frac{i}{\sqrt{2}} \vec{D}\left(\vec{F}\vec{F}'\right)^{1/2} + \frac{i}{\sqrt{2}} \vec{D} \times \left[\frac{\vec{F} \times \vec{F}'}{\left(\vec{F}\vec{F}'\right)^{1/2}}\right] = m\vec{F} \\ D_0\, \vec{F}' - \frac{i}{\sqrt{2}} \vec{D}\left(\vec{F}\vec{F}'\right)^{1/2} - \frac{i}{\sqrt{2}} \vec{D} \times \left[\frac{\vec{F} \times \vec{F}'}{\left(\vec{F}\vec{F}'\right)^{1/2}}\right] = -m\, \vec{F}'. \end{cases} \qquad (61)$$

6. DISCUSSION AND CONCLUSION

In this work, by using Cartan map, we derived tensor equations given by formulas (61). We proved that these non-linear tensor equations are exactly equivalent to spinor Dirac equation for electron. In this derivation, we used the standard representation of Dirac matrices. From formulas (61), we see that the form of Dirac equation in tensor formalism obtained in this work is different from that obtained in the previous works by using the spinor representation of Dirac matrices. Thus, we can conclude that, the form of Dirac equation in tensor form depends considerably on the used representation of Dirac matrices.

7. REFERENCES

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