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An Evaluation of a Reservoir Elevation with Markov Model in Continuous Time

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ABSTRACT — ***A three-state Markov model in continuous time was used to study the reservoir elevation of a dam. A non-stationary transition probability was assumed for the process. The result indicates a maximum probability of 0.3 and 0.5 in about four years to have middle reservoir and upper reservoir elevation given that it is low reservoir elevation at present. The model could be used to make a forecast of the reservoir elevation of the dam in the future given the present reservoir elevation. The information could be used to plan for the maximum use of the dam resources such as the HEP generation, fresh water fishing and the cultivation of the dam basin for crop production.***

Keywords – Markov Model, Continuous time, Non-stationary, Transition Probability, Reservoir, Elevation

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1. INTRODUCTION

Shiroro dam was built primarily for the production of Hydro electric power (H E P). The water level (Reservoir Elevation) of dam rises and falls seasonally and very unpredictably. Sometimes the reservoir elevation over rises and the run ways are opened for the excess water to overflow. Some other time the water level is so low so that some or all the turbines are shut down due to shortage of water. Some attempt have been made to study this reservoir process with mathematical models. Contained in [5] is the computed water reservoir release policy for Shiroro hydro-electric Dam using a probabilistic dynamic programming model. In the simulation, he used the State variable as the reservoir storage volume and uncertain nature of the inflow process is accounted for in the model by considering different possible inflow volume and their inflow probabilities. A related work is reported in [6], a case of two connected dams; a capture and supply dam with a regular demand of one unit from the supply dam and random inputs into the capture dam. Modeling the system as a Markov chain she derived a transition probability matrix with a general block structure to solve for the invariant state probability vector of the water stored in the dams. The amount of rainfall directly affect the reservoir elevation of the dam and consequently the crop production. Markov Models have been used to study the relationship between rainfall and crop production [8]. Also in [9] is the modified Chapman-Kolmogorov Equation (CKE) and applied it to model the daily precipitation data of Abeokuta, Ogun State, Nigeria. In [7], we have a non-homogeneous Markov chain model where it is used to study the reservoir elevation of Shiroro dam with the effect of the seasons on the transition probabilities and in discrete time unit. This paper is in contrast to [7] because the three state model is considered but for continuous time and non-stationary transition probabilities.. This is to provide for a higher estimate of the transition probabilities and information could be obtained from the model at any time both discrete and continuous.

2. FORMULATION OF MARKOV CHAIN MODEL

Consider a random variable X that is indexed by time parameter n, such a process is referred as stochastic process X_n , n = 0,1,2,… . Suppose that the stochastic process $\{X_n, n = 0, 1, 2, \dots\}$ takes on a finite or countable number of possible values such that:

$$p\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0\} = P_{ij} \quad (1)$$

for all states $i_0, i_1, \dots, i_{n-1}, i, j$ and all n≥0. Such a stochastic process is known as a *Markov chain,[4]*. Since probabilities are nonnegative and since the process must make a transition into some states, we have that.

$$P_{ij} \geq 0, \quad i, j, \geq 0; \quad \sum_{j=0}^{\infty} P_{ij} = 1, \quad i = 0,1, \dots..$$

It has been observed that the reservoir elevation of the Shiroro dam that can tolerate turbine operation has the highest and lowest reservoir elevation of 382.50m and 358.25m respectively. Any short fall below 358.25m in reservoir elevation will amount to shutdown of some or all turbine engines resulting to load shedding.

Similarly, if the water level rises above 382.50m, it incidentally calls for opening of windows for overflows which eventually affects the downstream farming and ecological activities.

Let us consider reservoir elevation of Shiroro dam in a month to be random variable X. Suppose that the random variable is collected for several months to constitute a stochastic process X_n of first order dependence presented in (1). It is assumed that this process is modeled by a three state Markov chain thus:

State 1: Elevation below 367.50m. Low reservoir elevation.

State 2: Elevation between 367.50m. – 376.25m. Middle reservoir elevation.

State 3: Elevation above 376.25m. Upper reservoir elevation.

The possible transitions between the states is described by the following transition diagram.

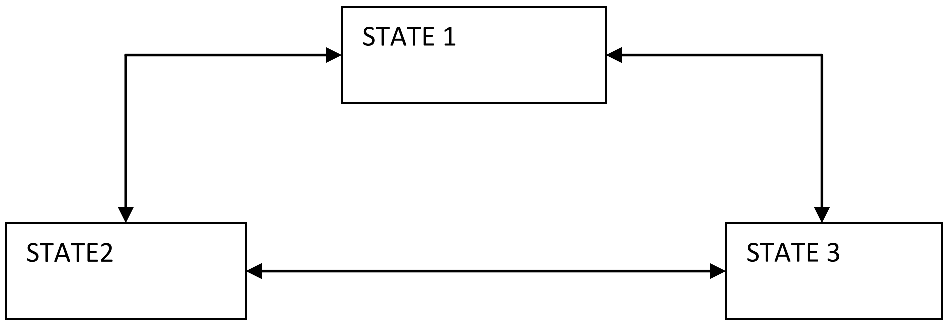

Figure 1: The transition diagram for the process.

Let P_{ij} be the probability that the reservoir elevation presently in state i will make a transition to state j in the next transition.

The transition between states is described by the following transition probability matrix P

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

$$0 \leq p_{ij} \leq 1 \qquad ij = 1,2 \; and \; 3$$

$\sum_{i=1}^{3} p_{ij} = 1 \quad i = 1,2, and\ 3$as contained in [7].

Following [3], we let a_{ij} represent the transition rate of the reservoir elevation from state i to state j, $i \neq j$ in a short time interval (t, t+Δt) the reservoir elevation currently in state i,will make a transition to state j with probability a_{ij} Δt, i≠j. If X_t is the state of the process at time t, then we have

$$P(X_{t+\Delta t=j|X_t=i}) = a_{ij\Delta t}$$

The probability of two or more state transitions is of order $(\Delta t)^2$ or more and it is negligible if Δt is sufficiently small.

Suppose that the transition rate do no change with time (a_{ij}'s are constants) and

$$a_{jj} = -\sum_{i \neq j} a_{ij} \qquad i, j = 1, 2, 3, \ldots \qquad (2)$$

We describe the process by a transition rate matrix A with component a_{ij}

Suppose $P_{ij}(t)$ is the probability that the reservoir elevation is in the state i at time t after the start of the process and let P_j (t + Δ t) be the probability that reservoir elevation will be in state j a short time Δt later. Then

$$P_j(t+\Delta t) = p_j(t)\left[1 - \sum_{i \neq j} a_{ij\Delta t}\right] + \sum_{i \neq j} p_i(t) a_{ij\Delta t} \qquad (3)$$

$$j = 1, 2, 3$$

Equation (3) is obtained by multiplying the probabilities and adding over all i that are not equal to j because the reservoir elevation could have entered j from any other state i putting (2) in (3) and rearranging the terms gives

$$P_j(t+\Delta t) - p_j(t) = \sum_{t=1}^{3} p_i(t) a_{ij} \ \Delta t$$

Thus, we have

$$\frac{dp_j(t)}{dt} = \sum_{i=1}^{3} P_i(t) a_{ij} \qquad j = 1,\ 2 \quad 3$$

in matrix form, we have

$$\frac{dp(t)}{dt} = P(t)A \qquad (4)$$

Infact, equation (4) is an exact (not approximate) differential equations for P_{ij} (t) in

$$\frac{dp_{ij}(t)}{dt} = \sum p_{ik} a_{kj}$$

(chapman kolmogorov differentiated equation) it is a linear, first order differential equation with constant coefficients a_{ij}'s

The elements of A may be further related by extending the properties of P (t). In particular since for each i

$$\sum_j p_{ij}(t) = 1$$

$$then \quad \frac{d}{dt}\left(\sum_j p_{ij}(t)\right)_{t=0} = \left| \frac{d(1)}{dt} \right|_{t=0}$$

$$\sum_j \frac{d}{dt} P_{ij}(t)\Big|_{t=0} = 0$$

$$\sum_j a_{ij} = 0 \qquad (5)$$

That is each row of A must sum to zero, since every off diagonal is non-negative, hence equation (2). The development of the equation that determine the $P_{ij}(t)$ functions, for this process can be simplified if the following assumptions are made:

(1) The process satisfies the Markov property

(2) The process is stationary

(3) The probability of a transition from one state to a different state in a short time interval is proportional to Δt

(4) The probability of two or more changes of state in a short interval Δt is zero

P (t) is a row vector of the state probabilities at time t. to obtain the solution of (4), the initial condition Pi (0); i= 1, 2, 3, must be specified. Taking the Laplace transform of (4)

$$we \quad have \quad P(s) = P(0)\left(SI - A\right)^{-1} \qquad (6)$$

Thus P (t) is obtained as the inverse transform of P (S) from [2].

3. APPLICATION

A summary statistics for 10 years for the monthly reservoir of Shiroro dam is contained in table1.

Table1. A summary of data of reservoir elevations of Shiroro dam (in meters) from 2001 – 2010.

CLASS INTERVAL (M)	STATES	FREQUENCY
358.25 – 367.50	1	50
367.50 – 376.25	2	36
376.25 – 382.50	3	34
TOTAL		120

Source; [7].

From table1, we obtain the transition count matrix of the stationary Markov chain [7]. We have

$$M = \begin{pmatrix} 39 & 7 & 3 \\ 10 & 21 & 6 \\ 0 & 8 & 26 \end{pmatrix}$$

Normalizing this matrix using equation (2) and (5), we have

$$A = \begin{pmatrix} -10 & 7 & 3 \\ 10 & -16 & 6 \\ 0 & 8 & -8 \end{pmatrix}$$

The matrix A can therefore be expressed as the expected value of the exponential distribution (1/λ) thus

$$B = \begin{pmatrix} -0.47 & 0.14 & 0.33 \\ 0.10 & -0.27 & 0.17 \\ 0 & 0.13 & -0.13 \end{pmatrix} \quad \text{corrected to 2 decimal places}$$

Using matrix B from equation (6) we have

$$P(s) = (SI - B)^{-1} = \left[\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} -0.47 & 0.14 & 0.33 \\ 0.10 & -0.27 & 0.17 \\ 0 & 0.13 & -0.13 \end{pmatrix}\right]^{-1}$$

$$P(s) = \begin{pmatrix} s+0.47 & -0.14 & 0.33 \\ 0.1 & s+0.27 & 0.17 \\ 0 & -0.13 & s+0.13 \end{pmatrix} \qquad (7)$$

Solving equation (7) by classical adjoint [1], we obtain the following equations

$P_{12}(t) = 0.3055 - 0.1747\, e^{-0.2972\, t} - 0.1310\, e^{-0.6728\, t}$

$P_{13}(t) = 0.5645 – 0.1326\ e^{-\ 0.2972\ t} - 0.4317\ e^{-\ 0.6728\ t}$

$P_{21}(t) = 0.065 + 0.1498\ e^{-\ 0.2972\ t} - 0.2148\ e^{-\ 0.6728\ t}$

$P_{23}(t) = 0.5645 – 0.5591\ e^{0.2972\ t} - 0.0059\ e^{-\ 0.6728\ t}$

$P_{31}(t) = 0.65 – 0.1165\ e^{-\ 0.2972\ t} + 0.0514\ e^{-\ 0.6728\ t}$

$P_{32}(t) = 0.3055- 0.2016\ e^{-\ 0.2972\ t} - 0.1045\ e^{-\ 0.6728\ t}$

The values of these functions for 12 years that is t = 0,1,2, …, 12 are presented in table2 and illustrated in the figure2 respectively.

Table 2.The values of P12(t), P13(t), P21(t), P23(t), P31(t) and P32(t) for t=0, 1, 2,…………………12

T	$P_{12}(t)$	$P_{13}(t)$	$P_{21}(t)$	$P_{23}(t)$	$P_{31}(t)$	$P_{32}(t)$
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.10887	0.245705	0.066671	0.146137	0.004681	0.102408
2	0.174974	0.37911	0.091738	0.2544	0.014088	0.167028
3	0.216467	0.452775	0.097874	0.334486	0.024064	0.208959
4	0.243407	0.494843	0.096061	0.393806	0.033001	0.23701
5	0.261437	0.51956	0.091463	0.437785	0.040417	0.256267
6	0.27382	0.534589	0.086388	0.470411	0.046324	0.269766
7	0.282503	0.544052	0.081771	0.494626	0.054428	0.279383
8	0.28869	0.550214	0.077909	0.512604	0.054428	0.286317
9	0.293152	0.554349	0.07482	0.525953	0.059097	0.29136
10	0.296398	0.557194	0.072412	0.535867	0.059097	0.295053
11	0.298775	0.559193	0.070566	0.54323	0.0606	0.297768
12	0.300523	0.560619	0.069166	0.548699	0.061724	0.29977

These results is as illustrated in figure2

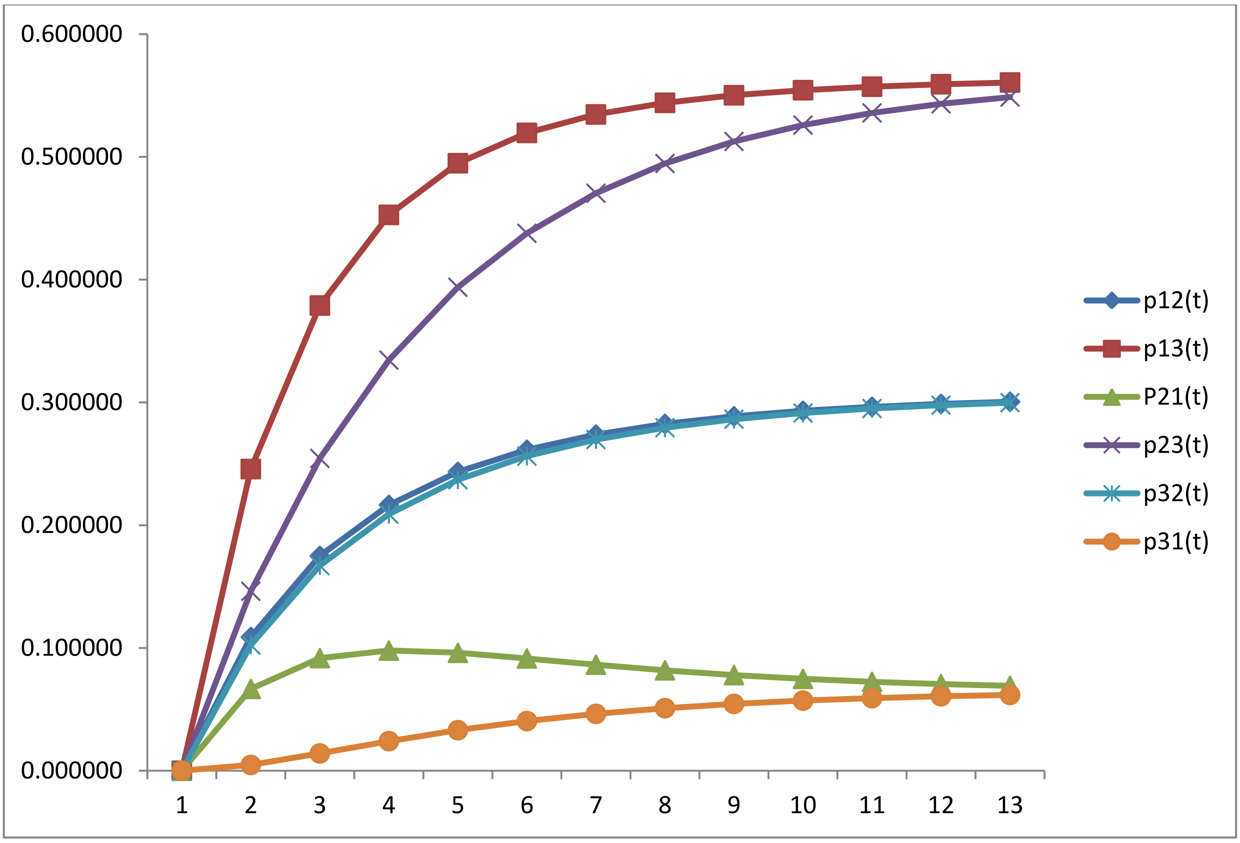

Figure2. The graph of transition probabilities

DISCUSSION OF RESULTS

The transition probabilities for the model in continuous time is presented in table2 and in figure 2 repectively. Table2 shows the values of $P_{ij}(t)$ for i,j = 1,2,3 and t= 1,2,…,12, and illustrated with graphs in figure2. $P_{13}(t)$ is the transition probability that the reservoir elevation presently in state1(low reservoir elevation) will make a transition to state3(upper reservoir elevation) in the next transition. $P_{13}(t)$, t=1,2,3 … provides higher values for the transition probabilities. The result indicates that, given that it is a low reservoir elevation in the preesnt year, it is expected to have a middle reservoir and upper reservoir elevation with an optimal probability of 0.3 and 0.5 respectively in about four years. The second higher transition probability is provided by $P_{23}(t)$ with a maximum value of 0.50 in about six years (ie t=6). Thus given the reservoir elevation at present, it is possible to determine the next reservoir elevation in the following year(s) and in the far future. The predictions could be useful for planning for the optimal use of the dam resources.

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